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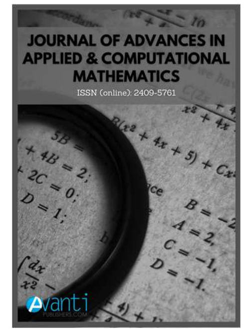

Fuzzy Best Dominants for Certain Fuzzy Differential Subordinations

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ABSTRACT

This paper aims to present a survey on certain fuzzy subordination properties for analytic functions defined in the open unit disk. The new results are derived by considering a certain differential operator. By making use of two differential properties of the operator we determine sufficient conditions to find the fuzzy best dominants for several fuzzy differential subordinations. Some interesting further fuzzy consequences are also considered.

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1. Introduction

The new notion of fuzzy subordination was defined and studied recently in the papers [18-20]. This theory was developed in order to extend the classical differential subordination theory introduced and studied by S.S. Miller and P.T. Mocanu in [14]. The basis of this fuzzy concept lies in the well known fuzzy set term introduced by Lotfi Zadeh. In 1965 Zadeh published his Pioneering paper on fuzzy sets and many examples have been supplied to understand the notion [23]. From here it derives many other important topics in mathematics. The theory of fuzzy logic appears in the fuzzy sets theory context. A real number belonging to the interval [0,1] is assigned to a specific element from a certain class. This is stated as a fuzzy set. The theory of fuzzy logic emerges by associating degrees of truth to different propositions. The interval [0,1] is the true-values set. The number 0 is allocated for " totally false" and the number 1 is allocated for "totally true". The rest of the numbers are associated with partial truth, which is the intermediate degrees of truth.

In this context, the substantiation of fuzzy differential subordination became a very natural one. Since its appearance, the theory of fuzzy differential subordination it developed at a very fast level as we can see in the recent papers [16,21,22]. The present study aims to lead to obtaining certain outcomes that involve both the notion of fuzzy differential subordination and that of differential operators. In this direction were outlined recently several papers [3,10,11,13]. Such works demonstrate once again the interest shown in this topic. Motivated by a joint earlier work of the author [12] and a recent paper [5] where it was introduced a differential operator, we establish in this article an interesting application of best fuzzy dominants for certain differential fuzzy subordinations. For future work, it can be useful to consider similar results as in [8] aiming an integral operator.

Further, we recall here some preliminary concepts and results which are used further. We are familiar with the well-known concepts from Geometric Function Theory.

Let us denote by $\mathcal{H}$ the set of analytic functions defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Consider also $\mathcal{H}[a,n]$ a subset of $\mathcal{H}$ with the following form of functions

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Consider $\mathcal{A}(p,n)$ the set of functions $f(z)$ that are normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, (p, n \in \mathbb{N} := \{1,2,3,\ldots\}).$$

We denote $\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$. Let $\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \ldots\}$ with $\mathcal{A}_1 := \mathcal{A}$.

We provide further, a brief description of basic elements of fuzzy differential subordination theory.

Definition 1.1. [23] A function $F : X \to 0;1]$ is named fuzzy subset, where X is a non-empty set. Another definition, would be the next one: A pair (A, F_A), with $F_A :\ X \to 0;1]$

$$A = \{x \in X : 0 < F_A(x) \leq 1\} = \sup(A, F_A),$$

is named a fuzzy subset of X. Set A represents the support of the fuzzy set (A, F_A). Also F_A is named the membership function of the fuzzy set (A, F_A). One can also denote $A = \sup(A, F_A)$.

Remark 1.1. [20] Let be the inclusion relation $A \subset X$. Then we have $F_A(x) = \{ll1if$ x A $0if$ x A..

The real number 0, for a fuzzy subset, is the smallest membership degree of $x \in X$ to A. Likewise, the real number 1 is the biggest membership degree of $x \in X$ to A.

The entire set X is associated with $F_X(x) = 1, x \in X$ and the empty set $\emptyset \subset X$ is associated with $F_{\emptyset}(x) = 0, x \in X$.

