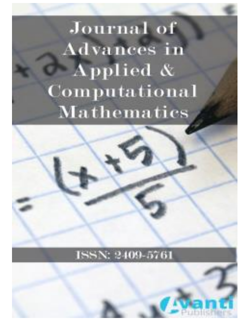
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761




# On the $G'/G$ Expansion Method Applied to (2+1)-Dimensional Asymmetric-Nizhnik-Novikov-Veselov Equation

S.M. Mabrouk[1] and A.S. Rashed[1,2,*]

[1]Department of Physics and Engineering Mathematics, Faculty of Engineering, Zagazig University, AlSharqia, Egypt
[2]Faculty of Engineering, Delta University for Science and Technology, Gamasa, Egypt

## ARTICLE INFO

*Article Type*: Research Article

*Academic Editor*: Vikas Gupta

*Keywords*:
ANNV equation
Soliton solution
$G'/G$ expansion method

*Timeline*:
Received: July 29, 2023
Accepted: August 30, 2023
Published: September 07, 2023

*Citation*: Mabrouk SM, Rashed AS. On the $G'/G$ expansion method applied to (2+1)-dimensional asymmetric-nizhnik-novikov-veselov equation. J Adv App Comput Math. 2023; 10: 39-49.

*DOI*: https://doi.org/10.15377/2409-5761.2023.10.4

## ABSTRACT

In this paper, the $G'/G$ expansion method is applied to the (2+1)-dimensional Asymmetric-Nizhnik-Novikov-Veselov equation (ANNV). The motivation is creating new families of solitary waves. The system of equations has been combined in one partial differential equation (PDE) and the traveling wave variable has been applied to transform the resultant equation into an ordinary differential equation (ODE). The homogenous balance condition has been applied to determine the truncation variable of the $G'/G$ expansion. Four cases are created according to the appropriate choice of the arbitrary constants relations. For each case, some new solitary wave solutions including solitons and kinks represented by trigonometric, hyperbolic, logarithmic, polynomial, and combinations of these functions.

*Corresponding Author
Email: ahmed.s.rashed@gmail.com
Tel: +(20) +1224330787

# 1. Introduction

Mathematical models that take into account nonlinearity in the dynamics of a system are referred to as nonlinear evolution equations. These models are used to represent the change that occurs in a system over time. These equations are very important in a variety of scientific fields, including engineering, biology, and physics, among others [1-4]. In these areas, the ability to understand and analyze the behavior of nonlinear evolution equations has major consequences for the ability to forecast and regulate complex processes. The purpose of this study article is to investigate the applications of one of the well-known nonlinear evolution equations Asymmetric - Nizhnik - Novikov - Veselov equation in a variety of fields and to emphasize the significance of these equations in terms of comprehending events that occur in the real world.

The Asymmetric - Nizhnik - Novikov - Veselov equation is a two-dimensional KdV equation described by the system of equations:

$$u_x - v_y = 0 \tag{1.1a}$$

$$u_t - 3(uv)_x + u_{xxx} = 0 \tag{1.1b}$$

This system of equations first derived by Boiti *et al*. [5] is a model for an incompressible fluid where $u$ and $v$ are the components of the dimensionless velocity [6]. ANNV equations are also obtained from a symmetry constraint of the Kadomtsev-Petviashvili (KP) equation [7, 8]. The system of equations (1.1a) and (1.1b) has been widely investigated from various perspectives, such as the study of its Painlevé property [9], Lie symmetries [10, 11] and solutions using arbitrary exponential functions [12]. The conservation laws forms of this equation were also studied in [13] while iterative solutions based on Darboux and Bäcklund transformations were presented in [14, 15]. Its exact solution using a separation of variable approach was also considered in [16-19]. Equations (1.1a) and (1.1b) are here reduced to a single equation through the transformations; $v = \omega_x$ and $u = \omega_y$ giving;

$$\omega_{yt} + \omega_{xxxy} - 3\omega_{xy}\omega_x - 3\omega_y\omega_{xx} = 0 \tag{1.2}$$

In [20], multi-periodic wave solutions were constructed for Eq. (1.2) using Hirota's bilinear method and Riemann theta function while in [21] new solutions were obtained through a Bäcklund transformation and a modified Clarkson direct method. New exact solutions of Eq. (1.2) were obtained using Bell exponential polynomial in [22] or through a linearizing function having a Miura form in [23]. Notice that most of the quoted previous work is concerned with the similarity reduction of Eq. (1.2) while the reduction of its Lax pair is much less frequent [11]. Generally, evolution equations were heavily discussed using numerous techniques such as Lie infinitesimals and hidden symmetries [24-33], Lax pair and group method [34-38], numerical techniques [39-44], direct traveling wave methods [26, 45-51].

