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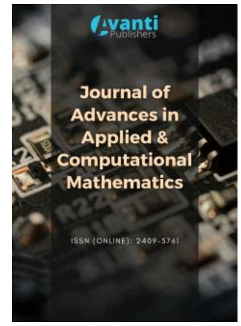

The Unique Solution to the Differential Equations of the Fourth Order with Non-Homogeneous Boundary Conditions

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ABSTRACT

This research paper aims to establish the uniqueness of the solution to fourth-order nonlinear differential equations

$$v^{(4)}(x) + f(x, v(x)) \quad = 0, \quad x \in [\mathrm{a}, \mathrm{b}],$$

with non-homogeneous boundary conditions

$$v(\mathrm{a}) = 0, \quad v^{'}(\mathrm{a}) = 0, \quad v^{''}(\mathrm{a}) = 0, \quad v^{''}(\mathrm{b}) - \alpha v^{''}(\zeta) = \mu,$$

where $0 \le \mathrm{a} < \zeta < \mathrm{b}$, the constants α, μ are real numbers and $f: [\mathrm{a}, \mathrm{b}] \times R \to R$ is a continuous function with $f(x, 0) \neq 0$. Using the sharper bounds on the integral of the kernel, the uniqueness of the solution to the problem is established based on Banach and Rus fixed point theorems on metric spaces.

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1. Introduction

The theory of differential equations has emerged as a valuable instrument for understanding and interpreting problems in various scientific fields. The laws concerning these situations can be expressed as differential equations of various orders satisfying certain conditions. In particular, the fourth-order differential equations arise in the modeling of the concepts in the inelastic flows, viscoelastic, electric circuits, theory of plate deflection, bending of beams, and various areas of applied mathematics, as well as the concepts in engineering [1-5]. Due to their immense importance in theory and applications, researchers have shown an interest in studying the existence of solutions to differential equations of different orders under certain conditions.

In 1988, Gupta [6] established the existing results for the bending of an elastic beam with entirely supported edges and given by

$$\frac{d^4u}{dx^4} - \pi^4 u + g(x,u) = e(x), \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0.$$

In 2003, Ma [7] considered the nonlinear fourth-order problem

$$u^{(4)}(x) = \lambda f(x, u(x), u'(x)) \quad 0 < x < 1,$$

$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0,$$

which describes the deformations of an elastic beam whose one end is fixed and the other free and studies the existence of multiple positive solutions using a fixed point theorem in cones. Following that, the researchers studied the existence and uniqueness of solutions to the boundary value problems of the third order [8-11] and fourth order [12-15]. In light of these works, we consider the fourth-order differential equations

$$v^{(4)}(x) + f(x, v(x)) = 0, \quad x \in [\mathrm{a}, \mathrm{b}], \tag{1}$$

with the boundary conditions involving a non-homogeneous term

$$v(\mathrm{a}) = 0, \quad v'(\mathrm{a}) = 0, \quad v''(\mathrm{a}) = 0, \quad v''(\mathrm{b}) - \alpha v''(\zeta) = \mu, \tag{2}$$

where $0 \leq a < \zeta < b$, the constants α, μ are real numbers, $f: [a,b] \times R \to R$ is a continuous function with $f(x,0) \neq 0$, and the existence results are established using fixed point theorems on metrics. By taking $a = 0, b = 1$ in (1) and (2), Sun and Zhu [16] established the existence of positive solutions by using Krasnosel'skii fixed point theorem. In the same way, Lakoud and Zenkoufi [17] studied the existence, uniqueness, and positivity of solutions through various fixed-point theorems for $\mu = 0$.

In real-world applications, the problem must be well-posed with certain constraints. If a problem has only one solution and certain ideal conditions, we can validate the problem's "well-posedness" using various methods. If the problem has two or more solutions or no solution, it needs to be better posed from a modeling standpoint and must be modified and created a new model [18].

For the sake of simplicity, the following notations are used.