Definition 1.2. [18] Consider two functions $f, g \in \mathcal{H}(D)$ and $D \subset \mathbb{C}$, $z_0 \in D$ being a fixed point. We say that the function f is a fuzzy subordinate to g and written as

$$f \prec_{\mathcal{F}} g \quad or \quad f(z) \prec_{\mathcal{F}} g(z), \quad (z \in D)$$

if the following relations are verified:

(1) $f(z_0) = g(z_0)$;

(2) $F_{f(D)} f(z) \le F_{g(D)} g(z), z \in D.$

Definition 1.3. [19] Consider $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be an univalent function in U, satisfying $\psi(a,0;0) = h(0) = a$. We say that p is named a fuzzy solution of the fuzzy differential subordination if p is an analytic function in U, such that $p(0) = a$ and verifies the next (second-order) fuzzy differential subordination:

$$F_{\psi(C^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \le F_{h(U)} h(z), \quad z \in U. \tag{1.1}$$

For all p satisfying (1.1), the univalent function q is named a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination, or a fuzzy dominant, if $F_{p(U)} p(z) \le F_{q(U)} q(z)$, $z \in U$. A fuzzy dominant $\tilde{q}$ which verifies $F_{\tilde{q}(U)} \tilde{q}(z) \le F_{q(U)} q(z), z \in U$, for all fuzzy dominants q of (1.1) represents the fuzzy best dominant of (1.1).

Definition 1.4. [20] Let Q be the set of all functions f that are analytic and injective on the set $\overline{U} - E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

such that $f'(\zeta) \ne 0$ for $\zeta \in \partial U - E(f)$.

Theorem 1.1. [20] Consider the function q is univalent in the open unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \ne 0$ when $w \in q(U)$. Set

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z).$$

Assume that

(1) $Q(z)$ is starlike univalent in Δ and

(2) $\mathrm{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic with $p(0)=q(0)$ and $p(U)\subseteq D$ and

$$F_{p(U)}\theta(p(z))+zp'(z)\phi(p(z))\leq F_{h(U)}\theta(q(z))+zq'(z)\phi(q(z))$$

then

$$F_{p(U)}p(z)\leq F_{q(U)}q(z)$$

and q is the fuzzy best dominant.

Definition 1.5. [5] Consider $f\in \mathrm{A}$. For the numbers $m,\beta\in \mathrm{N}_0=\mathrm{N}\cup\{0\}$, $\lambda\in\mathrm{R}$, $\lambda\geq 0$, $l\geq 0$, we consider differential operator $I^{m,\beta}(\lambda,l)$ defined on A and having the form

$$I^{m,\beta}(\lambda,l)f(z):=z+\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+kl}{1+l}\right]^m C(\beta,k)a_k z^k$$

where

$$C(\beta,k):=\binom{k+\beta-1}{\beta}=\frac{(\beta+1)_{k-1}}{(k-1)!}$$

and

$$(a)_n:=\begin{cases}1, & n=0\\ a(a+1)\ldots(a+n-1), & n\in\mathrm{N}=\mathrm{N}_0-\{0\},\end{cases}$$

is Pochhamer symbol.

Remark 1.2. We reobtain several operators obtained earlier by various researchers. Recall here the Ruscheweyh operator $I^{0,\beta}(\lambda,0)\equiv D_\beta$ defined in [15], the Sălăgean derivative operator $I^{m,0}(1,0)\equiv D^m$, studied in [17], the generalized Sălăgean operator $I^{m,0}(\lambda,0)\equiv D_\lambda^m$ defined by Al-Oboudi in [1], the generalized Ruscheweyh operator $I^{1,\beta}(\lambda,0)\equiv D_{\lambda,\beta}$ introduced in [9], the operator $I^{m,\beta}(\lambda,0)\equiv D_{\lambda,\beta}^m$ defined by K. Al-Shaqsi and M. Darus in [2] and for $\beta=0$ a similar operator introduced in [4]. The operator $I^{m,0}(\lambda,1-\lambda)\equiv I_\lambda^m$ (for $p=1$) was developed by Cho and Srivastava [6] and Cho and Kim [7].

By making use of a simple computation technique one obtains the following result.

Proposition 1.1. [5] Consider the numbers $m,\beta\in\mathrm{N}_0$, $\lambda\geq 0$, $l\geq 0$

$$(1+l)I^{m+1,\beta}(\lambda,l)f(z)=(1-\lambda)I^{m,\beta}(\lambda,l)f(z)+(\lambda+l)z(I^{m,\beta}(\lambda,l)f(z))'. \tag{1.2}$$

and regarding parameter β

$$z\left(I^{m,\beta}(\lambda,l)f(z)\right)'=(1+\beta)I^{m,\beta+1}(\lambda,l)f(z)-\beta I^{m,\beta}(\lambda,l)f(z). \tag{1.3}$$

The main object of the paper is to derive sufficient conditions required for analytic functions f which verify certain differential fuzzy subordination types.

