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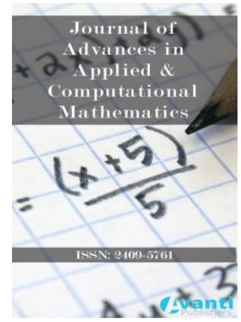

Generalized Legendre Polynomial Configuration Method for Solving Numerical Solutions of Fractional Pantograph Delay Differential Equations

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ABSTRACT

This paper develops a numerical approach for solving fractional pantograph delay differential equations using generalized Legendre polynomials. These polynomials are derived from generalized Taylor bases, which facilitate the approximation of the underlying analytical solutions, leading to the formulation of numerical solutions. The fractional pantograph delay differential equation is then transformed into a finite set of nonlinear algebraic equations using collocation points. Following this step, Newton's iterative method is applied to the resultant set of nonlinear algebraic equations to compute their numerical solutions. An error analysis for this methodology is subsequently presented, accompanied by numerical examples demonstrating its accuracy and efficiency. Overall, this study contributes a more streamlined and productive tool for determining the numerical solution of fractional differential equations.

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1. Introduction

Delay differential equations are a class of differential equations characterized by their dependence on not only the present state but also the past states. In real-world applications, most systems exhibit oscillatory or unstable behaviors, rendering time lags an omnipresent phenomenon.

Within the realm of population dynamics, Gurney, Blythe [1], and other researchers have introduced a delay differential equation model to characterize Nicholson's experimental findings on fly populations, which can be formulated as follows:

$$\frac{dN(t)}{dt} = qN(t-\tau)e^{-N(t-\tau)/N_0} - rN(t). \tag{1.1}$$

Where $N(t)$ denotes the number of adults, q signifies the number of larvae generated per adult, τ represents the time required for a larva to mature into an adult, N_0 corresponds to the environment's carrying capacity, and r represents the adult mortality rate.

In the field of epidemiology, Busenberg and Cooke [2] have developed a model to describe the proportion of patients infected with communicable diseases, which can be articulated as follows:

$$\frac{dy(t)}{dt} = q(t)y(t-\tau)(1-y(t)) - cy(t). \tag{1.2}$$

In this formulation, $y(t)$ denotes the infected population, while $1-y(t)$ corresponding to the susceptible population. Moreover, τ represents when an individual first became infected within the exposed population, and c signifies the recovery rate for infected persons.

Ockendon and Tayler [3] investigated the current collection within the conductive bow frame of an electric locomotive. They used pantograph delay differential equations to formulate the resulting model, expressed as follows:

$$\frac{dy(t)}{dt} = ry(t) + cy(qt). \tag{1.3}$$

In this formulation, $0<q<1$, while r and c represent parameters related to the coefficient of elasticity, spacing of lightweight springs, wire density, shock absorbers, bracing, and similar factors.

Delay differential equations have been widely applied in fields like automatic control, biology, and finance [4-12]. This has increased scholarly interest in fractional delay differential equations, distinguishing them from their integer counterparts due to their characteristic traits of 'nonlocality' and 'memory'. Consequently, using fractional-order equations for modeling complex physical problems results in outcomes that align more closely with natural phenomena. Both domestic and international researchers have extensively examined aspects such as the existence, uniqueness, asymptotic behavior, and stability of solutions for fractional delay differential equations [13-22].

Fractional derivatives in analytic solutions frequently present singularities. Combining singularities and delays significantly complicates obtaining analytical solutions for fractional delay differential equations. Consequently, it becomes imperative to investigate numerical solutions for fractional-order delay differential equations. Established numerical solution techniques applied to non-delay fractional differential equations consist of the restricted difference method, Adomian decomposition method, homotopy analysis method, and spectral method, among others [23-33]. Mittal and Pandit [34-36] employed the Scale-3 Haar wavelet method in the context of fractional differential equation systems, substantiating the algorithm's accuracy through numerical example applications. Hafshejani and Vanani [37] introduced the Legendre wavelet method, while Ghasemi [38] devised a semi-analytical solution for a class of nonlinear delay differential equations comprising Caputo derivatives in Hilbert function space.

The method's applicability and accuracy were substantiated by computing the absolute error values at chosen mesh points and assessing the technique by examining three distinct arithmetic cases. Saeed [39] approximated the delayed unknown function by employing Chebyshev wavelet levels and integrated the step-by-step approach with the Chebyshev wavelet method. Rabiei [40] developed a numerical approach for solving fractional order proportional delay differential equations by constructing fractional order Boubaker polynomials. In the same year, Chen and Gou [41] introduced the segmented Picard iteration method. Yang and Hou [42] converted a class of fractional pantograph delay differential equations into nonlinear Volterra integral equations, subsequently employing the Jacobi configuration method for their resolution. Singh [43] determined the numerical solution for fractional multiple proportional delay differential equations by employing the fractional differential transform method and subsequently conducted an error analysis of this approach. Elkot [44] also investigated the rescaled spectral configuration method as applied to a class of fractional nonlinear proportional delay differential equations. Many scholars have also proposed different numerical algorithms to solve the fractional pantograph delay differential equation [45-48].

