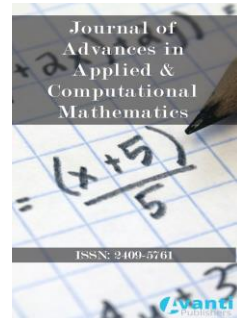
Publishers

Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



A New Approach of Milne-type Inequalities Based on Proportional Caputo-Hybrid Operator

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ARTICLE INFO

Article Type: Research Article

Academic Editor: Hüseyin Budak

Keywords:
Convex functions
Fractional integrals
Milne-type inequalities
Proportional caputo-hybrid operator

Timeline:
Received: November 03, 2023
Accepted: December 10, 2023
Published: December 28, 2023

Citation: Demir İ. A New Approach of Milne-type Inequalities Based on Proportional Caputo-Hybrid Operator. J Adv App Comput Math. 2023; 10: 102-119.

DOI: https://doi.org/10.15377/2409-5761.2023.10.10

ABSTRACT

In this study, we first offer a novel integral identity using twice-differentiable convex mappings for the proportional Caputo-hybrid operator. Next, we demonstrate many integral inequalities related to the Milne-type integral inequalities for proportional Caputo-hybrid operator with the use of this newly discovered identity. Also, we present several examples along with their corresponding graphs in order to provide a better understanding of the newly obtained inequalities. Finally, we observe that the obtained results improve and generalize some of the previous results in the area of integral inequalities.

2010 Mathematics Subject Classification. 26D07, 26D10, 26A33

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1. Introduction

The theory of convexity has several applications in classical analysis, which makes it both significant and appealing. This theory enables us to solve a wide range of problems that arise in both practical and pure mathematics. Furthermore, the usage of integral inequalities and their applications has expanded rapidly, and it has impacted many scientific and technological domains in addition to the numerous current mathematical subjects, such as measure theory, approximation theory, and information theory. It is also possible to determine the error bounds of numerical integration formulae for the differentiable mappings, by using the integral inequalities. Common inequality types, such as Grüss-type, Hermite-Hadamard-type, Ostrowski-type, Simpson-type, and Ostrowski-type, evaluate the remainder term and provide error boundaries for quadrature methods. Because of the close relationship between the theory of inequality and the theory of convexity, numerous researchers have been interested in merging them, which aids in developing and generalizing the integral inequalities [1-6].

Simpson's inequality is one of the most important and frequently required inequalities, which is as follows:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

where $f:[a,b]\to\mathrm{R}$ is four times continuously differantiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. This inequality establishes an upper limit on the error that occurs when estimating a definite integral using Simpson's rule. It is widely recognized that when the mapping f is not four times differentiable or when its fourth derivative $f^{(4)}$ is unbounded on the interval (a,b), the classical Simpson quadrature formula cannot be employed. In recent years, several authors have highlighted Simpson-type inequalities for various classes of mappings due to its abundant geometric importance and applications. For instance, in [7], Dragomir et al. demonstrated some recent advancements in Simpson's inequality, where the remaining part is expressed in terms of derivatives lower than the fourth order. In [8], Alomari introduced Simpson's type inequalities for s-convex functions. Sarikaya et al. gave some Simpson's type inequalities via twice differentiable functions in [9]. For the other results, one can refer to [10], [11-13].

Under conditions similar to those in Simpson inequality, the Milne inequality is the one that gives estimates of the error boundaries for the Milne formula:

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_\infty,$$

where $f:[a,b]\to\mathrm{R}$ is a four times differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. In the recent times, researchers' attention to the Milne inequality has been considerable. Alomari and Liu [14] established error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings. Rom a' n-Flores et al. [15] proved some Milne type inequalities for interval-valued functions. Budak et al. [16] investigated Milne-type inequalities for bounded functions, Lipschitz functions and functions of bounded variation. Ali et al. [17] gave the fractional version of Milne's formula-type inequalities for differentiable convex functions and Riemann–Liouville fractional integrals. Many recent articles have been published on this subject, as in [18-20].