In the present work, we deduce several interesting theorems regarding the best fuzzy best dominants for certain fuzzy differential subordinations.

2. Main Results

Theorem 2.1. Consider $a,b,c,\xi,\mu,\eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and the function q is a univalent one in the open unit disc U with $q(z) \neq 0$.

Assume that $\frac{zq'(z)}{q(z)}$ is a starlike univalent function in U. Consider

$$\operatorname{Re}\left\{\frac{b}{\xi}q(z)+\frac{2c}{\xi}(q(z))^2\right\}>0 \tag{2.1}$$

and

$$\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;z):=a+b\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{\mu}\cdot\left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{\eta}+ \tag{2.2}$$

$$+c\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{2\mu}\cdot\left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{2\eta}+$$

$$+\frac{\xi(l+1)}{\lambda+l}\cdot\left[\mu\left(\frac{I^{m+2,\beta}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right)+\eta\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)}-1\right)\right].$$

If q verifies the next fuzzy subordination

$$F_{\Lambda^m f(U)}\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;z)\le F_{q(U)}\left(a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)}\right) \tag{2.3}$$

then

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}\le F_{q(U)}q(z), \tag{2.4}$$

where $\Lambda^m f(U)=\left(\frac{I^{m+1,\beta}(\lambda,l)f(U)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(U)}{z}\right)^{\eta}$, $z\in U$, $z\neq 0$, $\eta\in\mathbb{C},\eta\neq 0$ and q is the fuzzy best dominant.

Proof. Define the function $p(z)$ by

$$p(z):=\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta},\quad z\in\mathbb{C}, z\neq 0, f\in\mathcal{A}. \tag{2.5}$$

By a straightforward computation, one obtains

$$\frac{zp^{'}(z)}{p(z)}=\mu\left(\frac{z[I^{m+1,\beta}(\lambda,l)f(z)]^{'}}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right)+\eta\left(\frac{z[I^{m,\beta}(\lambda,l)f(z)]^{'}}{I^{m,\beta}(\lambda,l)f(z)}-1\right).$$

Using the identity

$$(l+1)I^{m+2,\beta}(\lambda,l)f(z)=(1-\lambda)I^{m+1,\beta}(\lambda,l)f(z)+(l+\lambda)z(I^{m+1,\beta}(\lambda,l)f(z))^{'}$$

we obtain

$$\frac{zp^{'}(z)}{p(z)}=\frac{\mu(l+1)}{l+\lambda}\left(\frac{I^{m+2,\beta}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right)+\frac{\eta(l+1)}{l+\lambda}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)}-1\right). \quad (2.6)$$

By substituting the above equality into (2.3) we get

$$F_{p(U)}\left(a+bp(z)+c(p(z))^{2}+\xi\frac{zp^{'}(z)}{p(z)}\right)\leq F_{q(U)}\left(a+bq(z)+c(q(z))^{2}+\xi\frac{zq^{'}(z)}{q(z)}\right).$$

By setting

$$\theta(w):=a+bw+cw^{2}\ \ and\ \ \phi(w):=\frac{\xi}{w}$$

one can easily notice that the function θ is analytic in $\mathbb{C}$, ϕ is an analytic function in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\setminus\{0\}$. Considering

$$Q(z)=zq^{'}(z)\phi(q(z))=\xi\frac{zq^{'}(z)}{q(z)}$$

and

$$h(z):=\theta(q(z))+Q(z)=a+bq(z)+c(q(z))^{2}+\xi\frac{zq^{'}(z)}{q(z)},$$

is deduced that $Q(z)$ is a starlike univalent function defined U with

$$\operatorname{Re}\left\{\frac{zh^{'}(z)}{Q(z)}\right\}=\operatorname{Re}\left\{1-\frac{zq^{'}(z)}{q(z)}+\frac{zq^{''}(z)}{q^{'}(z)}+\frac{b}{\xi}q(z)+\frac{2c}{\xi}(q(z))^{2}\right\},$$

$$(a,b,c,\xi\in\mathbb{C},\xi\neq 0).$$

Knowing by the hypothesis that $\frac{zq^{'}(z)}{q(z)}$ is a function with the starlike univalent property in U and

$$\operatorname{Re}\left\{1-\frac{zq^{'}(z)}{q(z)}+\frac{zq^{''}(z)}{q^{'}(z)}\right\}>0$$

we find that $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0.$

Using Theorem 1.1, the assertion (2.4) of Theorem 2.1 is obtained. □

For the various form of the functions q, namely $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ and $q(z)=\left(\frac{1+z}{1-z}\right)^{\delta}$, $0\le\delta\le 1$ replaced in Theorem 2.1, one obtains the following two results.

