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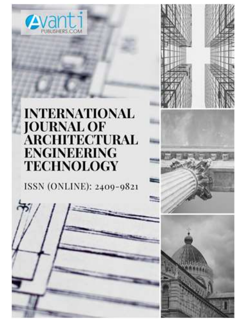

On Self-Motions of Planar Stewart-Gough Platforms

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ABSTRACT

Given five pairs of attachment points of a planar platform, there exists a sixth point pair so that the resulting planar architecturally singular platform has the same solution for the direct kinematics. This is a consequence of the Prix Vaillant problem posed in 1904 by the French Academy of Science. The theorem discusses the displacements of certain or all points of a rigid body that move on spherical paths. Borel and Bricard awarded the prizes for two papers in this regard, but they did not solve the problem completely. In this paper, the theorem is extended to the elliptic paths in order to determine the displacements of certain or all points of a rigid body that move on super-ellipsoid surfaces. The poof is based on the trajectories of moving points which are intersections of two implicit super-ellipsoid surfaces.

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1. Introduction

The French Academy wanted answers to some questions as: Is there a nontrivial motion for which all points move on spheres? How many of such motions exist? How do these motions look like? Which type are the paths? How many degrees of freedom have these motions? [1-3]. By taking two congruent rigid bodies and connecting them with equal length links with spherical joints on both ends, it appears a translation motion and every point of the moving body is constrained to move on a sphere. If a displacement has one fixed point then every point moves on a sphere.

The papers of Borel and Bricard say that one point b of the top moving platform P remains at a fixed distance to one point a belonging to the bottom fixed system P' [4-7]. The condition for rigid body displacements is that all points of the body move on planar paths, not necessarily parallel [8]. Mannheim [9, 10] added the inverse motion where all planes of a rigid body pass during the motion through a fixed point. Nawratil [11-13] shows that the Duporcq's theorem is not correct for the architecturally singular Stewart-Gough manipulators and present a revised version of this theorem. The revised form of the theorem is stated as If five pairs of anchor points (m_i, M_i) with $(m_i, M_i) \neq (m_j, M_j)$ for $i \neq j$ and $i, j \in \{1,2,3,4,5\}$ of a planar manipulator are given, then there exists, with exception of one case, a sixth point pair (m_6, M_6) with $(m_6, M_6) \neq (m_k, M_k)$ for $k = 1,2,...,5$ such that the resulting planar architecturally singular manipulator has the same solution for the direct kinematics as the given one. The pair (m_l, M_l) for $l = 1,2,...,6$ verifying the sphere condition, if m_j and M_l are finite, the Darboux condition, it m_l is finite and M_l is an ideal point, the Mannheim condition, if m_l is an ideal point and M_l i is finite, and the angle condition, if m_l and M_l are ideal points. If the point 6 or the point 6', respectively, does not exist from the Euclidean point of view, as it is a point at infinity. This is an *ideal point*. If the point pair (6, 6') does not exist, it coincides with one of the five-point pairs (i, i'), $i \in \{1,2,...,5\}$.

Bricard [14, 15] discusses three types of deformable octahedra. Each mechanism consists of twelve rigid rods, four of them meeting at a time in the six vertices of the octahedron. All connections of rods are spherical joints. The paths of the moving joint centers are spherical. The positions of P' depending on the time t and the points b and *a.* The coordinates (x, y, z) and (x', y', z') define a point in the moving system and in the fixed system, respectively. The transformation from P' to P is given by

$$\begin{aligned} x_1' &= a + \alpha x + \alpha' y + \alpha'' z \\ y_1 &= b + \beta x + \beta' y + \beta'' z \\ z_1' &= c + \gamma x + \gamma' y + \lambda'' z \end{aligned} \tag{1}$$

where a, b, c are components of the translation vector, and $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$ are the components of a proper orthogonal rotation matrix function of t. The point $b(x, y, z)$ remains at a fixed distance r to the point *a* if

$$\begin{aligned} &0.5(a^2 + b^2 + c^2) + (a\alpha + b\beta + c\gamma) + (a\alpha' + b\beta' + c\gamma')y + (a\alpha'' + b\beta'' + c\gamma'')z + \\ &+ax_1 + by_1 + cz_1 + \alpha x x_1 + \alpha' y x_1 + \alpha'' z x_1 + \beta x y_1 + \beta' y y_1 + \beta'' z y_1 + \gamma x z_1 + \gamma' y z_1 + \gamma'' z z_1 + \\ &+0.5(x_1^2 + y_1^2 + z_1^2 + x^2 + y^2 + z^2 - r^2) = 0 \end{aligned} \tag{2}$$