On the other hand, fractional calculus is a branch of mathematics that deals with derivatives and integrals of non-integer order. So, it plays a vital role in the generalization of classical calculus, modeling complex systems, solving fractional differential equations, analyzing fractal geometry and various scientific and engineering applications. Furthermore, it provides a framework for analyzing and understanding systems with fractional dynamics, allowing for a more comprehensive mathematical description of complex phenomena. Therefore, due to the new fractional integral and derivative such as Caputo-Fabrizio [21], Atangana-Baleanu [22] and tempered [23], this calculus has gained more importance and has found applications in various fields of science and engineering.

The Caputo derivative is defined as the application of a fractional integral to a standard derivative of the function whereas the Riemann-Liouville fractional derivative is obtained by differentiating the fractional integral of a function with respect to its independent variable of order n. The Caputo fractional derivative necessitates more suitable initial conditions in contrast to the conventional Riemann-Liouville fractional derivative considering fractional differential equations [24]. Accordingly, when evaluating other fractional derivatives, the Caputo derivative is advantageous since it yields solutions that are more meaningful in a physical sense for the specific problems. Besides, the operator of proportional derivative denoted as ${}^{P}D_{\alpha} f(x)$ is given by the equation [25]:

$$ {}^{P}D_{\alpha} f(x) = K_1(\alpha,t) f(t) + K_0(\alpha,t) f'(t), $$

where K_1 and K_0 are the functions with respect to $\alpha \in 0,1]$ and $t \in \mathbb{R}$ subject to certain conditions and also, the function f is differentiable with respect to $t \in \mathbb{R}$. This mathematical operator is commonly used in control systems and robotics. In recent years, there has been a notable increase in the importance of research conducted on both the Caputo derivative and the proportional derivative [26-28].

2. Preliminaries

To better understand the results obtained in this paper, we recall some basic concepts which we need in the sequel.

One of the significant definitions in fractional analysis is the following [29]:

Definition 1: Let $\alpha > 0$ and $\alpha \notin \{1,2,\ldots\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$, the space of functions having $n-th$ derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$ {}^{C}D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \;\; x > a $$

and

$$ {}^{C}D^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt, \;\; x < b. $$

If $\alpha = n \in \{1,2,3,\ldots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative ${}^{C}D^{\alpha}_{a^+}$ $f(x)$ coincides with $f^{(n)}(x)$ whereas ${}^{C}D^{\alpha}_{b^-}$ $f(x)$ with exactness to a constant multiplier $(-1)^n$. For $n=1$ and $\alpha = 0$, we have ${}^{C}D^{\alpha}_{a^+}$ $f(x)$ $=$ ${}^{C}D^{\alpha}_{b^-}$ $f(x) = f(x)$.

In [30], Baleanu et al. gave the following definition which they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann-Liouville fractional integral. gave the following definition which they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann-Liouville fractional integral.

Definition 2: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and f, f' are locally $L_1(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:

$$ {}^{C}D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left[K_1(\alpha,\tau) f(\tau) + K_0(\alpha,\tau) f'(\tau) \right] (t-\tau)^{-\alpha} d\tau $$

where $\alpha \in [0,1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$ \lim_{\alpha \to 0^+} K_0(\alpha,\tau) = 0; \ \lim_{\alpha \to 1} K_0(\alpha,\tau) = 1; \ K_0(\alpha,\tau) \neq 0, \ \alpha \in (0,1]; $$

$$ \lim_{\alpha \to 0} K_1(\alpha,\tau) = 0; \ \lim_{\alpha \to 1^-} K_1(\alpha,\tau) = 1; \ K_1(\alpha,\tau) \neq 0, \ \alpha \in [0,1). $$

Afterwards, Sarıkaya [31] presented a novel definition by employing distinct K_1 and K_0 functions based on Definition 2. Furthermore, Sarıkaya [31] derived the Hermite-Hadamard inequality utilizing his own new definition as presented below:

Definition 3: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and $f, f' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order α are defined respectively as follows:

$$ {}_{a^+}^{PC}D_b^{\alpha} f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, b-\tau) f(\tau) + K_0(\alpha, b-\tau) f'(\tau) \right] (b-\tau)^{-\alpha} d\tau $$

and

$$ {}_{b^-}^{PC}D_a^{\alpha} f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, \tau-a) f(\tau) + K_0(\alpha, \tau-a) f'(\tau) \right] (\tau-a)^{-\alpha} d\tau, $$

where $\alpha \in [0,1]$ and $K_0(\alpha,\tau) = (1-\alpha)^2 \tau^{1-\alpha}$ and $K_1(\alpha,\tau) = \alpha^2 \tau^{\alpha}$.