This research is organized as follows. Section 2 is devoted by describing the $(G'/G)$ method. Next, the method is applied to the ANNV equation in Section 3. Number of obtained cases are described and depicted in the section 4. Finally, the paper ends with the concluding remarks.

# 2. Description of $(G'/G)$ Expansion Method

The (2+1) nonlinear evolution equation represented by

$$P\left(u, u_t, u_x, u_y, u_{xt}, u_{yt}, u_{tt}, u_{xx}, u_{yy}, \ldots \ldots\right) = 0 \tag{2.1}$$

where $u = u(x, y, t)$ is an unknown function, $P$ is a polynomial in $u$ and its partial derivatives. The $(G'/G)$ expansion method can be summarized as:

First, the PDE (2.1) is transformed into an ODE:

$$P(u, u', u'', \dots \dots) = 0 \tag{2.2}$$

through introducing a traveling wave variable:

$$u(x, y, t) = u(\eta), \eta = x + y - ct \tag{2.3}$$

where c is a constant. If necessary, the ODE (2.2) can be integrated many times considering the constant of integration to be zero.

Second, the solution of the nonlinear differential equation is expressed in the form

$$u(\eta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i \tag{2.4}$$

where $G = G(\eta)$ satisfies the second-order linear ordinary differential equation

$$G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0 \tag{2.5}$$

where $G' = \frac{dG}{d\eta}$ , $G'' = \frac{d^2G}{d\eta^2}$ , $a_i$ , λ and μ are real constants to be determined.

The positive integer $m$ is determined through the homogeneous balance between the orders of the highest derivatives and highly nonlinear terms as follows:

$$\begin{cases} O\left[u^r \left(\frac{d^q u}{d\eta^q}\right)^s\right] = mr + s\,(q + m) \\ O\left(\frac{d^p u}{d\eta^p}\right) = m + p \end{cases} \tag{2.6}$$

Substituting (2.4) into (2.2), using (2.5), then collecting all terms with the same order of $(G'/G)$ and setting each coefficient to zero yields a set of algebraic equations for $a_i, c,$ μ and $\lambda$.

# 3. Mathematical Application

This section is motivated to find the explicit solutions of Eq. (1.2). First, inserting equation (2.3) into (1.2) confers

$$u^{(4)} - 6\,u'u'' - cu'' = 0 \tag{3.1}$$

where dashes refer to the derivatives with $\eta$. Integrating (3.1) with respect to $\eta$ yields

$$u''' - 3u'^2 - cu' = 0 \tag{3.2}$$

Letting $u' = v$, yields

$$v'' - cv - 3v^2 = 0 \tag{3.3}$$

Homogeneous balance between $v''$ and $v^2$ yields $m = 2$, then substituting into (2.4) yields,

$$v(\eta) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2 \tag{3.4}$$

Substituting from (3.4) using (2.5) into (3.3) yields,

$$(6a_2 - 3a_2^2)\left(\frac{G'}{G}\right)^4 + (10a_2\lambda - 6a_1a_2 + 2a_1)\left(\frac{G'}{G}\right)^3 + (3a_1\lambda + 4a_2\lambda^2 + 8a_2\mu - ca_2 - 3a_1^2 - 6a_0a_2)\left(\frac{G'}{G}\right)^2 + (a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu - ca_1 - 6a_0a_1)\left(\frac{G'}{G}\right) + (a_1\lambda\mu + 2a_2\mu^2 - ca_0 - 3a_0^2) = 0 \quad (3.5)$$

after collecting all terms with the same order of $(G'/G)$ with setting each coefficient to zero obtain a set of algebraic equations for $a_i, c, \mu$ and $\lambda$.

$$\begin{cases} 6a_2 - 3a_2^2 = 0 \\ 10a_2\lambda - 6a_1a_2 + 2a_1 = 0 \\ 3a_1\lambda + 4a_2\lambda^2 + 8a_2\mu - ca_2 - 3a_1^2 - 6a_0a_2 = 0 \\ a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu - ca_1 - 6a_0a_1 = 0 \\ a_1\lambda\mu + 2a_2\mu^2 - ca_0 - 3a_0^2 = 0 \end{cases} \quad (3.6)$$

Solving this system of equations reveal four cases.

# 4. Cases Study

In this section, many cases are studied according to the relations between the constants (Fig **1-4**).

