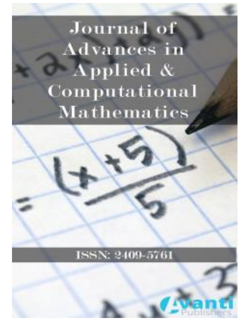
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



Arithmetic Mean-Geometric Mean Inequality for Convex Fuzzy Sets

Dashrath Singh [1] and Jai N. Singh [2,*]

[1] Ahmadu Bello University, Zaria, Nigeria

[2] Department of Mathematics and Computer Science, Barry University, 11300 NE Second Avenue, Miami Shores, FL 33161, USA

ARTICLE INFO

Article Type: Research Article

Academic Editor: Sarah Ghanim

Keywords:
Arithmetic mean
Geometric mean
Jensen inequality
Convex fuzzy sets

Timeline:
Received: July 08, 2023
Accepted: September 22, 2023
Published: October 23, 2023

Citation: Singh D, Singh JN. Arithmetic mean-geometric mean inequality for convex fuzzy sets. J Adv App Comput Math. 2023; 10: 71-76.

DOI: https://doi.org/10.15377/2409-5761.2023.10.7

ABSTRACT

Convex analysis is a discipline of mathematics dedicated to the explication of the properties of convex sets and convex functions. Convex functions are extremely useful in proving many famous inequalities in mathematics. It is widely known that inequalities defined by convex functions originated with the works done by Holder and Jensen among others and applied to modeling a variety of problems both in hard and soft sciences. It is said that the arsenal of an analyst is heavily stocked with inequalities. Studies related to convexity have kept occupying a central position in almost all areas of mathematics, especially in functional analysis and operations research. Following the emergence of fuzzy set theory, fuzzy convexity alongside a number of related concepts has been explicated. However, not much has been done regarding fuzzy inequalities defined by fuzzy convexity. In this paper, a novel attempt to study arithmetic mean-geometric mean is proposed. In this short note, we provide two proofs of the arithmetic mean-geometric mean inequality for convex fuzzy sets.

Mathematics Subject Classification 2020: 03E72, and 94D05.

*Corresponding Author
Email: jsingh@barry.edu
Tel: +(90) 5057729751

1. Introduction

Convex analysis is a branch of mathematics that deals with the explication of the properties of convex sets and convex functions. A set is called convex if the line joining any two points in this set also lies in the set. Convex functions also play a significant role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by several expedient properties. Various properties of the convex set and convex functions can be found in [1–12]. In 1965 Zadeh introduced the concept of fuzzy sets and after that several papers appeared in literature for example (see [4–6, 12–27] among others).

It is extensively known that inequalities defined by convex functions originated with Holder [28] and Jensen [29] and applied to modeling a variety of problems both in hard and soft sciences. Many famous inequalities in classical mathematics can be derived by applying Jensen's inequality [29] to some suitable convex function. Certainly, the notion of convexity and Jensen's inequality can be made the foundation for the development of the theory of inequalities in classical mathematics.

It is well known that Jensen [29] laid the foundation of convex functions and applied it to obtain several generalizations of arithmetic mean-geometric mean inequality (see [2, 21, 30] for details). The notion of fuzzy convexity was originally introduced by Zadeh [31]. In a number of subsequent studies related to the subject of fuzzy convexity, several interesting and useful results have been established. In fact, many fuzzy inequalities have been established in which convexity played a significant role ([4, 13, 20, 21, 25, 32], just to mention a few). Not very surprisingly, most of the results obtained for fuzzy convexity can be viewed as extensions of similar results for classical convexity. Recently, Jensen's inequality for convex fuzzy sets along with a couple of related fuzzy inequalities has been established [33]. In this short note, we provide two proofs of arithmetic mean-geometric mean inequality for convex fuzzy sets.

2. Preliminaries

Let E denote the n-dimensional Euclidean vector space $\mathbb{R}^n$, and I denotes the real unit interval [0, 1]. In general, we shall make no distinction between notions for a fuzzy set with constant value and that value itself, and, for simplicity, coordinate hyperplanes will be considered.