(C1) $\Gamma = \mu + \int_a^b (b - \Upsilon)\Phi(\Upsilon)d\Upsilon - \alpha \int_a^\zeta (\zeta - \Upsilon)\Phi(\Upsilon)d\Upsilon$, and

(C2) $\Delta = 3!\,[(b - a) - \alpha(\zeta - a)]$.

The summary of the remaining portion of the study is provided below. Section 2 describes the solution to problem (1)-(2) as the solution of the corresponding integral equation that includes the kernel; after that, the sharper estimates of the integral of the kernel are computed. The fixed point theorems on metrics are used to demonstrate the results for the existence and uniqueness of (1)-(2) using estimations on the integral of the kernel. Examples are constructed to support the conclusions in Section 3.

2. Results of Preparation

We first derive the solution of problem (1)-(2) by expressing it into an equivalent integral equation involving kernel. After that, the sharper estimates of the integral of the kernel are computed. This help demonstrates our major findings.

For this, let $\Phi(x) \in C([a,b],\ R)$ then the unique solution to the problem

$$v^{(4)}(x) + \Phi(x) \quad = 0, \quad x \in [\mathrm{a},\mathrm{b}], \tag{3}$$

satisfying the conditions specified in (2) is obtained.

Theorem 2.1 Let $\Delta \neq 0$. Then the unique solution to problem (3) with (2) is given by

$$v(x) = \frac{\mu(x-\mathrm{a})^3}{\Delta} + \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{H}(x,\Upsilon)\Phi(\Upsilon)d\Upsilon,$$

where

$$\mathrm{H}(x,\Upsilon) = \mathrm{M}(x,\Upsilon) + \frac{\alpha(x-\mathrm{a})^3}{\Delta}\mathrm{N}(\zeta,\Upsilon), \tag{4}$$

$$\mathrm{M}(x,\Upsilon) = \begin{cases} \dfrac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!\,(\mathrm{b}-\mathrm{a})} - \dfrac{(x-\Upsilon)^3}{3!}, & \mathrm{a} \le \Upsilon \le x \le \mathrm{b}, \\ \dfrac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!\,(b-a)}, & \mathrm{a} \le x \le \Upsilon \le \mathrm{b}, \end{cases} \tag{5}$$

and

$$\mathrm{N}(\zeta,\Upsilon) = \begin{cases} \dfrac{(\Upsilon-\mathrm{a})(\mathrm{b}-\zeta)}{(\mathrm{b}-\mathrm{a})}, & \mathrm{a} \le \Upsilon \le \zeta \le \mathrm{b}, \\ \dfrac{(\zeta-\mathrm{a})(\mathrm{b}-\Upsilon)}{(\mathrm{b}-\mathrm{a})}, & \mathrm{a} \le \zeta \le \Upsilon \le \mathrm{b}. \end{cases} \tag{6}$$

Proof: The corresponding integral equation of (3) is

$$v(x) = A_0 + A_1x + A_2x^2 + A_3x^3 - \frac{1}{3!}\int_a^x (x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon, \tag{7}$$

where $A_0, A_1, A_2, \text{and } A_3$ are constants. Using the conditions (2), we have the following set of equations

$$\begin{gathered} A_0 + A_1\mathrm{a} + A_2\mathrm{a}^2 + A_3\mathrm{a}^3 = 0, \\ A_1 + 2A_2\mathrm{a} + 3A_3\mathrm{a}^2 = 0, \\ A_2 + 3A_3\mathrm{a} = 0, \\ 2A_2(1-\alpha) + 6A_3(b-\alpha\zeta) = \Gamma, \end{gathered}$$

where Γ is given in $(\mathrm{C}1)$. On solving the above four equations, we get

$$A_0 = -\frac{\mathrm{a}^3\Gamma}{\Delta}, \quad A_1 = \frac{3\mathrm{a}^2\Gamma}{\Delta}, \quad A_2 = -\frac{3\mathrm{a}\Gamma}{\Delta} \quad \text{and} \quad A_3 = \frac{\Gamma}{\Delta},$$