**Case 1**

$$a_0 = 2\mu, a_1 = 2\lambda, a_2 = 2 \text{ and } c = 2\lambda - 4\mu = \alpha \quad (4.1)$$

$G$ is found through solution of equation (2.5) by setting $\alpha = 2\lambda - 4\mu$

i- for $\alpha > 0$

$$v = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right]^2 \quad (4.2)$$

For $C_1 = 0$ and $C_2 = 1$

$$v_1 = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right]^2 \quad (4.3)$$

$$u_1 = 2\mu\eta - \frac{\lambda^2\eta}{2} - \sqrt{\alpha}\coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) - \frac{\sqrt{\alpha}}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) + \frac{\sqrt{\alpha}}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \quad (4.4)$$

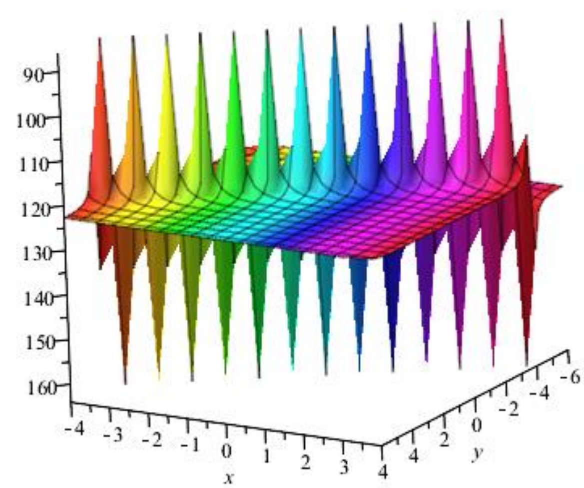



**Figure 1:** The soliton solution $u_1$ for $\lambda = 3, \mu = 1, t = 10, \alpha = 5$ and $c = 5$.

For $C_1 = 1$ and $C_2 = 0$

$$v_2 = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.5}$$

$$u_2 = 2\mu\eta - \frac{\lambda^2\eta}{2} - \sqrt{\alpha}\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) - \frac{\sqrt{\alpha}}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) + \frac{\sqrt{\alpha}}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \tag{4.6}$$

ii- for $\alpha < 0$

$$v = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right]^2 \tag{4.7}$$

For $C_1 = 0$ and $C_2 = 1$

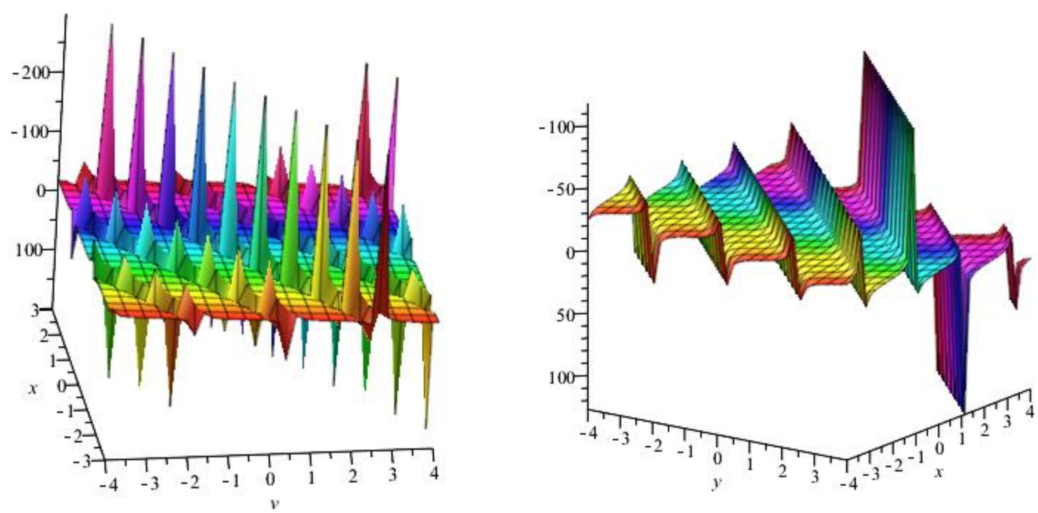
$$v_3 = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.8}$$

$$u_3 = 2\mu\eta - \frac{\lambda^2\eta}{2} + \frac{\alpha}{\sqrt{-\alpha}}\left(\cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) - \frac{\alpha\pi}{2\sqrt{-\alpha}} + \frac{\alpha}{\sqrt{-\alpha}}\cot^{-1}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) \tag{4.9}$$

For $C_1 = 1$ and $C_2 = 0$

$$v_4 = 2\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.10}$$

$$u_4 = 2\mu\eta - \frac{\lambda^2\eta}{2} - \frac{\alpha}{\sqrt{-\alpha}}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) + \frac{\alpha}{\sqrt{-\alpha}}\tan^{-1}\left(tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) \tag{4.11}$$



a. solution $u_3$ for $\boldsymbol{\lambda = 3, \mu = 4,}$ $\boldsymbol{t = 0.5, \alpha = -8}$ and $\boldsymbol{c = -8}$

b. multi kink solution $u_4$ for $\boldsymbol{\lambda = 3,}$ $\boldsymbol{\mu = 4, t = 0.5, \alpha = -8}$ and $\boldsymbol{c = -8}$