Let $\mu: E \to [0,1]$ denote a fuzzy set. We shall adopt the following definitions:

2.1. Definition of Convex Fuzzy Sets

The most commonly used definition of fuzzy convexity ([4, 31] and many others) is the following:

A fuzzy set μ on E is convex if and only if, for all $x, y \in E$, and $\lambda \in I$, we have

$$\mu\,(\lambda x + (1-\lambda)y) \geq Min\,\{\mu(x), \mu(y)\}$$

or, equivalently,

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x) \wedge \mu(y). \tag{1}$$

Moreover, μ is concave if -μ is convex.

We shall also make use of a more general definition of fuzzy convexity as given in [4, 25, 26]:

A fuzzy set μ is convex iff

$$\mu(\lambda x + (1-\lambda)y \geq \{\,\lambda\mu(x) + (1-\lambda)\mu(y)\} \tag{2}$$

for all $x, y \in supp\ \mu = \{x: \mu(x) > 0\},\ \lambda \in I$.

In other words, a fuzzy set μ is convex iff

$$\mu(\lambda_1 x + \lambda_2 y) \geq \lambda_1\mu(x) + \lambda_2\mu(y)$$

for all x, y ∈ $supp$ μ and $\lambda_1, \lambda_2 \in I$, $\lambda_1 + \lambda_2 = 1$.

It may be noted that definition (1) follows from definition (2) since

$$Max\,\{\mu(x), \mu(y)\} \geq \lambda\mu(x) + (1-\lambda)\mu(y) \geq Min\,\{\mu(x), \mu(y)\}. \quad (3)$$

See also [28, 2] for some comments on definition (1).

Furthermore, given a convex fuzzy set μ, it is immediate to see that the open chord

$\{(\lambda x + (1-\lambda)y,\ \lambda\mu(x) + (1-\lambda)\mu(y)\}, \lambda \in (0,1),\ x, y \in supp\ \mu\}$ joining the fuzzy points $(x, \mu(x))$ and $(y, \mu(y))$ on the graph of μ lies in its fuzzy hypograph (defined as the set $\{(x,t)|x \in E, t \in (0, \mu(x))\}$). Thus it follows from definition (2) that μ is convex if and only if its hypograph is convex.

It may be noted that for a fuzzy set μ: $E \rightarrow [0,1]$, we have for $x_1, x_2 \in E$, $x = \lambda x_1 + (1-\lambda)x_2 \in E$ and hence, $\mu(x) = \lambda\mu(x_1) + (1-\lambda)\mu(x_2)$, λ∈ [0,1], (see [26], definition 1).

2.2. Jensen's Inequality for Convex Fuzzy Sets [33]

Let $\mu: E \rightarrow I$ be a convex fuzzy set. Then it satisfies Jensen's Inequality in its generalized form:

$$\mu(\sum_{k=1}^{n} \lambda_k\ x_k) \geq \sum_{k=1}^{n} \lambda_k\ \mu(x_k) \geq \min_{1\leq k\leq n} \mu(x_k)$$

for any $x_{1,}\ x_{2,} \cdots, x_n \in E,\ and\ \ \lambda_{1,}\ \lambda_{2,} \cdots, \lambda_n \in (0,1)$ with

$$\sum_{k=1}^{n} \lambda_k\ = 1.$$

Note that at least one $\lambda_k \neq 1$ since $\sum_{k=1}^{n} \lambda_k\ = 1$

3. Main Results

3.1. Theorem

Let μ be a convex fuzzy set. Then for $x_1 \ldots, x_n \in R$, the following inequality holds:

$$\mu(\tfrac{1}{n}\sum_{k=1}^{n} x_k) \geq \tfrac{1}{n}\sum_{k=1}^{n} \mu(x_k) \geq (\prod_{k=1}^{n} \mu(x_k))^{\frac{1}{n}} \quad (*)$$

Proof

The proof outlined below is obtained by mimicking the proof of the AM-GM inequality in classical analysis attributed to A-L Cauchy (Cf. [14], for example).