Nevertheless, the extant numerical solution methods tend to be complex and computationally demanding, with room for improvement in terms of effectiveness and accuracy. The approach presented in this paper is more streamlined, ensuring the algorithm's efficiency while also offering a novel tool for resolving fractional pantograph delay differential equations. This paper presents a numerical solution for fractional pantograph delay differential equations based on the generalized Legendre polynomial configuration method. The generalized Legendre polynomials approximate the resolution of the required equations, which are subsequently transformed into a finite set of nonlinear algebraic equations using collocation points. The desired numerical solution is attained by solving the constrained nonlinear algebraic equations. In contrast to numerical results obtained through the Legendre wavelet method [37], the regenerative kernel method [38], the Chebyshev wavelet method [39], the PPIM method [41], the Jacobi spectral configuration method [42], and the rescaled spectral configuration method [44], our numerical results, utilizing the same or even fewer collocation points, reveal relatively more minor errors. This evidence corroborates that the proposed generalized Legendre polynomial configuration method offers enhanced efficiency for numerically solving fractional pantograph delay differential equations.

This paper is organized as follows: Section 2 briefly introduces the definition and properties of Caputo's fractional derivatives. Section 3 elaborates on the derivation of generalized Legendre polynomials, presents a function approximation involving the composition of these polynomials, and conducts an error analysis. In Section 4, a numerical algorithm based on generalized Legendre polynomials is introduced, and the efficiency of the configuration method is demonstrated using three numerical examples. Finally, Section 5 offers concluding remarks for the paper.

2. Preliminaries

In this section, we review some fundamental definitions and properties deemed relevant to the current study.

Definition 2.1. [49] Let $0<\alpha<1$ and $f(t)\in H^1([a,b])$ then the Riemann-Liouville integral of order α of $f(t)$ is defined as

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau. \tag{2.1}$$

Definition 2.2. [49] Let $\alpha\in R^+$ and satisfy $n-1\le\alpha<n$, where $n\in N$. If $f(t)\in C^n(R)$, then the Caputo derivative of order α of $f(t)$ is defined as

$$D^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f(\tau)}{(t-\tau)}d\tau. \tag{2.2}$$

Lemma 2.1. [49] Let $\alpha\ge 0$, $n-1\le\alpha\le n$ and $f\in A^n[a,b]$, then

$$J^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k}}{k!}D^{k}f(a). \tag{2.3}$$

The Caputo derivatives of the function and then the Riemann-Louvier integral of the process are used to provide technical support for the numerical calculations in Section 4 of this paper.

Lemma 2.2. (Generalized Taylor's formula [50]) Assuming, where $D^{k\alpha}f(t)\in C([0,1])$, $k=0,1,\ldots,n+1$, $0<\alpha\leq 1$, then we have

$$f(t)=\sum_{i=0}^{n}D^{i\alpha}f(0)\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}+R_{n}^{\alpha}(t,0). \tag{2.4}$$

where $R_{n}^{\alpha}(t,0)=D^{(n+1)\alpha}f(\xi)\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}$, $0<\xi<t$ and $t\epsilon$ [0,1]. The generalized Taylor formula holds significant importance in fractional calculus, and this lemma will be employed during the proof of Theorem 3.1.

3. Function Approximation and Error Analysis

In this section, the generalized Legendre polynomials are derived, the error arising from the use of their vector approximation function is analyzed, and the convergence of the error is proved.

Definition 3.1. [51] Let $H=L^{2}[0,1]$, $\forall x,y\in H$. Define the inner product on H as

$$(x,y)=\int_{0}^{1}x(t)y(t)dt. \tag{3.1}$$

Definition 3.2. [51] Let $H=L^{2}[0,1]$ and define the parametrization on H as

$$\|f(t)\|=\left(\int_{0}^{1}f^{2}(t)dt\right)^{\frac{1}{2}}. \tag{3.2}$$

On the interval $[0,1]$, we use the Schmidt orthogonalization process to orthogonalize a set of generalized Taylor bases $T_{n}(t)=[1,t_{1},t_{2},t_{3},\cdots,t_{n}]^{T}$ and unitary can be obtained by

$$\phi_{0}(t)=T_{0}(t), \tag{3.3}$$

$$\phi_{1}(t)=\frac{T_{1}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{1}(t)dt\right)\phi_{0}(t)}{\left\|T_{1}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{1}(t)dt\right)\phi_{0}(t)\right\|}, \tag{3.4}$$

$$\phi_{2}(t)=\frac{T_{2}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{2}(t)dt\right)\phi_{0}(t)-\left(\int_{0}^{1}\phi_{1}(t)T_{2}(t)dt\right)\phi_{1}(t)}{\left\|T_{2}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{2}(t)dt\right)\phi_{0}(t)-\left(\int_{0}^{1}\phi_{1}(t)T_{2}(t)dt\right)\phi_{1}(t)\right\|}, \tag{3.5}$$