**Figure 2:** The traveling wave solutions $u_3$ and $u_4$.

## Case 2

$$a_0 = \frac{\lambda^2}{3} + \frac{2}{3}\mu,\ a_1 = 2\lambda,\ a_2 = 2 \text{ and } c = -\alpha \tag{4.12}$$

i- for $\alpha > 0$

$$v = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right]^2 \tag{4.13}$$

For $C_1 = 0\ and\ C_2 = 1$

$$v_5 = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.14}$$

$$u_5 = \frac{2}{3}\mu\eta - \frac{\lambda^2\eta}{6} - \sqrt{\alpha}\coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) - \frac{\sqrt{\alpha}}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) + \frac{\sqrt{\alpha}}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \tag{4.15}$$

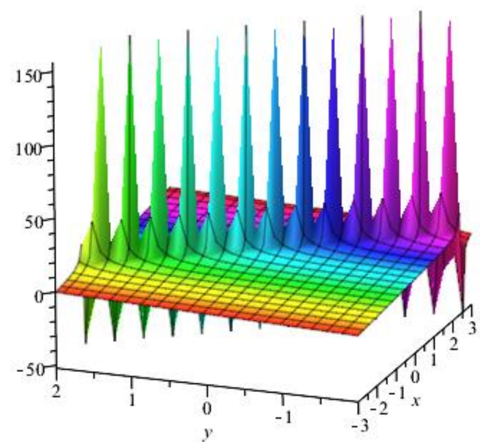



**Figure 3:** The soliton solution $u_5$ for $\boldsymbol{\lambda = 3, \mu = 1, t = 0.1, \alpha = 5}$ and $\boldsymbol{c = 5}$

For $C_1 = 1\ and\ C_2 = 0$

$$v_6 = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.16}$$

$$u_6 = \frac{2}{3}\mu\eta - \frac{\lambda^2\eta}{6} - \sqrt{\alpha}\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) - \frac{\sqrt{\alpha}}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) + \frac{\sqrt{\alpha}}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \tag{4.17}$$

ii- for $\alpha < 0$

$$v = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right]^2 \tag{4.18}$$

For $C_1 = 0\ and\ C_2 = 1$

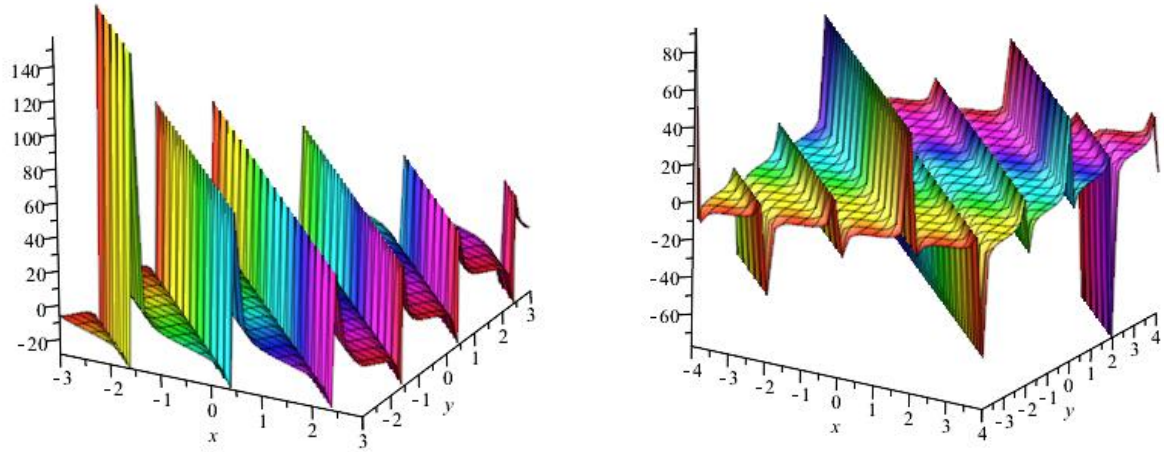
$$v_7 = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.19}$$

$$u_7 = \frac{2}{3}\mu\eta - \frac{\lambda^2\eta}{6} + \frac{\alpha}{\sqrt{-\alpha}}\left(\cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) - \frac{\alpha\pi}{2\sqrt{-\alpha}} + \frac{\alpha}{\sqrt{-\alpha}}\cot^{-1}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) \tag{4.20}$$