If the Stewart-Gough platform has six-leg lengths, then the platform is in general rigid, but under particular conditions, it can perform an *n*-parametric motion (*n* > 0), which is called *self-motion*. These are also the solutions to the Borel–Bricard problem [3-5]. It has been previously shown that singular parallel Stewart–Gough manipulators have either elliptic self-motions or pure translational self-motions. The elliptic self-motions are studied in [16]. It was shown that these necessarily 1-parameter self-motions have an instantaneously local second degree of freedom in each pose of the self-motion [17].

The self-motion of the architecturally singular planar Stewart-Gough platforms is extensively studied in [18, 19].

We are interested in self-motions of planar Stewart-Gough platforms, which are not architecturally singular. Nawratil [16] discussed the case where the base anchor points M_i and the platform anchor points m_i are related by a non-singular projectivity κ, i.e.

$$\kappa : \pi_m \to \pi_M \text{ with } m_i \mapsto M_i \ .$$

Nawratil shows that a non-singular projectivity κ is a bijective linear map between two planes. The ideal points of these planes are taken into consideration together to κ which preserves incidences and cross-ratios and map all ideal points again onto ideal points.

It is well known [20] that a planar projective Stewart-Gough platform is architecturally singular if and only if one set of anchor points is located on a conic section which is also reducible. It was shown by the author in [21] that one can attach a two-parametric set of additional legs to planar projective without changing the forward kinematics and singularity surface. The platform anchor points n_i and the base anchor points N_i of these additional legs are also related by κ, i.e. $n_{ik} = N_l$.

Under the assumption that s denotes the line of intersection of πM and πm in the projective extension of the Euclidean 3-space, elliptic self-motion can be defined as follows (cf. [9, def. 1]): Definition 1 A self-motion of a non-architecturally singular planar projective SG platform is called elliptic, if in each pose of this motions exists with s = sκ and the projectivity from s onto itself is elliptic, the projectivity from s onto itself has no real fixed points). It should be noted that Definition 1 implies that neither πM and πm nor two related points of the platform and the base coincide during elliptic self-motion. It was also shown in [9] that non-architecturally singular planar projective SG platforms can either have elliptic self-motions or pure translational ones. The latter is the only self-motions of non-architecturally singular planar affine SG platforms.1 In this case, the projectivity has to be an affinity a + *A*·x, where the singular values s1 and s2 of the 2 × 2 transformation matrix *A* with 0 < s1 ≤ s2 fulfill the condition s1 ≤ 1 ≤ s2. However, it should be mentioned that all planar affine SG platforms, which do not fulfill this condition, only have "ordinary" singularities causing a local shakiness of the manipulator but no actual self-motion. Note that self-motion is dangerous because it is uncontrollable and thus is a hazard to man and machine. Therefore being able to avoid manipulator designs that engender self-motion is of interest to engineers.

The singularity of the Stewart-Gough platforms represents a motivation for focusing on Duporcq's theorem within this article.

In the following let us consider a Stewart-Gough platform with six attachments on its moving platform (Figure **1**). The space of legs is a 6D space defined by $(x, y, z, \psi, \theta, \phi)$. The legs are centered at $a_i = (x_i, y_i, z_i)^T$, $i = 1,2...,6$, in P' , and $b_i = (\psi_i, \theta_i, \phi_i)^T$, $i = 1,2...,6$, in P . The ψ, θ, ϕ (pitch, roll and yaw) are the Euler angles describing the inclination P with respect to P'. Given $p = (p_x, p_y, p_z)^T$ and the rotation matrix $R = (i, j, k)$, the location of P' attachments are given by $b_i = p + R\tilde{b}_i$, $i = 1,2...,6$.

