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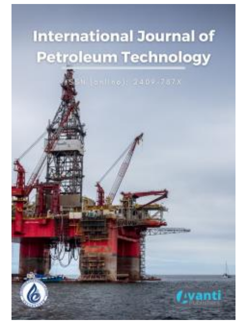


# On the Linear Stability Problem for a Three-Layer Displacement in a Porous Media

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## ABSTRACT

In this paper, we study the secondary oil recovery process. The oil from a porous reservoir at low pressure is pushed by a forerunner, less viscous fluid (a polymer solute). Then the well-known Saffman-Taylor instability appears. Some authors tried to minimize this instability by using a succession of intermediate liquids with constant viscosities - the multi-layer model. The surface tensions on the interfaces between liquid layers are a stabilizing factor. In some previous papers, we proved some contradictions of this multi-layer model. However, we considered that the corresponding stability problem has a solution. This model's first step (and the mathematical basis) is the three-layer model, with a single intermediate liquid. We prove that the linear stability problem for the three-layer model has no solution (in general) - the growth rates of perturbations may not exist. On the contrary, an intermediate liquid with a suitable variable viscosity can almost suppress the Saffman-Taylor instability, even if the surface tensions are missing [17].

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# 1. Introduction

When the oil in a porous reservoir is at low pressure, it can be obtained by pushing it by a second less viscous liquid (water or a polymer solute). This is the secondary oil recovery process. A useful model for porous media is the Hele-Shaw approximation [1, 18]. The Darcy flow in a porous medium can be approximated by the flow of a Stokes fluid between two very close parallel plates. When we have two displacing immiscible fluids, a sharp interface exists - this is the main element of the model. That means in each point of the Hele-Shaw cell, we have only one fluid. The interface is unstable when the displacing fluid is less viscous [25]. Then fingers of water can advance in the oil region, and we get a large quantity of water from the exit wells. Since the 1960s, it has been observed that a fluid with variable viscosity introduced between water and oil can improve the interface instability and reduce the water from the exit wells [16]. Some optimal viscosity profiles were obtained using numerical methods or theoretical approaches [2, 17]. However, it is difficult to obtain a prescribed variable viscosity profile. The control elements are the length of the intermediate region and a slow increase in viscosity.

To avoid the above difficulty, some authors [3-15] tried to replace the variable viscosity profile with a succession of constant viscosities, with positive viscosity jumps in the flow direction. This is the multi-layer Hele-Shaw model. The control element is the number of intermediate layers. We proved some contradictions in this model [19, 24]. For example, important improvement of stability can be obtained only by a huge number of intermediate layers - difficult to achieve in practice. This conclusion was obtained by assuming the existence of the growth rates of the stability problem.

The first step for the multi-layer model is the three-layer model [3,4]. Here a single intermediate layer exists between the initial displacing fluids. This is the subject of the present paper. The eigenfunctions of the stability system are the amplitudes of the linear perturbations. We have stability if the linear perturbations' growth rates (in time) are negative. Improved stability is given by small $\sigma$. In our case, $\sigma$ are contained in the two boundary conditions, then we get a compatibility condition (not pointed out in [3, 4]). This condition is the crux element of our main result: the stability problem has no solution (in general). The eigenfunctions do not verify the compatibility condition; thus, the growth rates may not exist. This is based on three new elements:

- we get a formula of the possible growth rates for large wavenumbers $k$;

- we also get the corresponding possible eigenfunctions;

- these eigenfunctions do not verify (in general) the compatibility condition.

The multi-layer Hele-Shaw model contains the same type of boundary conditions, and we get the same compatibility conditions. If the compatibility condition for the three-layer model is not fulfilled, then the same phenomenon appears for the multi-layer model. For this reason, the present non-existence result for the three-layer model leads us to the same result for the multi-layer case.