Corollary 2.1. Consider the numbers $a,b,c,\xi,\mu,\eta\in\mathbb{C}$, $\eta\neq 0$, $\xi\neq 0$, $-1\le B<A\le 1$ with

$$\operatorname{Re}\left\{\frac{b}{\xi}\frac{1+Az}{1+Bz}+\frac{2c}{\xi}\left(\frac{1+Az}{1+Bz}\right)^{2}\right\}>0. \tag{2.7}$$

If $f\in\mathcal{A}$, then fuzzy differential subordination

$$F_{\Lambda^m f(U)}\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)\le \tag{2.8}$$

$$\le F_{q(U)}\left(a+b\frac{1+Az}{1+Bz}+c\left(\frac{1+Az}{1+Bz}\right)^{2}+\xi\frac{(A-B)z}{(1+Az)(1+Bz)}\right)$$

implies

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}\le F_{q(U)}\frac{1+Az}{1+Bz}, \tag{2.9}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is given in (2.2) and $\frac{1+Az}{1+Bz}$ represents the fuzzy best dominant.

Corollary 2.2. Consider the numbers $a,b,c,\xi,\mu,\eta\in\mathbb{C}$, $\eta\neq 0$, $\xi\neq 0$, $0\le\delta\le 1$ and

$$\operatorname{Re}\left\{\frac{b}{\xi}\left(\frac{1+z}{1-z}\right)^{\delta}+\frac{2c}{\xi}\left(\frac{1+z}{1-z}\right)^{2\delta}\right\}>0. \tag{2.10}$$

If $f\in\mathcal{A}$, then differential fuzzy subordination

$$F_{\Lambda^m f(U)}\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)\le F_{q(U)}\left(a+b\left(\frac{1+z}{1-z}\right)^{\delta}+c\left(\frac{1+z}{1-z}\right)^{2\delta}+\frac{2\xi\delta z}{1-z^{2}}\right) \tag{2.11}$$

implies

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1}(\lambda,\beta,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m}(\lambda,\beta,l)f(z)}{z}\right)^{\eta}\le F_{q(U)}\left(\frac{1+z}{1-z}\right)^{\delta}, \tag{2.12}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is given in (2.2) and $\left(\frac{1+z}{1-z}\right)^{\delta}$ represents the fuzzy best dominant.

Replacing the function $q(z)=e^{\upsilon Az}$, with $|\upsilon A|<\pi$ in Theorem 2.1 we deduce the following corollary.

Corollary 2.3. Consider the numbers $a,b,c,\xi,\mu,\eta\in\mathbb{C}$, $\eta\neq 0$, $\xi\neq 0$, $|\upsilon A|<\pi$ and

$$\operatorname{Re}\left\{\frac{b}{\xi}e^{\upsilon Az}+\frac{2c}{\xi}e^{2\upsilon Az}\right\}>0. \tag{2.13}$$

If $f\in\mathcal{A}$, then fuzzy differential subordination

$$F_{\Lambda^m f(U)}\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)\leq F_{q(U)}\left(a+be^{\upsilon Az}+ce^{2\upsilon Az}+\xi\upsilon Az\right) \tag{2.14}$$

implies

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}\leq F_{q(U)}e^{\upsilon Az}, \tag{2.15}$$

where the function $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is given in (2.2) and $e^{\upsilon Az}$ represents the fuzzy best dominant.

Selecting $q(z)=(1+Bz)^{\frac{\delta(A-B)}{B}}$, $-1\leq B<A<1$, $B\neq 0$, one obtains the next known result.