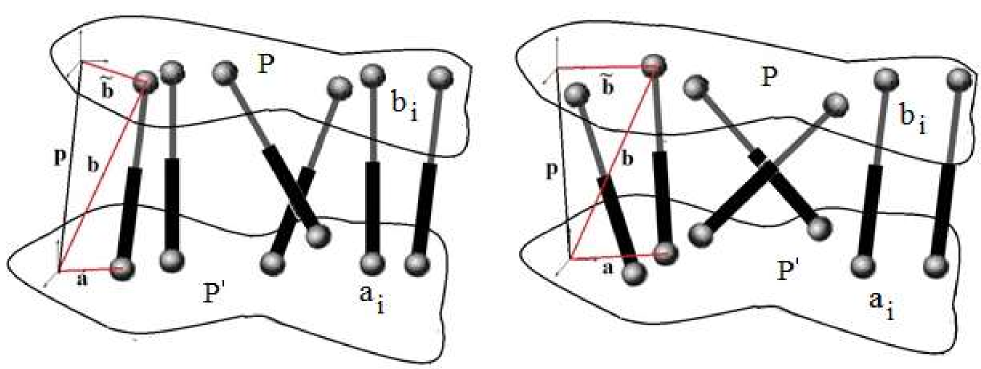

Figure 1: Two Stewart-Gough platforms with various legs arrangement.

The generalization of Duporcq's theorem uses the surface S of a super-ellipsoid [16-18]

$$F(x)=\left[\left(\frac{x}{r_1}\right)^{\frac{2}{\varepsilon_1}}+\left(\frac{y}{r_2}\right)^{\frac{\varepsilon_1}{\varepsilon_2}}\right]+\left(\frac{z}{r_3}\right)^{\frac{2}{\varepsilon_2}}-1\,, \tag{3}$$

where radii r_i, $i=1,2,3$, specify the dimensions of the super-ellipsoid and $\varepsilon_1,\varepsilon_2\in(0,2)$ are unknowns constants that correspond to a convex body and control the squareness of the superellipsoid in the (x_1,x_2) plane and x_3 direction ($\varepsilon_i\to 0$ yields a cuboid and $\varepsilon_i\to 2$ an octahedron).

The $x=(r,\theta,z)$ is the spatial cylindrical coordinates, centred in O, and $x_1=r\cos\theta$, $x_2=r\sin\theta$, $x_3=z$. The x_3-axis corresponds to the z-axis of inertia of the super ellipsoid. The point x is inside the super-ellipsoid $F(x)<0$. The point x lies on its surface $F(x)=0$. The minimum distance between P and P' belonging to S is

$$\min\left(\frac{1}{2}(x_1-x_2)^T(x_1-x_2)\right), \tag{4}$$

with x_1 and x_2 the position vectors of the points, respectively.

The theorem can be expressed as: Determine the displacements in which some or all points of a rigid body move on the surface S of a super-ellipsoid.

The goal of the paper is to find the unknowns r_i and ε_i, $i=1,2$, so that the displacements in which some or all points of a rigid body move on a super-ellipsoid surface S. When five pairs of attachment points $(b_i,a_i)\neq(b_j,a_j)$, $j\neq i$, $i,j\in\{1,2,...,5\}$ of a planar platform are given, then there exists a sixth point pair $(b_6,a_6)\neq(b_k,a_k)$, $k=1,2,...,5$, in a way that the resulting planar architecturally singular platform has the same solution for the direct kinematics. Conform to this statement, the sixth point pair can change with respect to different values for the lengths of the first five legs. The spheres can be viewed as super-ellipsoids whose curvature is distorted in all directions.

2. Formulation of the problem

A super-ellipsoid with center a trough b is defined as the set of points that verify (3). If the center *a* goes to infinity and b remains finite. The ellipsoid condition degenerates to a similar Darboux motion [6]. This means that b is located in a fixed plane orthogonal to the direction of the ideal point *a*. By changing the role of the top and bottom platform and if the point b goes to infinity and a remains finite, the ellipsoid condition degenerates to a similar Mannheim motion [6, 9, 10]. This means that a plane P, orthogonal to the direction of the ideal point b, slides through the point a. The Mannheim motion is the inverse of the Darboux motion [9, 10, 19-22].