An intermediate liquid with a suitable variable viscosity can almost suppress the Saffman-Taylor instability, even if the surface tensions on interfaces are zero [23]. This has already been done in practice. The used liquids could be polymers with specific properties already known in literature [16] and the references therein. The porous medium is approximated by a horizontal Hele-Shaw cell, parallel with the fix plane$x_1 0y$ [1, 18], as shown in Figure 1. The cell is saturated with three immiscible Stokes fluids with constant viscosities. The flow is given by the velocity $(U,0)$ of the displacin fluid far upstream. We use the Darcy law for the pressure $p$ and averaged veocity $(u, v)$, with a specific viscosity $\nu$ (divided by the constant permeability of the porous medium):

$$\nabla \cdot u = 0, u = (u, v), x \neq a, b; \tag{1}$$

$$\nabla p = -\nu u, x \neq a, b. \tag{2}$$

The basic viscosity is given by relationships

$$\nu = \mu, x \in (a,b); 0<\mu_L < \mu < \mu_R; b \leq 0;\ \nu = \mu_L, x < a; \nu = \mu_R, x > b. \tag{3}$$

See also the equations (1a), (1b) of [2]. The moving reference $x = x_1 - \mathrm{U}t$ is used.

A stationary solution for (1)-(3) exists, with two straight interfaces $x = a, x = b$ between the fluid layers [17]. The Laplace-Young law acts on both interfaces, where we have two positive surface tensions $T(a), T(b)$.

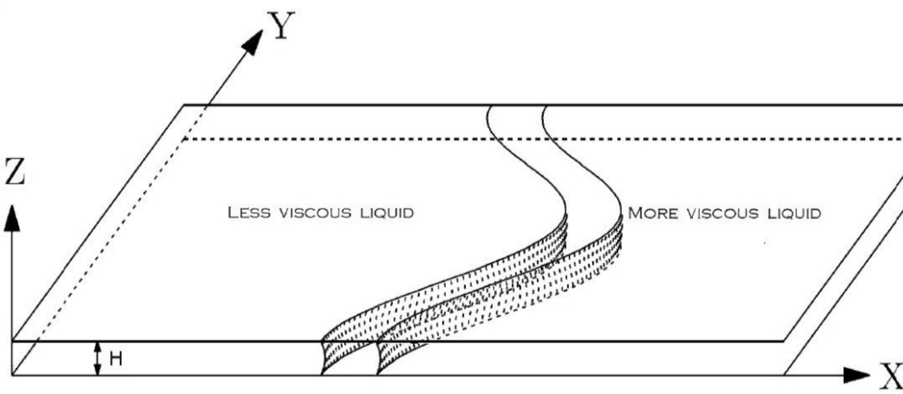


**Figure 1:** A three-layer (perturbed) Hele-Shaw displacement. For the basic flow, we have the interfaces $X = a, X = b$.

## 2. The stability System

We start with the perturbations of the velocity, denoted by $u'$, and consider the Fourier decomposition

$$u' = \varepsilon f(x)\exp(\mathrm{i}ky + \sigma t). \tag{4}$$

Here $\varepsilon$ is a small positive number; $k, \sigma, f$ are the wave numbers, growth rates, and amplitudes, respectively. The amplitude must decay to zero in the far-field and is continuous [11]. The following conditions are imposed in [19]:

$$f(x) = f(a)e^{k(x-a)}, \forall x \leq a;\ f(x) = f(b)e^{-k(x-b)}, \forall x \geq b. \tag{5}$$

Only small perturbations are allowed for linear perturbation; thus, we need also

$$-\infty < f(k,x) < \infty, \forall x \in [a,b], \forall k \geq 0. \tag{6}$$

The corresponding perturbations$p, (u,v)$ are inserted in (1)-(2). We follow the procedure given in given in [2, 9] and get the problem (2a), (2b), (3) of [2] (see also the relation (20) in [3]), where $\ _x$ is the partial derivative:

$$f_{xx} - k^2 f = 0, x \in (a,b), b \leq 0, \forall k \geq 0; \tag{7}$$

$$f_x(a) = [-\,kE_a(k)/\sigma + s]\, f(a); \tag{8}$$

$$f_x(b) = [kE_b(k)/\sigma + q]\, f(b); \tag{9}$$

$$E_b(k) := \frac{(\mu_R - \mu)\mathrm{U}k - T(b)k^3}{\mu}, q = -\frac{\mu_R k}{\mu};\ E_a(k) := \frac{(\mu - \mu_L)\mathrm{U}k - T(a)k^3}{\mu}, s = \frac{\mu_L k}{\mu}; \tag{10}$$

$$\sigma(k) = \frac{k\mathrm{E}_b(k)f(b)}{\mu f_x(b) + \mu_R k\mathrm{f}(b)} = \frac{k\mathrm{E}_a(k)f(a)}{\mu_L k\mathrm{f}(a) - \mu f_x(a)}, \forall k \geq 0. \tag{11}$$