where Δ is given in (C2). Substituting these values in (7), we have

$$v(x) = [-\mathrm{a}^3 + 3\mathrm{a}^2 x - 3\mathrm{a}x^2 + x^3]\frac{\Gamma}{\Delta} - \frac{1}{3!}\int_{\mathrm{a}}^{x}(x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{(x-\mathrm{a})^3}{\Delta}\left[\mu + \int_{\mathrm{a}}^{\mathrm{b}}(\mathrm{b}-\Upsilon)\Phi(\Upsilon)d\Upsilon - \alpha\int_{\mathrm{a}}^{\zeta}(\zeta-\Upsilon)\Phi(\Upsilon)d\Upsilon\right] - \frac{1}{3!}\int_{\mathrm{a}}^{x}(x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathrm{a})^3}{\Delta} + \frac{(x-\mathrm{a})^3[(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})+\alpha(\zeta-\mathrm{a})]}{3![(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})](\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}(\mathrm{b}-\Upsilon)\Phi(\Upsilon)d\Upsilon - \frac{\alpha(x-\mathrm{a})^3}{\Delta}\int_{\mathrm{a}}^{\zeta}(\zeta-\Upsilon)\Phi(\Upsilon)d\Upsilon - \frac{1}{3!}\int_{\mathrm{a}}^{x}(x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathrm{a})^3}{\Delta} + \frac{(x-\mathrm{a})^3}{3!(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}(\mathrm{b}-\Upsilon)\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathrm{a})^3(\zeta-\mathrm{a})}{(\mathrm{b}-\mathrm{a})\Delta}$$

$$\int_{\mathrm{a}}^{\mathrm{b}}(\mathrm{b}-\Upsilon)\Phi(\Upsilon)d\Upsilon - \frac{\alpha(x-\mathrm{a})^3}{\Delta}\int_{\mathrm{a}}^{\zeta}(\zeta-\Upsilon)\Phi(\Upsilon)d\Upsilon - \frac{1}{3!}\int_{\mathrm{a}}^{x}(x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathrm{a})^3}{\Delta} + \int_{\mathrm{a}}^{x}\left[\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!(\mathrm{b}-\mathrm{a})} - \frac{(x-\Upsilon)^3}{3!}\right]\Phi(\Upsilon)d\Upsilon$$

$$+ \int_{x}^{\mathrm{b}}\left[\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!(\mathrm{b}-\mathrm{a})}\right]\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathrm{a})^3}{\Delta}\left\{\int_{\mathrm{a}}^{\zeta}\left[\frac{(\Upsilon-\mathrm{a})(\mathrm{b}-\zeta)}{(\mathrm{b}-\mathrm{a})}\right]\Phi(\Upsilon)d\Upsilon + \int_{\zeta}^{\mathrm{b}}\left[\frac{(\zeta-\mathrm{a})(\mathrm{b}-\Upsilon)}{(\mathrm{b}-\mathrm{a})}\right]\Phi(\Upsilon)d\Upsilon\right\}$$

$$= \frac{\mu(x-\mathrm{a})^3}{\Delta} + \int_{\mathrm{a}}^{\mathrm{b}}\mathrm{M}(x,\Upsilon)\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathrm{a})^3}{\Delta}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{N}(\zeta,\Upsilon)\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathrm{a})^3}{\Delta} + \int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}(x,\Upsilon)\Phi(\Upsilon)d\Upsilon.$$

Let $u(x)$ be another solution of (3) and (2) to establish the uniqueness of the solution. Take $w(x) = v(x) - u(x)$. Then

$$w^{(4)}(x) = 0, \quad x \in [\mathrm{a},\mathrm{b}], \tag{8}$$

$$w(\mathrm{a}) = 0, \quad w'(\mathrm{a}) = 0, \quad w''(\mathrm{a}) = 0, \quad w''(\mathrm{b}) - \alpha w''(\zeta) = 0. \tag{9}$$