We state in this paper is the following version of Duporcq's theorem:

Theorem 1. Let S be the surface S of a super-ellipsoid defined by (3). Determine the unknowns r_i and ε_i, $i=1,2$, so that all displacements in which some or all points of a rigid body move on S. In addition, if five pairs of attachment points $(b_i,a_i)\neq(b_j,a_j)$, $j\neq i$ $i,j\in\{1,2,...,5\}$ of a planar platform, are given, then there exists a sixth point pair $(b_6,a_6)\neq(b_k,a_k)$, $k=1,2,...,5$, in a way that the resulting planar architecturally singular platform has the same solution for the direct kinematics.

To demonstrate this theorem in this problem we need to intersect two super-ellipsoids $F_1\geq 0$ and $F_2\geq 0$ in order to obtain the paths on which the displacements of distinct points of the body are moving. The intersection includes multiple components of different dimensionality: points, curves and surface patches. Written in terms of surface parameters $\psi \in [-\pi.\pi]$ and $\theta \in [-\pi/2, \pi/2]$, the surface of the superellipsoid is written as

$$x(\psi,\theta) = \begin{bmatrix} \operatorname{sgn}(\cos\psi) & r_1 |\cos\psi|^{\varepsilon_1} & |\cos\theta|^{\varepsilon_2} \\ \operatorname{sgn}(\sin\psi) & r_2 |\sin\psi|^{\varepsilon_1} & |\cos\theta|^{\varepsilon_2} \\ \operatorname{sgn}(\sin\theta) & r_3 |\sin\theta|^{\varepsilon_2} & \end{bmatrix}. \quad (5)$$

The surface S of a super-ellipsoid can be rewritten under the form

$$-F^2(x) \geq 0. \quad (6)$$

The set

$$F_{\text{int}} = (-F_1^2) \wedge (-F_2^2) \geq 0, \quad (7)$$

is obtained by intersecting two super-ellipsoids $F_1 \geq 0$ and $F_2 \geq 0$, where $\wedge$ is the intersection operator defined by R - functions

$$f_1 \wedge f_2 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}\ . \quad (8)$$

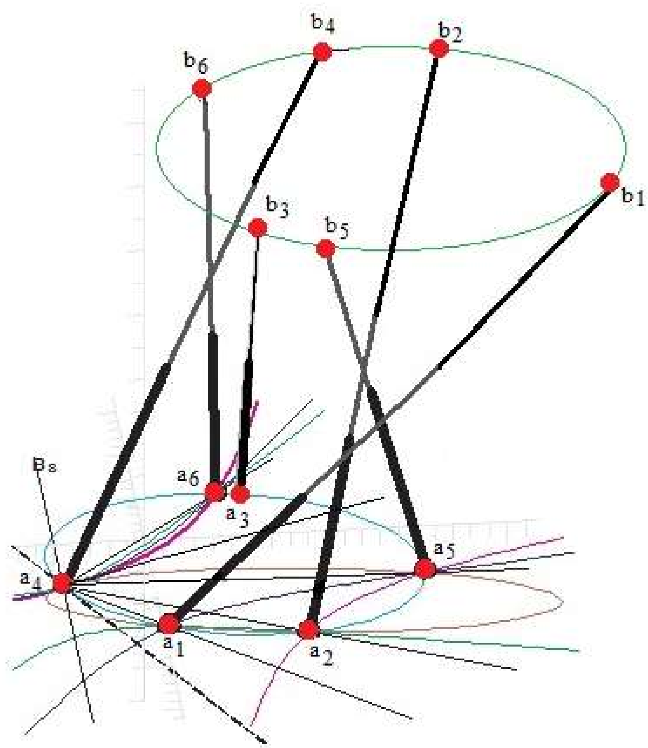

Figure 2: Scheme of trajectories obtained by the intersections of two super-ellipsoids.