Theorem 1: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a,b \in I^\circ$ with $a < b$ and let f, f' be the convex functions on I. Then, the following inequalities hold:

$$ \alpha^2 (b-a)^{\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) $$

$$ \leq \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_{a^+}^{PC}D_b^{\alpha} f(b) + {}_{b^-}^{PC}D_a^{\alpha} f(a) \right] $$

$$ \leq \alpha^2 (b-a)^{\alpha} \left[\frac{f(a)+f(b)}{2} \right] + (1-\alpha))(b-a)^{1-\alpha} \left[\frac{f'(a)+f'(b)}{4} \right]. $$

In [32], Sarıkaya also gave the following Simpson's type inequality using his own definition of the proportional Caputo operator:

Theorem 2: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be differantiable function on I°, the interior of the interval I where $a,b \in I^\circ$ with $a < b$, and $f', f'' \in L[a,b]$. Then, the following identity holds:

$$
\begin{aligned}
&S(a,b;\alpha)\\
&=\frac{\alpha^2(b-a)^\alpha}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}}{12}\left[f'(a)+4f'\left(\frac{a+b}{2}\right)+f'(b)\right]\\
&-\frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}_{a^+}^{PC}D_b^\alpha f(b)+{}_{b^-}^{PC}D_a^\alpha f(a)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
S(a,b;\alpha)=&\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^1 P(t)[f'(ta+(1-t)b)+f'(tb+(1-t)a)]dt\\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_0^1 Q(t)[f''(ta+(1-t)b)+f''(tb+(1-t)a)]dt,
\end{aligned}
$$

$$
P(t)=\begin{cases}\frac{1}{6}-t, & 0\le t<\frac{1}{2}\\ \frac{5}{6}-t, & \frac{1}{2}\le t\le 1,\end{cases}
$$

and

$$
Q(t)=\begin{cases}\frac{1}{6}-t^{2-2\alpha}, & 0\le t<\frac{1}{2}\\ \frac{5}{6}-t^{2-2\alpha}, & \frac{1}{2}\le t\le 1.\end{cases}
$$

The purpose of this work is to use the proportional Caputo-hybrid operator to study the similar forms of the Milne-type inequalities with respect to Riemann integrals. For this purpose, firstly, we give an identity by utilizing the newly found proportional Caputo-hybrid operator. Then, we present several Milne-type inequalities with the aid of convexity, the Hölder inequality and the power mean inequality. Moreover, these results enhance and generalize the inequalities derived in earlier works. Next, we provide some examples supported by graphical representations to verify the established inequalities.

3. Results

The following lemma is essential for demonstrating our other main results. Therefore, we will give the proof of this lemma.

Lemma 1: Let $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b\in I^o$ satisfying $a<b$ and let $f, f', f''\in L_1[a,b]$. Then, the following identity is satisfied:

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{t^{2-2\alpha}}{2}-\frac{2}{3}\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right].$$

Proof. Integrating by parts, we have

$$\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{4}{3(b-a)}f(a)-\frac{2}{6(b-a)}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$

and

$$\int_0^1\left(\frac{t^{2-2\alpha}}{2}-\frac{2}{3}\right)f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{4}{3(b-a)}f'(a)-\frac{2}{6(b-a)}f'\left(\frac{a+b}{2}\right)-\frac{2-2\alpha}{b-a}\int_0^1 t^{1-2\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.$$