For $C_1 = 1$ and $C_2 = 0$

$$v_8 = \frac{\lambda^2}{3} + \frac{2}{3}\mu + 2\lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] + 2\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right]^2 \tag{4.21}$$

$$u_8 = \frac{2}{3}\mu\eta - \frac{\lambda^2\eta}{6} + \frac{\alpha}{\sqrt{-\alpha}}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) + \frac{\alpha}{\sqrt{-\alpha}}\tan^{-1}\left(tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right) \tag{4.22}$$




a. Descending kink solution $u_7$ for $\boldsymbol{\lambda = 3}$, $\boldsymbol{\mu = 4, t = 0.5, \alpha = -8}$ and $\boldsymbol{c = -8}$

b. Multi kink solution $u_8$ for $\boldsymbol{\lambda = 3}$, $\boldsymbol{\mu = 4, t = 0.5, \alpha = -8}$ and $\boldsymbol{c = -8}$

**Figure 4:** The solution $u_5$ for $\boldsymbol{\lambda = 3, \mu = 1, t = 0.1, \alpha = 5}$ and $\boldsymbol{c = 5}$

## Case 3

$$a_0 = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right),\ a_1 = \lambda,\ a_2 = 0 \text{ and } c = \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu} \tag{4.23}$$

i- for $\alpha > 0$

$$v = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right) + \lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1 cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right] \tag{4.24}$$

For $C_1 = 0$ and $C_2 = 1$

$$v_9 = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right) + \lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] \tag{4.25}$$

$$u_9 = \left(-\frac{1}{3}\lambda^2 + \frac{1}{3}\mu - \frac{1}{6}\sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right)\eta - \frac{\lambda}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) - \frac{\lambda}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \tag{4.26}$$

For $C_1 = 1$ and $C_2 = 0$

$$v_{10} = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right) + \lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] \tag{4.27}$$

$$u_{10} = \left(-\frac{1}{3}\lambda^2 + \frac{1}{3}\mu - \frac{1}{6}\sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right)\eta - \frac{\lambda}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) - 1\right) - \frac{\lambda}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right) + 1\right) \tag{4.28}$$

ii- for $\alpha < 0$

$$v = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right) + \lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1 cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right) + C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right] \tag{4.29}$$

$C_1 = 0$ and $C_2 = 1$

$$v_{11} = \frac{1}{6}\left(\lambda^2 + 2\mu - \sqrt{(\lambda^2 - 2\mu)^2 - 4\lambda^2\mu}\right) + \lambda\left[\frac{-\lambda}{2} + \frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] \tag{4.30}$$

$$u_{11}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu-\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta-\frac{\lambda}{2}\ln\left(\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)^2+1\right) \tag{4.31}$$

For $C_1=1$ and $C_2=0$

$$v_{12}=\frac{1}{6}\left(\lambda^2+2\mu-\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] \tag{4.32}$$

$$u_{12}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu-\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta+\frac{\lambda}{2}\ln\left(\left(tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)^2+1\right) \tag{4.33}$$

## Case 4

$$a_0=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right),\ a_1=\lambda,\ a_2=0 \text{ and } c=\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu} \tag{4.34}$$

i- for $\alpha>0$

$$v=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{\alpha}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)+C_2cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}{C_1cosh\left(\frac{\sqrt{\alpha}}{2}\eta\right)+C_2\sinh\left(\frac{\sqrt{\alpha}}{2}\eta\right)}\right)\right] \tag{4.35}$$

For $C_1=0$ and $C_2=1$

$$v_{13}=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{\alpha}}{2}\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] \tag{4.36}$$

$$u_{13}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu+\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta-\frac{\lambda}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)-1\right)-\frac{\lambda}{2}\ln\left(coth\left(\frac{\sqrt{\alpha}}{2}\eta\right)+1\right) \tag{4.37}$$

For $C_1=1$ and $C_2=0$

$$v_{14}=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{\alpha}}{2}\left(\tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)\right)\right] \tag{4.38}$$

$$u_{14}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu+\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta-\frac{\lambda}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)-1\right)-\frac{\lambda}{2}\ln\left(tanh\left(\frac{\sqrt{\alpha}}{2}\eta\right)+1\right) \tag{4.39}$$

ii- for $\alpha<0$

$$v=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{-\alpha}}{2}\left(\frac{C_1\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)+C_2cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}{C_1cos\left(\frac{\sqrt{-\alpha}}{2}\eta\right)+C_2\sin\left(\frac{\sqrt{-\alpha}}{2}\eta\right)}\right)\right] \tag{4.40}$$

For $C_1=0$ and $C_2=1$

$$v_{15}=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{-\alpha}}{2}\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] \tag{4.41}$$

$$u_{15}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu+\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta-\frac{\lambda}{2}\ln\left(\left(cot\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)^2+1\right) \tag{4.42}$$