$$\phi_{3}(t)=\frac{T_{3}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{3}(t)dt\right)\phi_{0}(t)-\left(\int_{0}^{1}\phi_{1}(t)T_{3}(t)dt\right)\phi_{1}(t)-\left(\int_{0}^{1}\phi_{2}(t)T_{3}(t)dt\right)\phi_{2}(t)}{\left\|T_{3}(t)-\left(\int_{0}^{1}\phi_{0}(t)T_{3}(t)dt\right)\phi_{0}(t)-\left(\int_{0}^{1}\phi_{1}(t)T_{3}(t)dt\right)\phi_{1}(t)-\left(\int_{0}^{1}\phi_{2}(t)T_{3}(t)dt\right)\phi_{2}(t)\right\|}, \tag{3.6}$$

By the above procedure, we can construct generalized Legendre vectors. Where $\phi_{0}(t)=1$, $\phi_{1}(t)=t^{\alpha}-\frac{(2\alpha+1)(\alpha+1))^{2}}{\alpha^{2}(\alpha+1)}$, $\phi_{2}(t)=\frac{A_{t^{\alpha}}^{\alpha}}{B^{\alpha}}$.

$$A^{\alpha}_{t^{\alpha}} = (6t^{2\alpha}\alpha^4 + 3\sqrt{2\alpha+1}t^{\alpha}\alpha^2 + 5t^{2\alpha}\alpha^3 - 2t^{\alpha}\alpha^3 + \sqrt{2\alpha+1}t^{\alpha}\alpha + 2\sqrt{2\alpha+1}\alpha^2 + t^{2\alpha}\alpha^2 - t^{\alpha}\alpha^2 - 3\alpha^2 + \sqrt{2\alpha+1}\alpha - 7\alpha^2 - 5\alpha - 1) \times\sqrt{\alpha+1}\sqrt{4\alpha+1}\sqrt[4]{2\alpha+1}, \tag{3.7}$$

$$B^{\alpha} = \sqrt{36\alpha^9 + 28\alpha^8 + 44\alpha^7 + 474\alpha^6 + 892\alpha^5 + 750\alpha^4 + 351\alpha^3 + 97\alpha^2 + 15\alpha + 1} \times \sqrt{\sqrt{2\alpha+1} + 96\alpha^8 - 200\alpha^7 - 1056\alpha^6 - 1418\alpha^5 - 916\alpha^4 - 316\alpha^3 - 56\alpha^2 - 4\alpha}. \tag{3.8}$$

Definition 3.3. Let $V_n(t) = \left[\phi_0(t), \phi_1(t), \phi_2(t), \phi_3(t), \ldots, \phi_n(t)\right]^T$, $n = 0,1,2,3,\cdots,n$, $V_n(t)$ is called a generalized Legendre vector.

Regarding function approximation, suppose $H = L^2[0,1]$ and is a finite-dimensional vector subspace of H and $\phi_n(t)$ is a generalized Legendre vector, then every $f(t) \in H$ has a unique best square approximation $s_n^*(t) \in W$ such that

$$\left\| f(t) - s_n^*(t) \right\|^2 = \min_{\hat{f}(t)\in W} \left\| f(t) - \hat{f}(t) \right\|^2 = \min_{\hat{f}(t)\in W} \int_0^1 \left[f(t) - \hat{f}(t) \right]^2 dt. \tag{3.9}$$

Therefore, the function $f(t)$ can be expanded as

$$f(t) \simeq \lim_{n\to\infty} \sum_{i=0}^{n} c_i \phi_i(t). \tag{3.10}$$

Truncating the series in equation (3.10) to a finite number of terms, we obtain

$$f(t) \simeq s_n^*(t) = \sum_{i=0}^{n} c_i \phi_i(t). \tag{3.11}$$

To analyze the error of the best square approximation posed by the generalized Legendre polynomials, we first analyze the error of the best approximation posed by the generalized Taylor base.

Theorem 3.1. Let $s_n(t)$ be the best approximation of $f(t)$ on W, $W = span\left\{1, t^{\alpha}, t^{2\alpha}, t^{3\alpha}, \ldots, t^{n\alpha}\right\}$, and $D^{k\alpha} f(t) \in C([0,1])$ with $k = 0,1,\ldots,$ then

$$\left\| f(t) - s_n(t) \right\| \le \frac{M_{\alpha}}{\Gamma((n+1)\alpha+1)} \sqrt{\frac{1}{2n\alpha + 2\alpha + 1}}, \tag{3.12}$$

$$\left\| J^{\alpha} f(t) - J^{\alpha} s_n(t) \right\| \le \frac{1}{\Gamma(\alpha)} \frac{M_{\alpha}}{\Gamma((n+1)\alpha+1)} \sqrt{\frac{1}{2n\alpha + 2\alpha + 1}}. \tag{3.13}$$

Where $M_{\alpha} = \max_{t\in[0,1]} \left| D^{(n+1)\alpha} f(t) \right|$.