It is immediate to set that the first half of $(*)$ follows from Jensen's inequality (2.1, [33]). In the following, we provide a proof of the second half of $(*)$.

Let us assume that $(*)$ holds for *n = 2*.

In fact, this can be seen as follows:

$$(\mu(x_1) - \mu(x_2))^2 \geq 0$$

$$\Rightarrow (\mu(x_1) + \mu(x_2))^2 \geq 4\, \mu(x_1)\mu(x_2)$$

$$\Rightarrow \frac{\mu(x_1)+\mu(x_2)}{2} \geq (\mu(x_1)\mu(x_2))^{\frac{1}{2}}\,. \qquad (\#)$$

Equality holds iff $\mu(x_1) = \mu(x_2) \Rightarrow x_1 = x_2$.

Now we suppose that ($*$) holds for *n* = *m*, and show that it holds for n = 2m. This can be seen as follows:

Let $x_1, \dots, x_m, y_1, \dots y_m \in \mathbb{R}^+$. Then

$$(\mu(x_1) \dots \mu(x_m)\mu(y_1) \dots \mu(y_m))^{\frac{1}{2m}} = \left[(\mu(x_1) \dots \mu(x_m))^{\frac{1}{m}} (\mu(y_1) \dots \mu(y_m))^{\frac{1}{m}}\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\left[(\mu(x_1) \dots \mu(x_m))^{\frac{1}{m}} + (\mu(y_1) \dots \mu(y_m))^{\frac{1}{m}} \right], \text{ (by (\#))}$$

$$\leq \frac{1}{2}\left[\frac{\mu(x_1)+\cdots+\mu(x_m)}{m} + \frac{\mu(y_1)+\cdots+\mu(y_m)}{m}\right], \text{ (by our supposition)}$$

$$= \frac{\mu(x_1) + \cdots + \mu(x_m) + \mu(y_1) + \cdots + \mu(y_m)}{2m}$$

Equality holds for $\mu(x_1) = \cdots = \mu(x_m) = \mu(y_1) = \cdots = \mu(y_m)$.

Thus ($*$) holds for n = 2m i. e., whenever n is a power of 2. In order to complete the proof of ($*$), we need to show that it holds for any arbitrary integer n.

Let $n < 2^r = N$, and $\mu(x) = \frac{1}{n}\sum_{k=1}^{n} \mu(x_k)$.

Let $\mu(x_{n+1}) = \cdots = \mu(x_N) = \mu(x)$.

Then we have $\prod_{k=1}^{n} \mu(x_k) = (\mu(x))^{N-n} \prod_{k=1}^{n} \mu(x_k)$

$$\leq \left(\frac{1}{N}\sum_{k=1}^{N} \mu(x_k)\right)^N = (\mu(x))^N$$

$$\Rightarrow \prod_{k=1}^{n} \mu(x_k) \leq (\mu(x))^n$$

$$\Rightarrow \left(\prod_{k=1}^{n} \mu(x_k)\right)^{\frac{1}{n}} \leq \mu(x) = \frac{1}{n}\sum_{k=1}^{n} \mu(x_k).$$

Equality holds iff $x_1 = \cdots = x_N$ i. e., iff $x_1 = \cdots = x_n$.

3.2. Proof of Theorem ($*$) from Jensen's Inequality for Convex Fuzzy Sets

It is immediate to see that the first half of the inequality ($*$) is a direct consequence of the first half of Jensen's inequality. We outline the proof of the second half as follows:

For brevity, let the arithmetic mean $\frac{1}{n}\sum_{k=1}^{n} \mu(x_k)$ be denoted by $A(\mu(x))$ and the geometric mean $(\prod_{k=1}^{n} \mu(x_k))^{\frac{1}{n}}$ by $G(\mu(x))$ where $x = (x_1, \dots, x_n)$ be a sequence of points in $\mathbb{R}$, and non-negative n[th] root is being taken. Note that each $\mu(x)$ is positive. We need to show that $A(\mu(x)) \geq G(\mu(x))$. ($**$)

Proof of ($**$) can be straightforwardly obtained from Jensen's inequality by exploiting the convexity of $-\log \mu(x)$ on $\mathbb{R}^+$.

$$\text{I.e.,} \quad \frac{1}{n}\sum_{k=1}^{n} \log\, \mu(x_k) \leq \log\left(\sum_{k=1}^{n} \frac{\mu(x_k)}{n}\right)$$

which is equivalent to (∗).