Therefore, the solution of (8) is

$$w(x) = D_0 + D_1 x + D_2 x^2 + D_3 x^3,$$

where D_0, D_1, D_2 and D_3 are the arbitrary constants. By applying the conditions in (9), it can be written as a matrix form $\boldsymbol{AD} = \boldsymbol{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & \mathrm{a} & \mathrm{a}^2 & \mathrm{a}^3 \\ 0 & 1 & 2\mathrm{a} & 3\mathrm{a}^2 \\ 0 & 0 & 1 & 3\mathrm{a} \\ 0 & 0 & 2(1-\alpha) & 6(\mathrm{b}-\alpha\zeta) \end{bmatrix},$$

$$\mathbf{D} = [D_0 \quad D_1 \quad D_2 \quad D_3]^T$$

and

$$\mathbf{O} = [0 \quad 0 \quad 0 \quad 0]^T$$

with $|A| = 3!\,[(b-a) - \alpha(\zeta - a)]$ and is non-zero. So, the matrix system $\boldsymbol{AD} = \boldsymbol{O}$ has only a trivial solution. As a result, $w(x) \equiv 0$ for all $x \in [a,b]$. Thus, the uniqueness of the solution is proved.

Lemma 2.2 The kernel $\mathrm{M}(x,\Upsilon)$ given in (5) is non-negative for all $x,\Upsilon \in [\mathrm{a},\mathrm{b}]$.

Proof: Simple algebraic computations can be used to determine the positivity of $M(x,\Upsilon)$.

Lemma 2.3 The kernel $\mathrm{M}(x,\Upsilon)$ in (5) satisfies the following integral inequality.

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{M}(x,\Upsilon)d\Upsilon \leq \frac{9}{128}(\mathrm{b}-\mathrm{a})^4, \quad \text{for all} \quad x \in [\mathrm{a},\mathrm{b}]. \tag{10}$$

Proof: For all $x \in [a,b]$, consider

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{M}(x,\Upsilon)d\Upsilon &= \int_{\mathrm{a}}^{x}\left[\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!\,(\mathrm{b}-\mathrm{a})} - \frac{(x-\Upsilon)^3}{3!}\right]d\Upsilon + \int_{x}^{\mathrm{b}}\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)}{3!\,(\mathrm{b}-\mathrm{a})}d\Upsilon \\ &= \left[-\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)^2}{12(\mathrm{b}-\mathrm{a})} + \frac{(x-\Upsilon)^4}{24}\right]_{\mathrm{a}}^{x} + \left[-\frac{(x-\mathrm{a})^3(\mathrm{b}-\Upsilon)^2}{12(\mathrm{b}-\mathrm{a})}\right]_{x}^{\mathrm{b}} \\ &= \frac{(x-\mathrm{a})^3(\mathrm{b}-\mathrm{a})}{12} - \frac{(x-\mathrm{a})^4}{24}.\end{aligned}$$

Let $\Psi(x) = \frac{(x-a)^3(b-a)}{12} - \frac{(x-a)^4}{24}$. Then, by the application of results in basic calculus, the maximum value $\Psi(x)$ is attained at $x = \frac{3b-a}{2}$ and is given by

$$\max_{x\in[\mathrm{a},\mathrm{b}]}\Psi(x) = \max_{x\in[\mathrm{a},\mathrm{b}]}\left[\frac{(x-\mathrm{a})^3(\mathrm{b}-\mathrm{a})}{12} - \frac{(x-\mathrm{a})^4}{24}\right] = \frac{9}{128}(\mathrm{b}-\mathrm{a})^4.$$

Hence, the inequality (10).

Lemma 2.4 The kernel $\mathrm{N}(\zeta,\Upsilon)$ in (6) satisfies the following integral inequality.