For $C_1=1$ and $C_2=0$

$$v_{16}=\frac{1}{6}\left(\lambda^2+2\mu+\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)+\lambda\left[\frac{-\lambda}{2}+\frac{\sqrt{-\alpha}}{2}\left(\tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)\right] \tag{4.43}$$

$$u_{16}=\left(-\frac{1}{3}\lambda^2+\frac{1}{3}\mu+\frac{1}{6}\sqrt{(\lambda^2-2\mu)^2-4\lambda^2\mu}\right)\eta+\frac{\lambda}{2}\ln\left(\left(tan\left(\frac{\sqrt{-\alpha}}{2}\eta\right)\right)^2+1\right) \tag{4.44}$$

# 5. Conclusions

Solitary waves of the ANNV equation in its (2+1)-dimensional form have been investigated by exploiting the $G'/G$ method. This method had the ability to create new forms of solitary waves after getting the homogenous balance required for this method. Four cases were formulated according to the appropriate choice of the relations between the arbitrary constants. The solutions included trigonometric, hyperbolic, logarithmic, polynomial, and combinations of these functions. The attained soliton and kink solutions are very useful in describing the behavior of the solitary wave in different engineering and physical applications including plasma explosions and ocean waves.

# Conflict of Interest

The authors have no competing interests to declare that are relevant to the content of this article.

# Funding

No funding was received for conducting this study.

# References

[1] Columbu A, Frassu S, Viglialoro G. Refined criteria toward boundedness in an attraction–repulsion chemotaxis system with nonlinear productions. Appl Anal. 2023: 1-17. https://doi.org/10.1080/00036811.2023.2187789

[2] Li T, Frassu S, Viglialoro G. Combining effects ensuring boundedness in an attraction–repulsion chemotaxis model with production and consumption. Zeitschrift Für Angewandte Mathematik Und Physik. 2023; 74: Article number 109. https://doi.org/10.1007/s00033-023-01976-0

[3] Li T, Pintus N, Viglialoro G. Properties of solutions to porous medium problems with different sources and boundary conditions. Zeitschrift Für Angewandte Mathematik Und Physik. 2019; 70: Article number 86. https://doi.org/10.1007/s00033-019-1130-2

[4] Li T, Viglialoro G. Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. Differential and Integral Equations. 2021; 34: 315-36. https://doi.org/10.57262/die034-0506-315

[5] Boiti M, Leon JJ-P, Manna M, Pempinelli F. On the spectral transform of a Korteweg-de Vries equation in two spatial dimensions. Inverse Probl. 1986; 2: 271-9. https://doi.org/10.1088/0266-5611/2/3/005

[6] Estévez PG, Leble SB. A kdv equation in 2+1 dimensions: Painlevé analysis, solutions, and similarity reductions. Acta Appl Math. 1995; 39: 277-94. https://doi.org/10.1007/BF00994637

[7] Lou S-Y. (2+1)-Dimensional Integrable Models from the Constraints of the KP Equation. Commun Theor Phys. 1997; 27: 249-52. https://doi.org/10.1088/0253-6102/27/2/249

[8] Lou S-Y, Hu X-B. Infinitely many Lax pairs and symmetry constraints of the KP equation. J Math Phys. 1997; 38: 6401-27. https://doi.org/10.1063/1.532219

[9] Clarkson PA, Mansfield EL. On a shallow water wave equation. Nonlinearity. 1994; 7: 975-1000. https://doi.org/10.1088/0951-7715/7/3/012

[10] Li-Hua Z, Xi-Qiang L, Cheng-Lin B. Symmetry, reductions and new exact solutions of ANNV equation through lax pair. Commun Theor Phys. 2008; 50: 1-6. https://doi.org/10.1088/0253-6102/50/1/01

[11] Ma H-C, Lou S-Y. Finite symmetry transformation groups and exact solutions of lax integrable systems. Commun Theor Phys. 2005; 44: 193-6. https://doi.org/10.1088/6102/44/2/193

[12] Lü Z-S. Special bi-solitons for asymmetric nizhnik-novikov-veselov equation. Commun Theor Phys. 2011; 55: 85-8. https://doi.org/10.1088/0253-6102/55/1/17

[13] Ling W, Zhong-Zhou D, Xi-Qiang L. Symmetry reductions, exact solutions and conservation laws of asymmetric nizhnik-novikov-veselov equation. Commun Theor Phys. 2008; 49: 1-8. https://doi.org/10.1088/0253-6102/49/1/01