Remark. Proof of (∗) can also be obtained from Jensen's inequality (in section 2.2) by proving

$$\left(\prod_{k=1}^{n} \mu(x_k)\right)^{\frac{1}{n}} \leq min_{1\leq k\leq n}\ \mu(x_k)$$

which in turn, can be obtained by the induction principle.

4. Concluding Remarks

The notion of fuzzy convexity as an extension of classical convexity was originally introduced by Zadeh in his seminal work [31]. It is well-known that Jensen [29] laid the foundation for convex functions and used it to establish several generalizations of arithmetic mean-geometric mean inequality. In [33] authors have established Jensen's Inequality for convex fuzzy set. Possibly this is the first attempt of this kind. In the classical convex analysis, many inequalities can be derived by applying Jensen's inequality to some suitable convex function. We have not been able to include many fuzzy counterparts of many fundamental results related to arithmetic mean-geometric mean inequalities, that exist in classical mathematics, and trust this will be an exciting area for future research.

Acknowledgments

The authors are thankful to the editor and the two referees for their constructive comments and useful suggestions that have improved the quality of the presentation of the paper.

Conflict of Interest

The authors declare no conflict of interest.

Funding

The authors declare that no funding was available for conducting this research work.

References

[1] Avriel M, Diewert WE, Schaible S, Zang I. Generalized concavity. Boston, MA: Plenum Press; 1988. https://doi.org/10.1007/978-1-4684-7600-2

[2] Boyd S, Vandenberghe L. Convex optimization. Cambridge University Press; 2004. https://doi.org/10.1017/CBO9780511804441

[3] Dubois D, Esteva F, Godo L, Prade H. An information-based discussion of vagueness. IEEE International Conference on Fuzzy Systems, vol. 2, Melbourne: 2001, p. 781-4. https://doi.org/10.1109/FUZZ.2001.1009071

[4] Drewniak J. Convex and strongly convex fuzzy sets. J Math Anal Appl. 1987; 126: 292-300. https://doi.org/10.1016/0022-247X(87)90093-X

[5] Foo N, Low BT. A Note on prototypes, convexity and fuzzy sets. Stud Log. 2008; 90: 125-37. https://doi.org/10.1007/s11225-008-9146-1

[6] Gadjiev D, Kochetkov I, Rustanov A. The convex fuzzy sets and their properties with application to the modeling with fuzzy convex membership functions. In: Murgual V, Pukhkal V, editors. Advances in Intelligent Systems and Computing, EMMFT: Springer; 2021, p. 276-84. https://doi.org/10.1007/978-3-030-57453-6_24

[7] Jain R. Tolerance analysis using fuzzy sets. Int J Syst. 1976; 7: 1393-401. https://doi.org/10.1080/00207727608942013

[8] Matłoka M. A Note on convex fuzzy processes. Adv Fuzzy Syst. 2012; 2012: 1-4. https://doi.org/10.1155/2012/290845

[9] Rockafellar RT. Convex analysis. Princeton University Press; 1970. https://doi.org/10.1515/9781400873173

[10] Syau Y-R. Closed and convex fuzzy sets. Fuzzy Sets Syst. 2000; 110: 287-91. https://doi.org/10.1016/S0165-0114(98)00082-7

[11] Tremblay J, Manohar R. Discrete mathematical structures with applications to computer science. McGraw Hill; 1975.