Proof. $\left\| f(t) - s_n(t) \right\|^2 = \left\| D^{(n+1)\alpha} f(\xi) \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right\|^2 = \int_0^1 \left(\frac{D^{(n+1)\alpha} f(\xi)}{\Gamma((n+1)\alpha+1)} \right)^2 t^{2n\alpha+2\alpha} dt$

$$\le \left(\frac{D^{(n+1)\alpha} f(\xi)}{\Gamma((n+1)\alpha+1)} \right)^2 \int_0^1 t^{2n\alpha+2\alpha} dt \le \left(\frac{M_{\alpha}}{\Gamma((n+1)\alpha+1)} \right)^2 \frac{1}{2n\alpha + 2\alpha + 1}. \tag{3.14}$$

Therefore

$$\left\| f(t) - s_n(t) \right\| \le \frac{M_{\alpha}}{\Gamma((n+1)\alpha+1)} \sqrt{\frac{1}{2n\alpha + 2\alpha + 1}}, \tag{3.15}$$

$$\left\|J^{\alpha}f(t)-J^{\alpha}s_n(t)\right\|=\left\|J^{\alpha}\left(f(t)-s_n(t)\right)\right\|=\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\tau)-s_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau\right\|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t\left\|\frac{f(\tau)-s_n(\tau)}{(t-\tau)^{1-\alpha}}\right\|d\tau\ \le\frac{1}{\Gamma(\alpha)}\int_0^1\left\|f(\tau)-s_n(\tau)\right\|d\tau\ \le\frac{1}{\Gamma(\alpha)}\frac{M_\alpha}{\Gamma((n+1)\alpha+1)}\sqrt{\frac{1}{2n\alpha+2\alpha+1}}.\tag{3.16}$$

From Theorem 3.1. we can see that

$$\lim_{n\to\infty}\|f(t)-s_n(t)\|=0.\tag{3.17}$$

Given that the generalized Legendre polynomials are derived through the orthogonal transformation of the generalized Taylor base, these polynomials can be regarded as the optimal approximation of the generalized Taylor base. Consequently, the error associated with the best-squared approximation formulated using the generalized Legendre polynomials is smaller than that of the best approximation generated with the generalized Taylor base. This evidence supports the convergent nature of the proposed method.

4. Numerical Calculation

In this paper, we use generalized Legendre polynomials to approximate the numerical solution of the fractional pantograph delay differential equation, which in turn yields the numerical solution format $\sum_{i=0}^{n}c_i\phi_i(t)$, which is substituted into the equation, and using the properties of the Riemann-Liouville fractional integrals as well as Caputo fractional derivatives, we act on both sides of the equation with the simultaneous action of the J^{α} operator, followed by transforming the fractional differential equation into a solution of a system of $n+1$ nonlinear algebraic equations using the collocation point t_i, applying Newton's iterative method to solve the system of $n+1$ nonlinear equations, solving for c_i, and thus obtaining an approximate solution to the fractional pantograph delay differential equation. Fig. (**1**) shows the flowchart of the algorithm.

Consider the fractional pantograph delay differential equation as follows:

$$\begin{cases}D^{\alpha}f(t)=G\left(f(t),f(bt),t\right)\\ f(0)=f_0,\alpha\in(0,1]\end{cases}\tag{4.1}$$

Where $b\in(0,1],G\left(f(t),f(bt),t\right)$ is a polynomial in $f(t),f(bt),t$. $D^{\alpha}f(t)$ is the Caputo derivative.

The action of the J^{α} operator on both sides of equation (4.1) yields

$$J^{\alpha}D^{\alpha}f(t)=J^{\alpha}G\left(f(t),f(bt),t\right)\tag{4.2}$$

From Lemma 2.1, Acting on (2.3) with equation (3.11) and acting with equidistant nodes t_i

$$t_i=\frac{i}{n},\ \ i=0,1,2,\ldots,n\tag{4.3}$$

We can obtain a system of $n+1$ nonlinear equations

$$\begin{cases}\sum_{i=0}^{n}c_i\phi_i-f_0=J^{\alpha}G\left(f(t_0),f(bt_0),t_0\right)\\ \sum_{i=0}^{n}c_i\phi_i-f_0=J^{\alpha}G\left(f(t_1),f(bt_1),t_1\right)\\ \vdots\\ \sum_{i=0}^{n}c_i\phi_i-f_0=J^{\alpha}G\left(f(t_n),f(bt_n),t_n\right)\end{cases}\tag{4.4}$$

The numerical solution $f(t) \simeq s_n^*(t) = \sum_{i=0}^{n} c_i \phi_i$ of the fractional pantograph delay differential equation is obtained by solving c_i by Newton's iterative method.