The trajectories of moving points are intersections of two surfaces S_1 and S_2 of a super-ellipsoid. The intersection includes multiple components of different dimensionality: points, curves and surface patches defined by C_1 -continuous in the entire space with the possible exception of the intersection points. The resulting set F_{int} takes zero values at the intersection of two surfaces and negative values elsewhere. The closed surfaces are considered and all intersection components with their shape mainly defined by real functions.

The intersection problem has to find if two points $P_1 \in F_1$ and $P_2 \in F_2$ are in contact. The distance of penetration $d = \|P_2 - P_1\|$ and the contact direction c verify $n_1 = \mu^2 c$ and $n_2 = -\nu^2 c$, $d \times c = 0$ where n_1 and n_2 are the outward surface normal at P_1 and P_2, and μ, ν are arbitrary real numbers.

A scheme of trajectories is shown in Figure **2**, where the bottom platform consists of six legs located at a_i, $i = 1,2,...,6$. The corresponding moving platform has the attachments b_i, $i = 1,2,...,6$. In contrast to known Stewart-Gough platforms which six constraints in distances between points, the theorem permits to find Stewart-Gough platforms with six distance or/and angular constraints between six pairs of points, lines, and/or planes in the base and platform, respectively. The sets of points, planes and lines in 3D Euclidean space are subjected to geometric constraints: the distance between point and point, point and line, point and plane, line and line and the angular constraints between line and line, line and plane, plane and plane. Therefore, the manipulator position and orientation are determined by six geometric constraints.

The robot includes four types of constraints: three distances and three angular constraints, four distances and two angular constraints and five distances and one angular constraint, and six distance constraints. In this context, the literature reports 1120 motions of the first type, 1260 of the second type, 1008 of the third type, and 462 of the last type [14]. The resulting architecturally singular planar Stewart-Gough platforms have the same solution for the direct kinematics as the given one and the pair (b_l, a_l), $l = 1,2,...,6$, verifies the super-ellipsoid condition.

In contrast to the known Stewart-Gough platforms with six constraints in distances between points, the theorem permits to define a Stewart-Gough platform with six distance or/and angular constraints between six pairs of points, lines, and/or planes in both platforms.

By the intersection of two super-ellipsoids, different sectional curves are obtained. If P' moves parallelly to P so that one of its points traces out a fixed-line perpendicular to P, and another point is on a fixed super-ellipsoid with center on the same plane, then all points of the space attached to P' move on super-ellipsoidal curves. Such self-motion curves of a singular Stewart-Gough platform are represented in Figure **3**.

The stability of the platform is analyzed by taking into account the distance constraint of each leg corresponding to the intersection with a hyperplane. The platform moves along the sectional curves obtained by the intersection of two super-ellipsoids keeping the same relative rotation with respect to the base. Sectional curves are correlated with each other by a reciprocal interaction which can lead to chaos behavior if one of the curves is perturbed [7]. The Euclidean distance in the phase space between a perturbed curve and another curve is given by

$$D(\tau) = \sqrt{\sum_{i=1}^{6}(x_i^0 - x_i^1)^2 + \sum_{i=1}^{6}(x_i'^0 - x_i'^1)^2 \tau^2}\,, \tag{9}$$

where τ is the time, the superscript 0 indicates the non-perturbed curve and $1$the perturbed one, respectively.

Simulation results show that the initial perturbation applied to a curve affects the stability of the platform. The trajectories become unstable in the least two directions in the vicinity of a transition point, namely x_2 and x_3. Figure **4** plots the orbits (x_2, x_3) that exhibit the riddling bifurcation in the form of tongues. The generation of tongues represents the first sign of losing the stability of the robot. The increasing number of tongues means chaos when the robot losses its stability [23].

3. Conclusions

In this paper, a generalization of the Prix Vaillant problem is presented for Stewart-Gough platforms: *determine all displacements in which some or all points of a rigid body move on a super-ellipsoid surface*. This problem is of interest for finding the self-motions of the planar Stewart-Gough platforms. The poof is based on the trajectories of moving points which are intersections of two implicit surfaces of a super-ellipsoid. The intersection includes multiple components of different dimensionality: points, curves and surface patches.