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{N}(\zeta,\Upsilon)d\Upsilon \leq \frac{1}{2}(\mathrm{b}-\mathrm{a})^2.$$

Proof: For all $x \in [a,b]$, consider

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{N}(\zeta,\Upsilon)d\Upsilon &= \int_{\mathrm{a}}^{\zeta}\frac{(\Upsilon-\mathrm{a})(\mathrm{b}-\zeta)}{(\mathrm{b}-\mathrm{a})}d\Upsilon + \int_{\zeta}^{\mathrm{b}}\frac{(\zeta-\mathrm{a})(\mathrm{b}-\Upsilon)}{(\mathrm{b}-\mathrm{a})}d\Upsilon \\ &= \left[\frac{(\Upsilon-\mathrm{a})^2(\mathrm{b}-\zeta)}{2(\mathrm{b}-\mathrm{a})}\right]_{\mathrm{a}}^{\zeta} + \left[-\frac{(\zeta-\mathrm{a})(\mathrm{b}-\Upsilon)^2}{2(\mathrm{b}-\mathrm{a})}\right]_{\zeta}^{\mathrm{b}} \\ &= \frac{1}{2}(\zeta-\mathrm{a})(\mathrm{b}-\zeta) \\ &\leq \frac{1}{2}(\mathrm{b}-\mathrm{a})^2.\end{aligned}$$

Lemma 2.5 The kernel $\mathrm{H}(x,\Upsilon)$ in (4) satisfies the following integral inequality.

$$\int_{\mathrm{a}}^{\mathrm{b}} |\mathrm{H}(x,\Upsilon)|\,d\Upsilon \leq (\mathrm{b}-\mathrm{a})^4\left[\frac{9}{128} + \frac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})|}\right].$$

Proof: For all $x \in [a,b]$, consider

$$\begin{aligned}\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|d\Upsilon &= \int_{\mathrm{a}}^{\mathrm{b}}\left|\mathrm{M}(x,\Upsilon)+\frac{\alpha(x-\mathrm{a})^3}{\Delta}\mathrm{N}(\zeta,\Upsilon)\right|d\Upsilon\\ &\le \int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{M}(x,\Upsilon)|\,d\Upsilon+\left|\frac{\alpha(x-\mathrm{a})^3}{\Delta}\right|\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{N}(\zeta,\Upsilon)|\,d\Upsilon\\ &\le \frac{9}{128}(\mathrm{b}-\mathrm{a})^4+\frac{|\alpha|(\mathrm{b}-\mathrm{a})^3}{|\Delta|}\times\frac{(\mathrm{b}-\mathrm{a})^2}{2}\\ &= (\mathrm{b}-\mathrm{a})^4\left[\frac{9}{128}+\frac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})|}\right].\end{aligned}$$

The Banach and Rus fixed point theorems stated below are the primary tools for establishing our results.

Theorem 2.6 [19] Let $\mathbf{d}$ be a metric on a nonempty set D, and the pair $(\mathrm{D},\mathbf{d})$ form a complete metric space. If the function $\mathrm{F}:\mathrm{D}\to\mathrm{D}$ satisfies the following inequality for $v,w\in\mathrm{D}$,

$$\mathbf{d}(\mathrm{F}v,\mathrm{F}w)\le\beta\mathbf{d}(v,w),\text{ where } 0<\beta<1,$$

then there is a unique point $\vartheta^*\in D\ with\ F\vartheta^*=\vartheta^*$.

Theorem 2.7 [20] Let $\mathbf{d}$ and ρ be two metrics on a nonempty set D, and the pair $(\mathrm{D},\mathbf{d})$ form a complete metric space. If the function $\mathrm{F}:\mathrm{D}\to\mathrm{D}$ is continuous concerning the metric $\mathbf{d}$ on D and satisfies the following inequalities $v,w\in\mathrm{D}$,

$$\mathbf{d}(\mathrm{F}v,\mathrm{F}w)\le\theta\rho(v,w),\text{ where } \theta>0, \tag{11}$$

and

$$\rho(\mathrm{F}v,\mathrm{F}w)\le\kappa\rho(v,w),\text{ where } 0<\kappa<1, \tag{12}$$

then there is a unique point $\vartheta^*\in D\ with\ F\vartheta^*=\vartheta^*$.