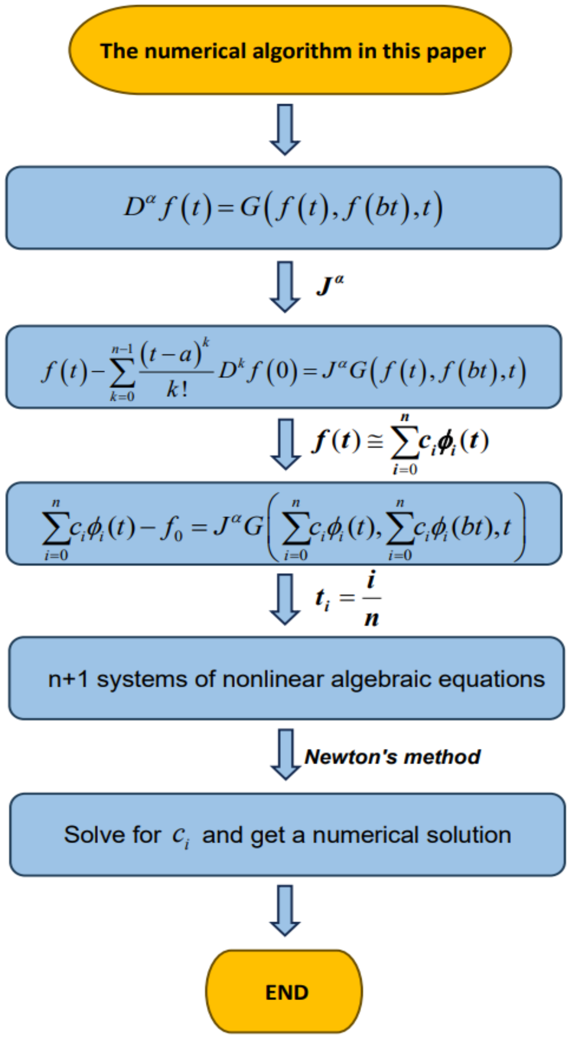

Figure 1: Numerical algorithm flowchart.

We verified the effectiveness of the generalized Legendre configuration method through multiple numerical experiments, and all simulations in this paper were performed on the Matlab platform.

Example 1 [38, 41]

$$\begin{cases} D^\alpha f(t) = -f(t) + f(bt) + \frac{7}{8}t^3 + \frac{16}{5\Gamma(0.5)} t^{\frac{5}{2}} \\ f(0) = 0, \alpha \in [0,1] \end{cases} \tag{4.5}$$

When $\alpha = \frac{1}{2}, b = \frac{1}{2}$, $f(t) = t^3$ is the analytical solution of equation (4.5). To compare with the regenerative kernel method with $n = 10$, the PPIM method, we selected a smaller number of matching points $(n = 3)$ when using the generalized Legendre polynomial configuration method.

The images of numerical and analytical solutions in Fig. (**2**) show that it is feasible to solve the numerical solution of equation (4.5) by the generalized Legendre polynomial configuration method, and the error curves of numerical and analytical solutions in Fig. (**3**) show more clearly that the maximum error of our proposed method is

6.380×10^{-15}, which is very small and proves that our approach is practical. In addition, a comparison of the errors of the regenerative kernel method, the PPIM method, and the generalized Legendre polynomial configuration method can be seen in Table **1**, which shows that our proposed method is more efficient for solving the numerical solution problem of equation (4.5).

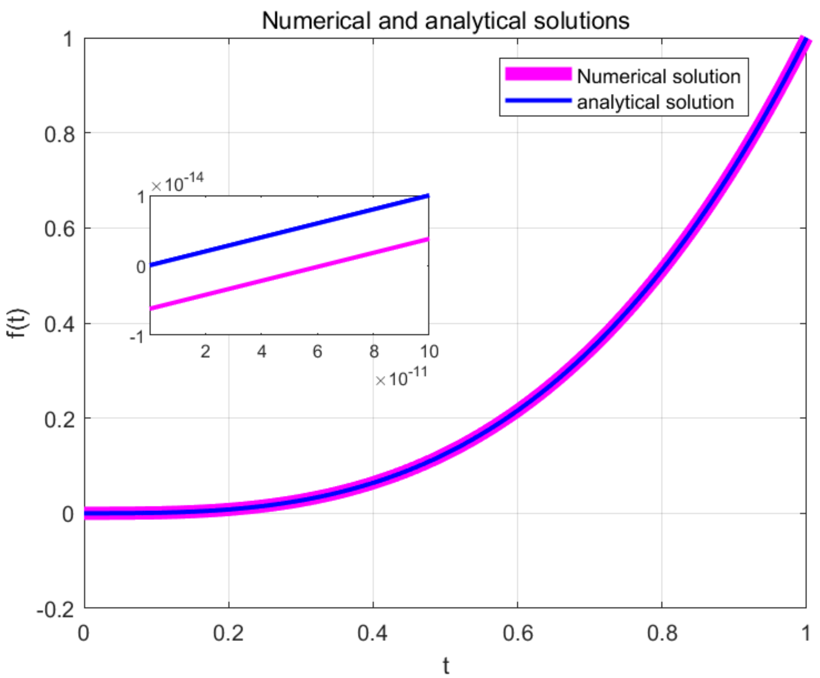

Figure 2: Example 1 Numerical and analytical solutions.