[12] Youness EA. E-Convex Sets, E-Convex Functions, and E-Convex Programming. J Optim Theory Appl. 1999; 102: 439-50. https://doi.org/10.1023/A:1021792726715

[13] Ammar EE. Some properties of convex fuzzy sets and convex fuzzy cones. Fuzzy Sets Syst. 1999; 106: 381-6. https://doi.org/10.1016/S0165-0114(97)00273-X

[14] Ammar E, Metz J. On fuzzy convexity and parametric fuzzy optimization. Fuzzy Sets Syst. 1992; 49: 135-41. https://doi.org/10.1016/0165-0114(92)90319-Y

[15] Bellman RE, Zadeh LA. Decision-Making in a Fuzzy Environment. Manage Sci. 1970; 17: 141-64. https://doi.org/10.1287/mnsc.17.4.b141

[16] Chalco-Cano Y, Rojas-Medar MA, Osuna-Gómez R. S-convex fuzzy processes. Comput Math Appl. 2004; 47: 1411-8. https://doi.org/10.1016/S0898-1221(04)90133-2

[17] Dubois D, Prade H. Fuzzy Sets and Systems: Theory and Applications. Academic Press 1980; 33. https://doi.org/10.2307/2581310

[18] Proctor CH, Kaufman A, Swanson DL. Introduction to the theory of fuzzy subsets. vol. 1. NY: 1975. https://doi.org/10.2307/2286947

[19] Klir G, Yuan B. Fuzzy sets and fuzzy logic: Theory and applications. Prentice Hall; 1995.

[20] Yang X. Some properties of convex fuzzy sets. Fuzzy Sets Syst. 1995; 72: 129-32. https://doi.org/10.1016/0165-0114(94)00285-F

[21] Lowen R. Convex fuzzy sets. Fuzzy Sets Syst. 1980; 3: 291-310. https://doi.org/10.1016/0165-0114(80)90025-1

[22] Matłoka M. Convex fuzzy processes. Fuzzy Sets Syst. 2000; 110: 109-14. https://doi.org/10.1016/S0165-0114(98)00053-0

[23] Matłoka M. Some properties of Φ-convex fuzzy processes. J Fuzzy Math. 1999; 7: 706-26. https://doi.org/10.25291/VR/1999-1-VR-706

[24] Syau Y-R, Low C-Y, Wu T-H. A note on convex fuzzy processes. Appl Math Lett. 2002; 15: 193-6. https://doi.org/10.1016/S0893-9659(01)00117-3

[25] Wu S-Y, Cheng W-H. A note on fuzzy convexity. Appl Math Lett. 2004; 17: 1127-33. https://doi.org/10.1016/j.aml.2003.11.003

[26] Yang X. Some properties of convex fuzzy sets. Fuzzy Sets Syst. 1995; 72: 129-32. https://doi.org/10.1016/0165-0114(94)00285-F

[27] Yatabe S, Inaoka H. On evans's vague object from set theoretic viewpoint. J Philos Logic. 2006; 35: 423-34. https://doi.org/10.1007/s10992-005-9022-7

[28] Hölder O. Ueber einen Mittelwertsatz Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen. Band (in German) 1889; (2): 38-47.

[29] Jensen JLWV. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. Acta Math. 1906; 30: 175-93. https://doi.org/10.1007/BF02418571

[30] Bollobás B. Linear analysis: An introductory course. 2nd ed. Cambridge University Press; 1999. https://doi.org/10.1017/CBO9781139168472

[31] Zadeh LA. Fuzzy sets. Inf Control. 1965; 8: 338-53. https://doi.org/10.1016/S0019-9958(65)90241-X

[32] Yang X. Convexity of semicontinuous functions. J Oper Res Soc Ind. 1994; 31: 309-17.

[33] Singh D, Singh J. Jensen's inequality for convex fuzzy sets. J Fuzzy Math. 2019; 27: 743–8.

[34] Cvetkovski Z. Inequalities theorems, techniques, and selected problems. Berlin, Heidelberg: Springer Berlin Heidelberg; 2012. https://doi.org/10.1007/978-3-642-23792-8