The general solution of (7) is

$$f(x) = A(k)e^{\mathrm{kx}} + B(k)e^{-\mathrm{kx}}. \tag{12}$$

We also use the notation $A, B$. Condition (11) gives us

$$E_b/E_a = [\mu F(k,b) + \mu_R]/[\mu_L - \mu F(k,a)], \tag{13}$$

$$F(k,x) = \frac{\mathrm{A}e^{kx} - \mathrm{B}e^{-kx}}{\mathrm{A}e^{kx} + \mathrm{B}e^{-kx}}. \tag{14}$$

From the boundary conditions (8)-(9), we obtain the relations

$$[\mu f_x(a) - \mu_L f(a)] = -f(a)\,E_a/\sigma,\ \ [\mu f_x(b) + \mu_R f(b)] = f(b)\,E_b/\sigma\,.$$

Therefore, $A, B$ are verifying the equations

$$\mu\left(\mathrm{A}e^{ka} - \mathrm{B}e^{-ka}\right) = (-E_a/\sigma + \mu_L)\left(\mathrm{A}e^{ka} + \mathrm{B}e^{-ka}\right),\ \ \mu\left(\mathrm{A}e^{kb} - \mathrm{B}e^{-kb}\right) = (E_b/\sigma - \mu_R)\left(\mathrm{A}e^{kb} + \mathrm{B}e^{-kb}\right).$$

We get the homogeneous system with variable coefficients

$$\begin{gathered}\mathrm{A}e^{ka}c + \mathrm{B}e^{-ka}d{=}0, \mathrm{A}e^{kb}g + \mathrm{B}e^{-kb}h{=}0;\ c = (\mu_L - \mu - E_a/\sigma), d = (\mu_L + \mu - E_a/\sigma),\\ g = (\mu_R + \mu - E_b/\sigma), h = (\mu_R - \mu - E_b/\sigma).\end{gathered} \tag{15}$$

A solution $(A,B) \neq (0{,}0)$ exists if the following condition is verified

$$e^{2k(a-b)}\mathrm{ch} - \mathrm{gd}{=}0, \forall k \geq 0. \tag{16}$$

# 3. The Possible Growth Rates for Large $k$

In this section, we obtain the values of the growth rates $\sigma$ for very large wavenumbers $k$. To this end, we introduce the notations $Q, \Delta, i, j, m, n$ below. The existence condition (16) gives us a second-order equation:

$$\sigma^2(k)(\mathrm{Qij} - \mathrm{mn}) + \sigma(k)\left[(\mathrm{m}E_b + \mathrm{n}E_a) - Q\left(\mathrm{i}E_b + \mathrm{j}E_a\right)\right] + \left(\mathrm{Q}E_a E_b - E_a E_b\right){=}0; Q = e^{2k(a-b)}. \tag{17}$$

$$i = \mu_L - \mu; j = \mu_R - \mu; m = \mu_L + \mu; n = \mu_R + \mu;\ \Delta := +Q^2 \tag{18}$$

The crucial point of our analysis is based on the following estimates: for $k \to \infty$ we have

$$e^{-2k}k^3 \approx 0, e^{-2k} << k^3;\ e^{-2k}k^6 \approx 0, e^{-2k}k^6 << k^6.$$

Of course, these estimates are questionable from a strictly mathematical point of view. Nevertheless, we consider that they are physically acceptable. So all the terms that contain $Q$ (as a factor) will converge to zero. As a consequence, we get a much simpler expression of $\Delta$ for large enough $k$:

$$\Delta \approx (mE_b + nE_a)^2 - 4mnE_aE_b = (mE_b - nE_a)^2.$$

Finally, from (17) & (18), we get the following formula (with a very good approximation for large $k$):

$$\sigma(k) \approx \frac{-(mE_b\ +\ nE_a) \pm\ \ |mE_b - nE_a|}{-2mn}. \tag{19}$$