Corollary 2.4. Consider the numbers $a,b,c,\xi,\mu,\eta,\delta\in\mathbb{C}$, $\eta\neq 0,\delta\neq 0$, $\xi\neq 0$, $-1\leq B<A<1$, $B\neq 0$ with

$$\operatorname{Re}\left\{\frac{b}{\xi}(1+Bz)^{\frac{\delta(A-B)}{B}}+\frac{2c}{\xi}(1+Bz)^{\frac{2\delta(A-B)}{B}}\right\}>0. \tag{2.16}$$

If $f\in\mathcal{A}$, then fuzzy differential subordination

$$F_{\Lambda^m f(U)}\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)\leq \tag{2.17}$$

$$\leq F_{q(U)}\left(a+b(1+Bz)^{\frac{\delta(A-B)}{B}}+c(1+Bz)^{\frac{2\delta(A-B)}{B}}+\xi\frac{z\delta(A-B)}{1+Bz}\right)$$

implies

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}\leq F_{q(U)}\left((1+Bz)^{\frac{\delta(A-B)}{B}}\right), \tag{2.18}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is given in (2.2) and $(1+Bz)^{\frac{\delta(A-B)}{B}}$ represents the fuzzy best dominant.

One noticed that the function $q(z)=(1+Bz)^{\frac{\delta(A-B)}{B}}$ is univalent if and only if either

$$\left|\frac{\delta(A-B)}{B}-1\right|\leq 1 \; or \; \left|\frac{\delta(A-B)}{B}+1\right|\leq 1.$$

Regarding parameter β we derive the next result.

Theorem 2.2. Consider q a univalent function in the open unit disc U with $q(z)\neq 0$, $a,b,c,\xi,\mu,\eta\in \mathbb{C}$, $\eta\neq 0$, $\xi\neq 0$ and $\lambda>0$.

Assume that $\frac{zq'(z)}{q(z)}$ defined a starlike univalent function in U and inequality (2.1) holds. Let the function

$$\Upsilon_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f):= \tag{2.19}$$

$$=\xi(\beta+1)\left[\mu\left(\frac{I^{m+1,\beta+1}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right)+\eta\left(\frac{I^{m,\beta+1}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)}-1\right)\right]$$

$$+a+b\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{\mu}\cdot\left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{\eta}+$$

$$+c\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{2\mu}\cdot\left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{2\eta}.$$

If the function q verifies the next fuzzy subordination

$$F_{\Lambda^m f(U)}\Upsilon_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;z)\leq F_{q(U)}\left(a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)}\right) \tag{2.20}$$

then

$$F_{\Lambda^m f(U)}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}\leq F_{q(U)}q(z), \tag{2.21}$$

$z\in U,\, z\neq 0, \eta\in \mathbb{C}, \eta\neq 0$, and q represents the fuzzy best dominant.

Proof. Consider p being given as in (2.5)

By using the identity

$$z\left(I^{m+1,\beta}(\lambda,l)f(z)\right)'=(1+\beta)I^{m+1,\beta+1}(\lambda,l)f(z)-\beta I^{m+1,\beta}(\lambda,l)f(z)$$

we obtain

$$\frac{zp'(z)}{p(z)}=\mu(\beta+1)\left[\frac{I^{m+1,\beta+1}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right]+$$

$$+\eta(\beta+1)\left[\frac{I^{m,\beta+1}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)}-1\right]. \quad (2.22)$$

By substituting the last equality into (2.20) we get

$$F_{p(U)}\left(a+bp(z)+c(p(z))^2+\xi\frac{zp'(z)}{p(z)}\right)\le F_{q(U)}\left(a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)}\right).$$

By setting

$$\theta(w):=a+bw+cw^2 \ and \ \phi(w):=\frac{\xi}{w}$$

we notice that the function θ is analytic one in $\mathbb{C}$, also ϕ is an analytic function in $\mathbb{C}\setminus\{0\}$ with $\phi(w)\neq 0$, $w\in\mathbb{C}\setminus\{0\}$. Considering

$$Q(z)=zq'(z)\phi(q(z))=\xi\frac{zq'(z)}{q(z)}$$

and

$$h(z):=\theta(q(z))+Q(z)=a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)},$$

we deduce that $Q(z)$ represent a starlike univalent function in U with

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{\frac{b}{\xi}q(z)+\frac{2c}{\xi}(q(z))^2\right\}>0,$$

$$(a,b,c,\xi\in\mathbb{C},\xi\neq 0).$$

Using Theorem 1.1, the relation (2.21) of Theorem 2.2 is obtained. □

Remark 2.1. One can notice here that Theorem 2.2 can be reformulated for various forms of the functions q (as in Corollaries 2.1-2.4).

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