3. Main Results Based on Metrics

This section establishes the uniqueness of the solution to the problem (1)-(2) based on metrics. Let D be the set of real-valued continuous functions on $[a,b]$. $For\ v(x),w(x)\in D$, define the metrics on D as follows:

$$\mathbf{d}(v,w)=\max_{x\in[\mathrm{a},\mathrm{b}]}|v(x)-w(x)|, \tag{13}$$

and

$$\rho(v,w)=\left(\int_{\mathrm{a}}^{\mathrm{b}}|v(x)-w(x)|^p dx\right)^{\frac{1}{p}},\qquad p>1. \tag{14}$$

Here the ordered pair $(D,\boldsymbol{d})$ forms a complete metric space, whereas (D,ρ) is metric space. The following helpful relation [9] between the two metrics $\boldsymbol{d}$ and ρ on D is given by

$$\rho(v,w)\le(\mathrm{b}-\mathrm{a})^{\frac{1}{p}}\mathbf{d}(v,w),\text{ for all } v,w\in\mathrm{D}. \tag{15}$$

Let us consider the operator $F:D\to D$ as

$$\mathrm{F}v(x)=\frac{\mu(x-\mathrm{a})^3}{\Delta}+\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}(x,\Upsilon)\,f(\Upsilon,v(\Upsilon))d\Upsilon,\text{ for all } x\in[\mathrm{a},\mathrm{b}],$$

where the kernel $H(x,\Upsilon)$ is mentioned in (4).

It is evident that $v(x)$ a solution of (1)-(2) if and only if $v(x)$ satisfies the following

$$v(x)=\frac{\mu(x-\mathrm{a})^3}{\Delta}+\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}(x,\Upsilon)\,f(\Upsilon,v(\Upsilon))d\Upsilon,\text{ for all } x\in[\mathrm{a},\mathrm{b}]. \tag{16}$$

Assume that the following condition is true.

(E1) $|f(x,v)-f(x,w)|\le\lambda|v-w|$, for all $(x,v),(x,w)\in[\mathrm{a},\mathrm{b}]\times R$, where λ is a Lipschitz constant.

Theorem 3.1 Suppose the condition (E1) is fulfilled. Let the function $f:[\mathrm{a},\mathrm{b}]\times R\to R$ be continuous with $f(x,0)\neq 0$, for all $x\in[\mathrm{a},\mathrm{b}]$. If a,b satisfies the following inequality

$$(\mathrm{b}-\mathrm{a})^4\left[\frac{9}{128}+\frac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})|}\right]<\frac{1}{\lambda} \tag{17}$$

then there is a unique solution to the problem (1)-(2).

Proof: We to prove that problem (1)-(2) has a unique solution. It is enough to prove that the operator$F$ has a unique fixed point $\vartheta^*\in D\ such\ that\ F\vartheta^*=\vartheta^*$. Every such fixed point will also lie in $C^{(4)}([a,b])$, as can be directly shown by differentiating (16).

Consider, for any $v,w\in D$ and for $x\in[a,b]$, we obtain

$$\begin{aligned}|\mathrm{F}v(x)-\mathrm{F}w(x)| &= |\tfrac{\mu(x-\mathrm{a})^3}{\Delta}+\textstyle\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}(x,\Upsilon)f(\Upsilon,v(\Upsilon))d\Upsilon-\tfrac{\mu(x-\mathrm{a})^3}{\Delta}+\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}(x,\Upsilon)f(\Upsilon,w(\Upsilon))d\Upsilon| \\ &\le \textstyle\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|\,|f(\Upsilon,v(\Upsilon))-f(\Upsilon,w(\Upsilon))|d\Upsilon \\ &\le \lambda\textstyle\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|\,|v(\Upsilon)-w(\Upsilon)|d\Upsilon \\ &\le \lambda\textstyle\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|\,\mathbf{d}(v,w)d\Upsilon \\ &\le \lambda(\mathrm{b}-\mathrm{a})^4\left[\tfrac{9}{128}+\tfrac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})|}\right]\mathbf{d}(v,w),\end{aligned}$$

Using $(E1)$. It is evident from the fact that

$$\mathbf{d}(\mathrm{F}v,\mathrm{F}w)\le\beta\mathbf{d}(v,w),$$

where

$$\beta=\lambda(\mathrm{b}-\mathrm{a})^4\left[\frac{9}{128}+\frac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a})-\alpha(\zeta-\mathrm{a})|}\right].$$

Using (17), we have $\beta<1$ and hence, the operator F has fulfilled the condition of Theorem 2.6. This implies that the operator$F$ has a unique fixed point and is the solution of (1)-(2).