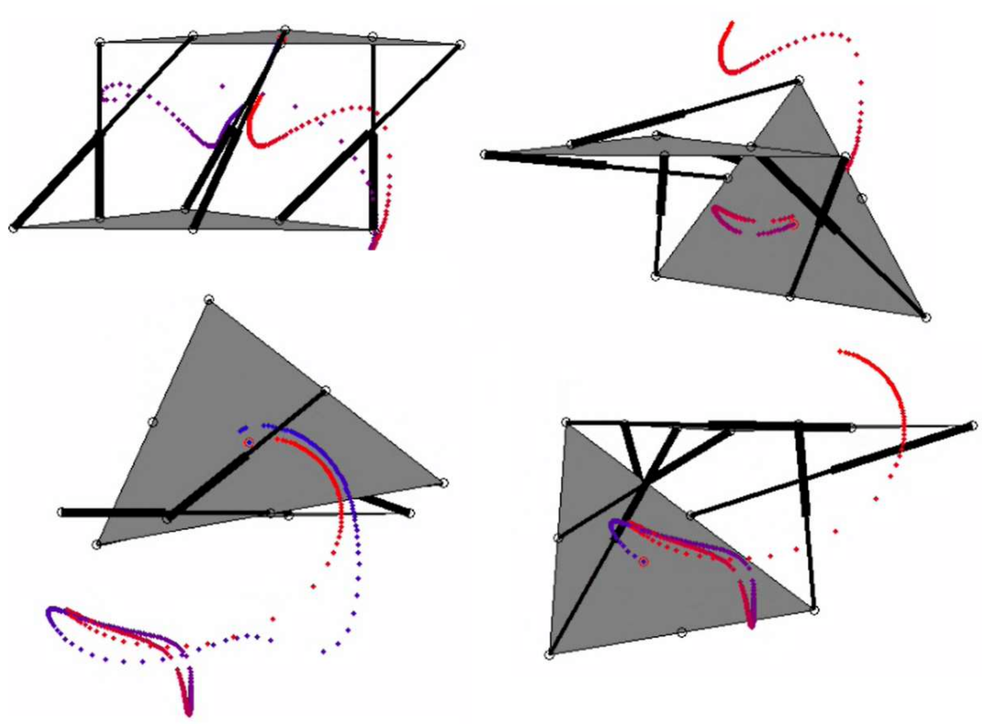

Figure 3: Sketches of self-motions of a singular Stewart-Gough platform with plane-body component.

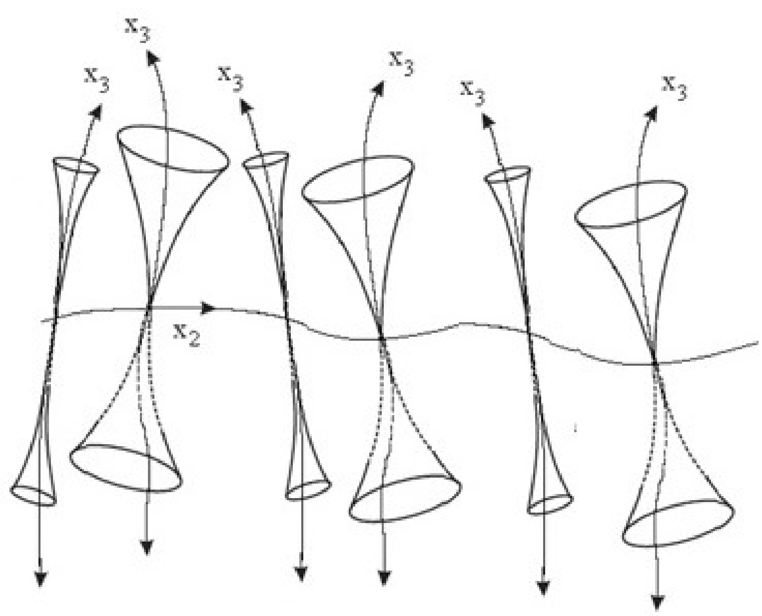

Figure 4: Riddling bifurcations of the unstable trajectories with respect to x_2 and x_3.

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