[14] Yu G-F, Tam H-W. A vector asymmetrical nnv equation: Soliton solutions, bilinear bäcklund transformation and lax pair. J Math Anal Appl. 2008; 344: 593-600. https://doi.org/10.1016/j.jmaa.2008.02.057

[15] Bai C-L, Zhao H. The study of soliton fission and fusion in (2+1)-dimensional nonlinear system. Eur Phys J D. 2006; 39: 93-9. https://doi.org/10.1140/epjd/e2006-00080-8

[16] Hang-Yu R, Zhi-Fang L. Interaction between line soliton and algebraic soliton for asymmetric nizhnik novikov-veselov-equation. Commun Theor Phys. 2008; 49: 1547-52. https://doi.org/10.1088/0253-6102/49/6/41

[17] Ruan H, Chen Y. Interaction between a line soliton and a y-periodic soliton in the (2+1)-dimensional nizhnik-novikov-veselov equation. Zeitschrift Für Naturforschung A. 2002; 57: 948-54. https://doi.org/10.1515/zna-2002-1207

[18] Qian X, Lou S, Hu X. Variable separation approach for a differential-difference asymmetric nizhnik-novikov-veselov equation. Zeitschrift Für Naturforschung A. 2004; 59: 645-58. https://doi.org/10.1515/zna-2004-1005

[19] Lou SY, Ruan HY. Revisitation of the localized excitations of the (2 + 1)-dimensional KdV equation. J Phys A Math Gen. 2001; 34: 305. https://doi.org/10.1088/0305-4470/34/2/307

[20] Lin L. Quasi-periodic waves and asymptotic property for boiti-leon-manna-pempinelli equation. Commun Theor Phys. 2010; 54: 208-14. https://doi.org/10.1088/0253-6102/54/2/02

[21] Li Y, Li D. New exact solutions for the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. Appl Math Sci. 2012; 6(9-12): 579-87.

[22] Luo L. New exact solutions and bäcklund transformation for boiti-leon-manna-pempinelli equation. Phys Lett A. 2011; 375: 1059-63. https://doi.org/10.1016/j.physleta.2011.01.009

[23] Song-Hua M, Jian-Ping F. Multi dromion-solitoff and fractal excitations for (2+1)-dimensional boiti-leon-manna-pempinelli system. Commun Theor Phys. 2009; 52: 641-5. https://doi.org/10.1088/0253-6102/52/4/18

[24] Rashed AS, Kassem MM. Hidden symmetries and exact solutions of integro-differential Jaulent–Miodek evolution equation. Appl Math Comput. 2014; 247: 1141-55. https://doi.org/10.1016/j.amc.2014.09.025

[25] Kassem MM, Rashed AS. N-solitons and cuspon waves solutions of (2 + 1)-dimensional Broer–Kaup–Kupershmidt equations via hidden symmetries of Lie optimal system. Chinese J Phys. 2019; 57: 90-104. https://doi.org/10.1016/j.cjph.2018.12.007

[26] Mabrouk SM, Rashed AS. N-Solitons, kink and periodic wave solutions for (3 + 1)-dimensional Hirota bilinear equation using three distinct techniques. Chinese J Phys. 2019; 60: 48-60. https://doi.org/10.1016/j.cjph.2019.02.032

[27] Rashed AS. Analysis of (3+1)-dimensional unsteady gas flow using optimal system of lie symmetries. Math Comput Simul. 2019; 156: 327-46. https://doi.org/10.1016/j.matcom.2018.08.008

[28] Saleh R, Rashed AS. New exact solutions of (3 + 1)-dimensional generalized Kadomtsev-Petviashvili equation using a combination of lie symmetry and singular manifold methods. Math Methods Appl Sci. 2020; 43: 2045-55. https://doi.org/10.1002/mma.6031

[29] Saleh R, Rashed AS, Wazwaz A-M. Plasma-waves evolution and propagation modeled by sixth order Ramani and coupled Ramani equations using symmetry methods. Phys Scr. 2021; 96: 085213. https://doi.org/10.1088/1402-4896/ac0075

[30] Rashed AS. Interaction of two long waves in shallow water using Hirota-Satsuma model and similarity transformations. Delta University Sci J. 2022; 5: 93-101. https://doi.org/10.21608/dusj.2022.233925

[31] Rashed AS, Mabrouk SM, Wazwaz A-M. Forward scattering for non-linear wave propagation in (3 + 1)-dimensional Jimbo-Miwa equation using singular manifold and group transformation methods. Waves Random Complex Media. 2022; 32: 663-75. https://doi.org/10.1080/17455030.2020.1795303

[32] Rashed AS, Inc M, Saleh R. Extensive novel waves evolution of three-dimensional Yu–Toda–Sasa–Fukuyama equation compatible with plasma and electromagnetic applications. Mod Phys Lett B. 2023; 37: 22501950. https://doi.org/10.1142/S0217984922501950

[33] Rashed AS, Mostafa ANM, Wazwaz AM, Mabrouk SM. Dynamical behavior and soliton solutions of the jumarie's space-time fractional modified benjamin-bona-mahony equation in plasma physics. Rom Rep Phys. 2023; 75: Article no.104.