We use two metrics by Rus theorem to establish the uniqueness of solutions of (1)-(2).

Theorem 3.2 Suppose the condition (E1) is fulfilled. Let the function$f:[\mathrm{a},\mathrm{b}]\times R\to R$ be continuous with $f(x,0)\neq 0$, for every $x\in[\mathrm{a},\mathrm{b}]$. If there are two positive numbers $p>1$, $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ with the inequality

$$\lambda\left(\int_a^b\left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < 1, \tag{18}$$

then there is a unique solution to the problem (1)-(2).

Proof: We to prove that problem (1)-(2) has a unique solution. It is enough to prove that the operator F has a unique fixed point $\vartheta^* \in D \; such \; that \; F\vartheta^* = \vartheta^*$. Every such fixed point will also lie in $C^{(4)}([a,b])$, as can be directly shown by differentiating (16). We first show that the inequality (11) of Theorem 2.7 is fulfilled. Consider, for any $v, w \in D$ and for $x \in [a,b]$, we obtain

$$\begin{aligned}|\mathrm{F}v(x) - \mathrm{F}w(x)| &\leq \int_a^b |\mathrm{H}(x,\Upsilon)||f(\Upsilon, v(\Upsilon)) - f(\Upsilon, w(\Upsilon))| \, d\Upsilon \\ &\leq \int_a^b |\mathrm{H}(x,\Upsilon)| \, \lambda \, |v(\Upsilon) - w(\Upsilon)| \, d\Upsilon \\ &\leq \left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{1}{q}} \lambda \left(\int_a^b |v(\Upsilon) - w(\Upsilon)|^P \, d\Upsilon\right)^{\frac{1}{P}} \\ &\leq \lambda \max_{x\in[a,b]} \left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{1}{q}} \rho(v,w)\end{aligned}$$

Using $(E1)$ and Holder's inequality [21]. Now, we define

$$\theta = \lambda \max_{x\in[a,b]} \left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{1}{P}}$$

We conclude that

$$\boldsymbol{d}(Fv, Fw) \leq \theta\rho(v,w), \text{ for some } \theta > 0 \text{ and for all } v, w \in \mathrm{D}. \tag{19}$$

Thus, the inequality (11) of Theorem 2.7 is fulfilled. Now, we apply (15) to (19), we get

$$\mathbf{d}(\mathrm{F}v, \mathrm{F}w) \leq \theta\rho(v,w) \leq \theta(\mathrm{b}-\mathrm{a})^{\frac{1}{p}}\mathbf{d}(v,w), \text{ for all } v, w \in \mathrm{D}.$$

Thus, for any given $\varepsilon > 0$, we can take $\delta = \frac{\varepsilon}{\theta(b-a)^{\frac{1}{p}}}$ such that $\boldsymbol{d}(Fv, Fw) < \varepsilon$ whenever $\boldsymbol{d}(v,w) < \delta$. Hence the operator F is continuous on D concerning the metric $\boldsymbol{d}$ given in (13).

Further, we show that the inequality (12) of Theorem 2.7 is fulfilled. Consider for any $v, w \in D$ and for $x \in [a,b]$, we obtain

$$\begin{aligned}\left(\int_a^b |\mathrm{F}v(x) - \mathrm{F}w(x)|^p \, dx\right)^{\frac{1}{p}} &\leq \left(\int_a^b \left[\left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{1}{q}} \lambda \left(\int_a^b |v(\Upsilon) - w(\Upsilon)|^p \, d\Upsilon\right)^{\frac{1}{p}}\right]^p dx\right)^{\frac{1}{p}} \\ &\leq \lambda \left(\int_a^b \left(\int_a^b |\mathrm{H}(x,\Upsilon)|^q \, d\Upsilon\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \rho(v,w).\end{aligned}$$