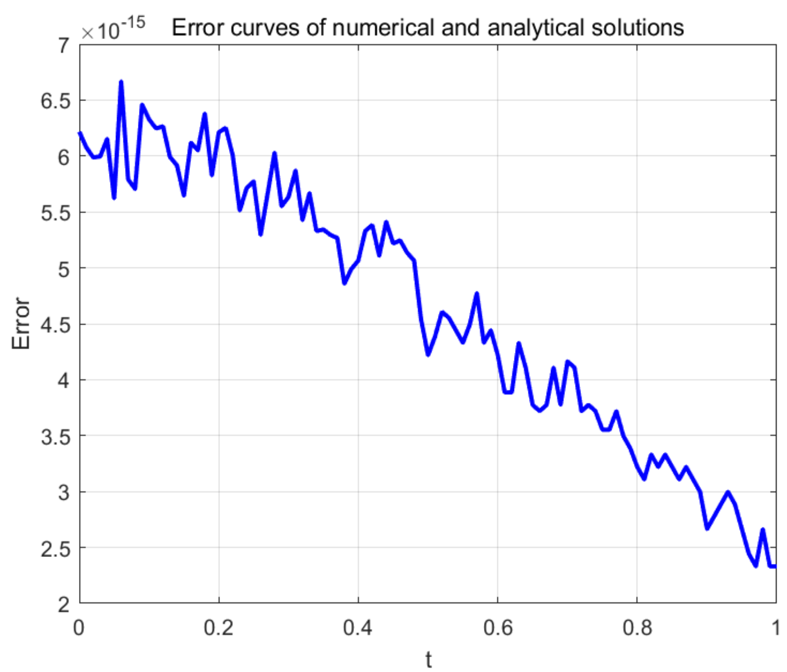

Figure 3: Example 1 Error curves of numerical and analytical solutions

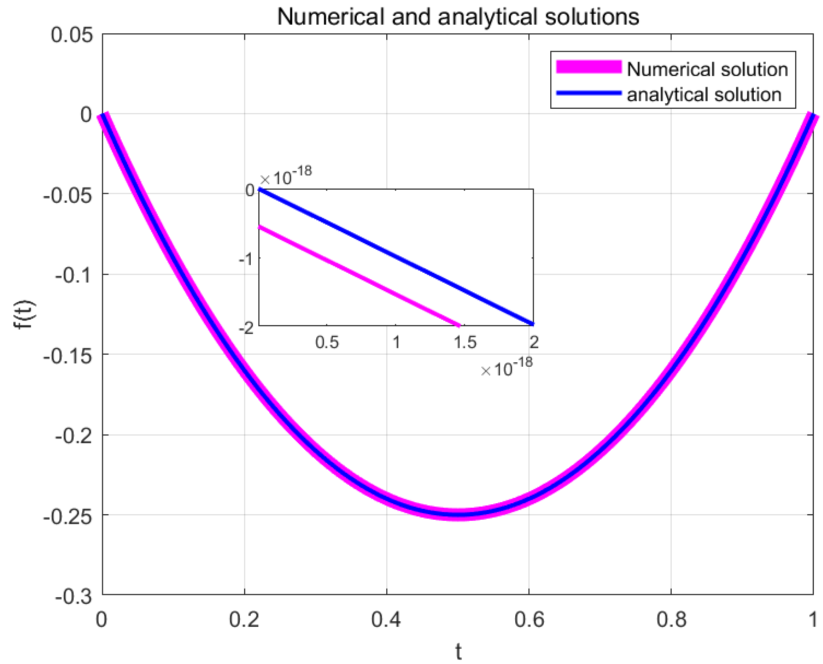

Figure 4: Example 2 numerical and analytical solutions.

Table 1: Example 1 Comparison of the errors of the regenerative kernel method, the PPIM method, and the generalized Legendre polynomial configuration method.

t_i	Regenerative Kernel [38]	PPIM [41]	Present Method
0.1	1.475×10^{-7}	1.303×10^{-13}	6.328×10^{-15}
0.2	4.492×10^{-6}	3.731×10^{-11}	6.212×10^{-15}
0.3	2.361×10^{-5}	1.171×10^{-9}	5.648×10^{-15}
0.4	8.239×10^{-5}	1.351×10^{-8}	2.914×10^{-15}
0.5	2.107×10^{-4}	2.107×10^{-4}	5.065×10^{-15}
0.6	4.403×10^{-4}	4.403×10^{-4}	4.219×10^{-15}
0.7	8.037×10^{-4}	8.037×10^{-4}	4.163×10^{-15}
0.8	1.333×10^{-3}	1.333×10^{-3}	3.220×10^{-15}
0.9	2.064×10^{-3}	2.064×10^{-3}	2.665×10^{-15}
1	1.303×10^{-3}	1.303×10^{-3}	2.331×10^{-15}

Example 2 [42]

$$\begin{cases} D^{\alpha} f(t) = f^2(t) + f(bt) + \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t^4 + 2t^3 - \frac{9}{10} t^2 + \frac{1}{3} t \\ f(0) = 0, \alpha \in [0,1] \end{cases} \quad (4.6)$$

When $\alpha = \frac{1}{2}, b = \frac{1}{3}$, $f(t) = t^2 - t$ is the analytical solution of equation (4.6). To compare with the numerical results obtained by the Jacobi spectral configuration method ($n = 10$), we have chosen the generalized Legendre polynomial configuration method with fewer collocation points $n = 3$ to solve equation (4.6).