[34] Mabrouk SM, Rashed AS. Analysis of (3 + 1)-dimensional boiti - leon -manna-pempinelli equation via lax pair investigation and group transformation method. Comput Math Appl. 2017; 74: 2546-56. https://doi.org/10.1016/j.camwa.2017.07.033

[35] Shang Y. Bäcklund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation. Appl Math Comput. 2007; 187: 1286-97. https://doi.org/10.1016/j.amc.2006.09.038

[36] He Y, Tam H-W. Bilinear backlund transformation and lax pair for a coupled ramani equation. J Math Anal Appl. 2009; 357: 132-6. https://doi.org/10.1016/j.jmaa.2009.04.006

[37] Cheng-Lin B. Extended homogeneous balance method and lax pairs, backlund transformation. Commun Theor Phys. 2002; 37: 645-8. https://doi.org/10.1088/0253-6102/37/6/645

[38] Mabrouk S, Kassem M. Group similarity solutions of (2 + 1) boiti-leon-manna-pempinelli lax pair. Ain Shams Eng J. 2014; 5: 227-35. https://doi.org/10.1016/j.asej.2013.06.004

[39] Mohamed NA, Rashed AS, Melaibari A, Sedighi HM, Eltaher MA. Effective numerical technique applied for Burgers' equation of (1 + 1)-, (2 + 1)-dimensional, and coupled forms. Math Methods Appl Sci. 2021; 44: 10135-53. https://doi.org/10.1002/mma.7395

[40] Arora G, Joshi V. A computational approach using modified trigonometric cubic B-spline for numerical solution of Burgers' equation in one and two dimensions. Alex Eng J. 2018; 57: 1087-98. https://doi.org/10.1016/j.aej.2017.02.017

[41] Biazar J, Aminikhah H. Exact and numerical solutions for non-linear Burger's equation by VIM. Math Comput Model. 2009; 49: 1394-400. https://doi.org/10.1016/j.mcm.2008.12.006

[42] Shi F, Zheng H, Cao Y, Li J, Zhao R. A fast numerical method for solving coupled burgers' equations fast numerical method, Numer. Numer Methods Partial Differ Equ. 2017; 33: 1823-38. https://doi.org/10.1002/num.22160

[43] Fang J, Nadeem M, Habib M, Akgül A. Numerical investigation of nonlinear shock wave equations with fractional order in propagating disturbance. Symmetry. 2022; 14(6), 1179. https://doi.org/10.3390/sym14061179

[44] Kaya D, El-Sayed SM. A numerical method for solving jaulent-miodek equation. Phys Lett A. 2003; 318: 345-53. https://doi.org/10.1016/j.physleta.2003.08.033

[45] Wazwaz A-M. The tanh and the sine–cosine methods for a reliable treatment of the modified equal width equation and its variants. Commun Nonlinear Sci Numer Simul. 2006; 11: 148-60. https://doi.org/10.1016/j.cnsns.2004.07.001

[46] Wazwaz A-M. The sine–cosine and the tanh methods: Reliable tools for analytic treatment of nonlinear dispersive equations. Appl Math Comput. 2006; 173: 150-64. https://doi.org/10.1016/j.amc.2005.02.047

[47] Wazwaz A-M. Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh–coth method. Appl Math Comput. 2007; 190: 633-40. https://doi.org/10.1016/j.amc.2007.01.056

[48] Wazwaz A-M. The tanh–coth method for solitons and kink solutions for nonlinear parabolic equations. Appl Math Comput. 2007; 188: 1467-75. https://doi.org/10.1016/j.amc.2006.11.013

[49] Wazwaz A-M. The tanh-coth and the sine-cosine methods for kinks, solitons, and periodic solutions for the pochhammer-chree equations. Appl Math Comput. 2008; 195: 24-33. https://doi.org/10.1016/j.amc.2007.04.066

[50] Darvishi MT, Najafi M. A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-ytsf equation. Chin Phys Lett. 2011; 28: 040202. https://doi.org/10.1088/0256-307X/28/4/040202

[51] Darvishi MT, Najafi M, Najafi M. Application of multiple exp-function method to obtain multi-soliton solutions of (2 + 1)- and (3 + 1)-dimensional breaking soliton equations. Am J Comput Appl Math. 2012; 1: 41-7. https://doi.org/10.5923/j.ajcam.20110102.08