We see that $E_a(k), E_b(k) \neq 0$ if $k$ is large enough. This is important: $\sigma(k)$ cannot be equal to zero. So we have two possible growth rates:

$$\sigma_1(k) \approx \frac{E_b(k)}{\mu_R + \mu}, \tag{20}$$

$$\sigma_2(k) \approx \frac{E_a(k)}{\mu_L + \mu}. \tag{21}$$

# 4. The Non-Existence Result

In this section, we prove that eigenfunctions (12), with $A(k)$ and $B(k)$ given by the system (15), do not check the compatibility condition (13). Indeed, *if $\sigma(k)$ is a growth rate*, then (11), (13) & (14) give us

$$\sigma(k) = \frac{E_b(k)}{\mu F(k,b) + \mu_R} = \frac{E_a(k)}{\mu_L - \mu F(k,a)}, \forall k \geq 0. \tag{22}$$

We use the following notations

$$\lim_{k\to\infty} F(k,b) = F(b), \lim_{k\to\infty} F(k,a) = F(a).$$

From (20), (21), & (22), we obtain the existence and the values of the limits $F(b), F(a)$. We get:

$$\text{red}\sigma(k) = \sigma_1(k), F(b) \neq 1 \Rightarrow \lim_{k\to\infty} \frac{E_b(k)}{\sigma_1(k)} \neq \mu + \mu_R; \tag{23}$$

$$\text{red}\sigma(k) = \sigma_2(k), F(a) \neq -1 \Rightarrow \lim_{k\to\infty} \frac{E_a(k)}{\sigma_2(k)} \neq \mu_L + \mu. \tag{24}$$

Both last relations contradict (20) and (21). Therefore, we get $F(b)$=1,$F(a) = -1$, and the relation (22) for large $k$ gives us

$$\lim_{k\to\infty} \frac{E_b(k)}{E_a(k)} = \frac{T(b)}{T(a)} = \frac{\mu + \mu_R}{\mu_L + \mu}. \tag{25}$$

Thus, we obtain the following unexpected restrictions, which were not initially imposed:

$$\frac{T(b)}{T(a)} = \frac{\mu_R + \mu}{\mu_L + \mu} \Leftrightarrow \mu = \frac{\mu_R T(a) - \mu_L T(b)}{T(b) - T(a)}, \tag{26}$$

$$0 < \mu_L < \frac{\mu_R T(a) - \mu_L T(b)}{T(b) - T(a)} < \mu_R. \tag{27}$$

The above restrictions can give us some contradictions. For example:

$$0 < T(b) < T(a) \Rightarrow \mu_R T(a) < \mu_L T(b) < \mu_L T(a)$$

So we get $\mu_R < \mu_L$, in contradiction with the hypothesis (3). Thus, compatibility condition (13) is generally not fulfilled. As a consequence, the growth rates of the considered stability problem may not exist.

# 4. Conclusions

The eigenfunctions (12) do not check the compatibility condition (13). So the growth rates of the problem (6)-(11) do not exist, in general. Therefore, the linear stability problem for a three-layer displacement with constant viscosity fluids has no solution. However, in [3] are given some estimations for the growth rates $\sigma$by using relation (16). Moreover, the results obtained in [3] are used in the papers [4-15]. In [19-24], we pointed out some weak points in the papers [3-15] concerning the practical implementation of the multi-layer model. Our result is an improvement of [19-24] or he case $A = A(k), B = B(k)$. I have proved in the introduction the close connection between the three-layer and multi-layer models. Therefore, the multi-layer stability problem has no solution either, in general. As a consequence, this model is not so useful for the minimization of the Saffman-Taylor instability. We proved in [23] that a single intermediate liquid with a suitable variable viscosity (which increases from $\mu_L$ to $\mu_R$) could almost suppress the Saffman-Taylor instability, even if the surface tensions on the two interfaces are zero. Two elements are necessary in this case: a large enough intermediate region and a slow increase of the intermediate viscosity.

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