The images of numerical and analytical solutions in Fig. (**4**) show that it is feasible to solve the numerical solution of equation (4.6) with our proposed method, and the error curve of numerical and analytical solutions in Fig. (**5**) shows more clearly that the maximum error of our proposed generalized Legendre polynomials configuration method is 5.343×10^{-16}, which is a very small error and proves the validity of our method. In addition, a comparison of the errors between the Jacobi spectral configuration method and the generalized Legendre polynomial configuration method by taking different values of n in Table **2** shows that our proposed method is more efficient for solving the numerical solution problem of equation (4.6).

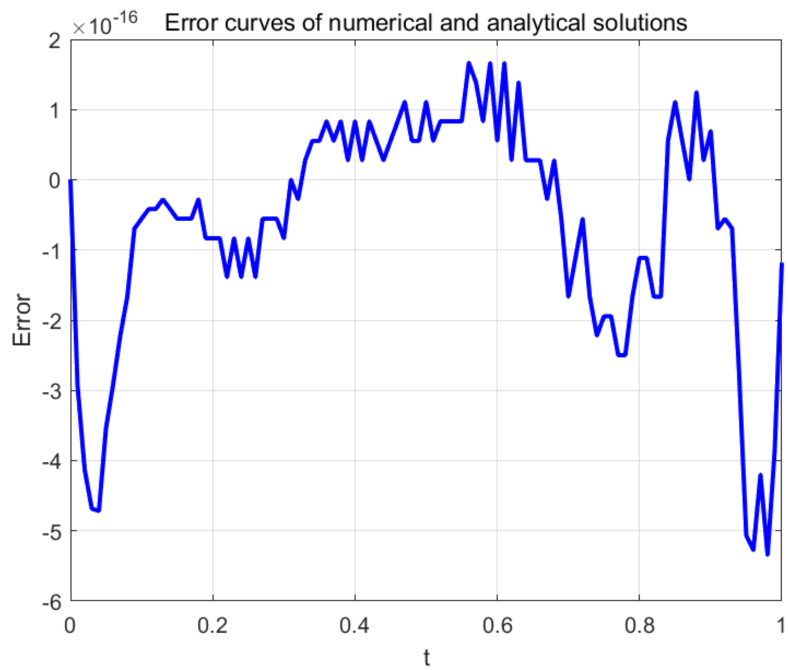

Figure 5: Example 2 Error curves of numerical and analytical solutions.

Table 2: Example 2 Comparison of errors between the Jacobi spectral configuration method [42] and the generalized Legendre polynomial configuration method for different n values.

Maximum Error	$n=6$	$n=8$	$n=10$
Jacobi spectral configuration (L^2)	3.6822×10^{-6}	2.7460×10^{-8}	3.9471×10^{-11}
Jacobi spectral configuration (L^∞)	1.0832×10^{-5}	1.2072×10^{-8}	1.3049×10^{-11}
present method	4.5103×10^{-16}	4.7184×10^{-16}	5.3429×10^{-16}

Example 3 [44]

$$\begin{cases} D^\alpha f(t) = 1 - 2f^2(bt) \\ f(0) = 0, \alpha \in [0,1] \end{cases} \tag{4.7}$$

When $\alpha = 1, b = \frac{1}{2}$ the analytical solution of equation (14) is $f(t) = sint$. For comparison with other numerical methods, we choose the generalized Legendre polynomial configuration method with the number of collocation points $n = 9$ to solve equation (4.7).

The images of numerical and analytical solutions in Fig. (**6**) show that it is feasible to solve the numerical solution of equation (4.7) by the generalized Legendre polynomial configuration method, and the error curves of the numerical and analytical solutions in Fig. (**7**) show more clearly that the maximum error of our proposed generalized Legendre polynomial configuration method is 1.737×10^{-12}, which is very small and proves that our method is effective. In addition, Table **3** shows that the generalized Legendre polynomial configuration method has a smaller