Therefore

$$\rho(\mathrm{F}v,\mathrm{F}w) \quad \leq \lambda\left(\int_{\mathrm{a}}^{\mathrm{b}}\left(\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|^{q}\,d\Upsilon\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\rho(v,w)$$

$$= \kappa\rho(v,w),$$

where

$$\kappa = \lambda\left(\int_{\mathrm{a}}^{\mathrm{b}}\left(\int_{\mathrm{a}}^{\mathrm{b}}|\mathrm{H}(x,\Upsilon)|^{q}\,d\Upsilon\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}.$$

Using (18), we have $\kappa < 1$ and hence, the operator F satisfies all the conditions of Theorem 2.7. This implies that the operator F has a unique fixed point and is the solution of (1)-(2).

As an application, the above conclusions are supported with examples.

Example 3.1 Consider

$$v^{(4)}(x) + 3 + 2x + cos\, v = 0, \qquad x \in \left[\frac{1}{4},1\right], \tag{20}$$

with

$$v\left(\frac{1}{4}\right) = 0, \quad v'\left(\frac{1}{4}\right) = 0, \quad v''\left(\frac{1}{4}\right) = 0, \quad v''(1) - \frac{1}{4}\,v''\left(\frac{3}{4}\right) = \mu. \tag{21}$$

Clearly, $f(x,0) \neq 0$ and $\Delta = \frac{15}{4} \neq 0$. Then

$$\left|\frac{\partial f(x,v)}{\partial v}\right| = sin\, v \leq 1$$

and

$$(\mathrm{b}-\mathrm{a})^{4}\left[\frac{9}{128} + \frac{|\alpha|(\mathrm{b}-\mathrm{a})}{12|(\mathrm{b}-\mathrm{a}) - \alpha(\zeta-\mathrm{a})|}\right] = \frac{4941}{163840} = 0.03015747 < \frac{1}{\lambda}.$$

So, all the assumptions of the Theorem 3.1 are satisfied, and thus the problem (20)-(21) has a unique solution.

Example 3.2 Consider

$$v^{(4)} + 3 + 2x + cos\, v = 0, \qquad x \in [0,1], \tag{22}$$

with

$$v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 0, \quad v''(1) - v''\left(\frac{1}{2}\right) = \mu. \tag{23}$$

Clearly, $f(x,0) \neq 0$ and $\Delta = 3 \neq 0$. Then

$$\left|\frac{\partial f(x,v)}{\partial v}\right| = sin\, v \leq 1.$$

For simplicity, we take $p = 2$ and $q = 2$ then, by algebraic computation, one can obtain

$$\int_0^1 |\mathrm{H}(x,\Upsilon)|^2\, d\Upsilon = \frac{1}{56}x^7 + \frac{1}{27}x^6(1-x)^3,$$

$$\left(\int_0^1 \left(\int_0^1 |\mathrm{H}(x,\Upsilon)|^2\, d\Upsilon\right) dx\right) = \frac{59}{25920} = 0.0022762$$

and so

$$\left(\int_0^1 \left(\int_0^1 |\mathrm{H}(x,\Upsilon)|^2\, d\Upsilon\right) dx\right)^{\frac{1}{2}} = 0.0477095 < \frac{1}{\lambda}.$$

So, all the assumptions of Theorem 3.2 are satisfied, and thus the problem (22)-(23) has a unique solution.

Remark 3.1 Rus theorem involves two metrics that may not be necessarily equivalent. In particular, the space in Rus theorem is assumed to be complete concerning the first metric but not necessarily complete for the second metric. Also, the operator is assumed to be contractive concerning the second metric. Hence, the Rus theorem applies to a larger class of problems than the Banach theorem.

4. Conclusion

We established the existence of a unique solution to the fourth-order nonlinear differential equations with non-homogeneous three-point boundary conditions by employing Banach fixed point theorem on a metric space and also established the unique solution to the problem in the larger intervals with different assumptions by applying Rus fixed point theorem on metric spaces.

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