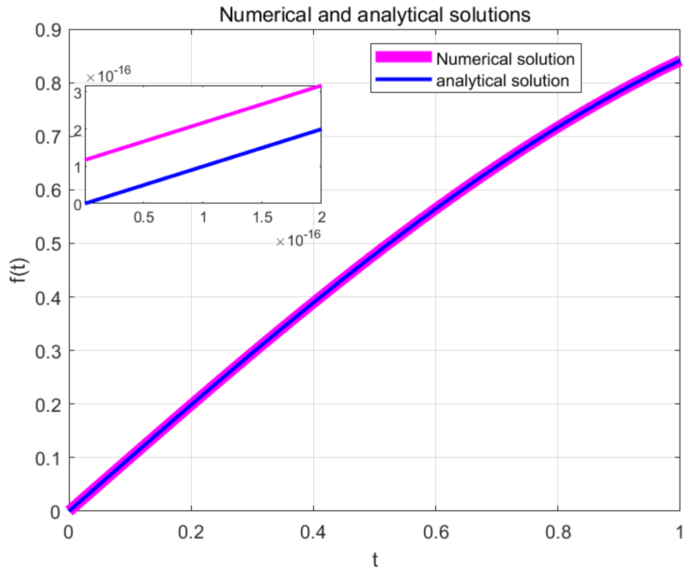

Figure 6: Example 3 Numerical and analytical solutions.

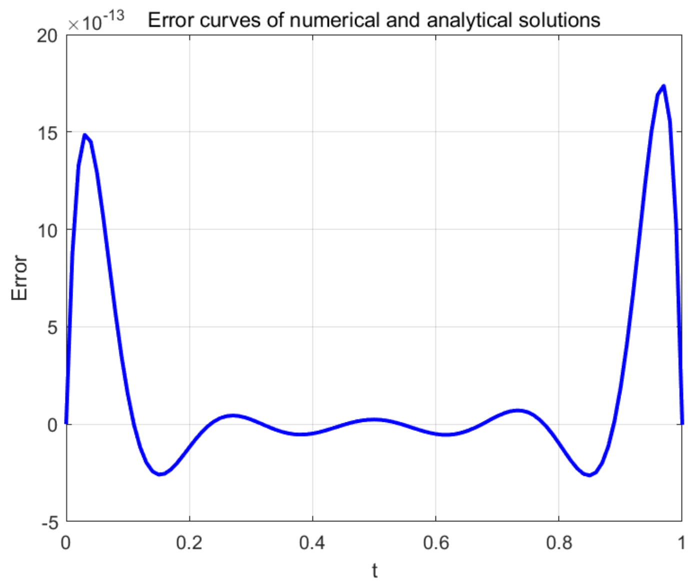

Figure 7: Example 3 Error curves of numerical and analytical solutions.

Table 3: Example 3 Comparison of the errors of other methods with the error of the generalized Legendre polynomial configuration method.

t_i	Legendre Wavelet [37] $k = 1, n = 9$	Jacobi Spectral Configuration [42] $n = 9$	Chebyshev Wavelet [39] $k = 2, n = 9$	Rescaled Spectral Configuration [44] $n = 9$	Present Method $n = 9$
0.125	2.39×10^{-9}	2.05×10^{-12}	1.71×10^{-12}	1.81×10^{-13}	1.61×10^{-13}
0.250	2.25×10^{-9}	1.61×10^{-12}	2.47×10^{-12}	1.34×10^{-13}	3.14×10^{-14}
0.375	2.09×10^{-9}	3.48×10^{-11}	9.36×10^{-11}	1.50×10^{-13}	5.29×10^{-14}
0.500	1.60×10^{-9}	5.31×10^{-11}	1.78×10^{-11}	1.61×10^{-13}	2.38×10^{-14}
0.625	1.94×10^{-9}	2.07×10^{-11}	1.88×10^{-11}	1.69×10^{-13}	5.35×10^{-14}
0.750	1.02×10^{-9}	2.47×10^{-11}	3.04×10^{-12}	1.30×10^{-13}	6.05×10^{-14}
0.875	1.24×10^{-9}	1.59×10^{-11}	1.64×10^{-12}	1.38×10^{-13}	1.61×10^{-14}
1	8.08×10^{-10}	3.85×10^{-10}	4.51×10^{-10}	8.98×10^{-13}	3.11×10^{-15}

error when $n = 9$, compared with the Legendre wavelet method [37], the Chebyshev wavelet method [39], the Jacobi spectral configuration method [42], and the rescaled spectral configuration method [44], and therefore our proposed method is more efficient for solving the numerical solution problem of equation (4.7).

5. Conclusion

This paper proposes a straightforward method for determining the numerical solution to the fractional pantograph delay differential equation. The error associated with the proposed method is analyzed, and numerical experiments are conducted using arithmetic examples to validate the method's accuracy. The obtained numerical results are then compared with those from existing literature to ascertain the efficiency of the proposed method. Consequently, the generalized Legendre polynomial configuration method proves suitable for solving the numerical solution of pantograph delay differential equations in a fractional order context. Even with this, the process may be more appropriate for certain fractional differential equations. Future work will focus on improving and optimizing the algorithm to address a broader range of numerical solution problems for fractional differential equations.

Conflict of Interest

The authors declare that there is no conflict of interest to disclose.

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