Using a variable change, multiplying the results by $\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}$ and $(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}$, and combining them side by side, we get the following result:

$$\begin{aligned}&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{t^{2-2\alpha}}{2}-\frac{2}{3}\right)f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\&=\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha+1}}{3}f(a)-\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha-1}}{3}f\left(\frac{a+b}{2}\right)\\&+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-1}}{3}f'(a)-\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-3}}{3}f'\left(\frac{a+b}{2}\right)\\&-\frac{2^{-\alpha}}{(b-a)^{1-\alpha}}\int_a^{\frac{a+b}{2}}\left[\alpha^2(\tau-a)^{\alpha}f(\tau)+(1-\alpha)^2(\tau-a)^{1-\alpha}f'(\tau)\right](\tau-a)^{-\alpha}d\tau.\end{aligned}\tag{1}$$

By following similar steps, we reach

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \tag{2}$$

$$=\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha+1}}{3}f(b)-\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha-1}}{3}f\left(\frac{a+b}{2}\right)$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-1}}{3}f'(b)-\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-3}}{3}f'\left(\frac{a+b}{2}\right)$$

$$-\frac{2^{-\alpha}}{(b-a)^{1-\alpha}}\int_{\frac{a+b}{2}}^{b}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}d\tau.$$

Thus, by putting (1) and (2) side by side, we obtain

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{t^{2-2\alpha}}{2}-\frac{2}{3}\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^{+}}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^{-}}D_a^{\alpha}f(a)\right],$$

which completes the proof.

Remark 1: Letting the limit as $\alpha\to 1$ in Lemma 1, it follows that

$$\frac{(b-a)}{2}\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$=\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(x)dx.$$

By using a variable substitution with the condition $\alpha = 1$, it is identical to Lemma 1 given by Budak et al. [16].

Corollary 1: In the limiting case $\alpha = 0$ in Lemma 1, we have

$$\frac{(b-a)}{2}\int_0^1\left(\frac{t^2}{2}-\frac{1}{3}\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)-f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt$$

$$=\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b}f(x)dx-\int_a^{\frac{a+b}{2}}f(x)dx\right).$$

Theorem 3: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b \in I^o$ satisfying $a<b$ and let $f, f', f'' \in L_1[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q \geq 1$, then the following inequality holds:

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.$$
$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right) \tag{3}$$
$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|$$
$$\leq\frac{5\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}}{12}\left[\left(\frac{|f'(a)|^q+4|f'(b)|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{4|f'(a)|^q+|f'(b)|^q}{5}\right)^{\frac{1}{q}}\right]$$
$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(A(\alpha)|f''(a)|^q+B(\alpha)|f''(b)|^q\right)^{\frac{1}{q}}+\left(B(\alpha)|f''(a)|^q+A(\alpha)|f''(b)|^q\right)^{\frac{1}{q}}\right]$$

where

$$A(\alpha)=\frac{1}{6}-\frac{1}{4(4-2\alpha)}$$

and

$$B(\alpha)=\frac{1}{2}-\frac{1}{2(3-2\alpha)}+\frac{1}{4(4-2\alpha)}.$$

Proof. Firstly, let $q=1$. By the convexity of $|f'|$ and $|f''|$, we obtain from Lemma 1 it follows that

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|$$

$$\leq\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)\left[\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left[\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt$$

$$\leq\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)dt$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)dt$$

$$=\frac{5\alpha^2(b-a)^{\alpha+1}2^{-\alpha-3}}{3}\left(|f'(a)|+|f'(b)|\right)$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

Now, consider $q>1$. In view of Lemma 1, using the property of the power mean inequality and the convexity of $|f'|^q$, $|f''|^q$, we get

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|$$

$$\leq \alpha^2 (b-a)^{\alpha+1} 2^{-\alpha-1}\left[\left(\int_0^1 \left(\frac{2}{3}-\frac{t}{2}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{2}{3}-\frac{t}{2}\right)\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left(\frac{2}{3}-\frac{t}{2}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{2}{3}-\frac{t}{2}\right)\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+(1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-3}\left[\left(\int_0^1 \left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \alpha^2 (b-a)^{\alpha+1} 2^{-\alpha-1}\left(\frac{5}{12}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{12}|f'(a)|^q+\frac{4}{12}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{4}{12}|f'(a)|^q+\frac{1}{12}|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

$$+(1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-3}\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\left(\frac{1}{6}-\frac{1}{4(4-2\alpha)}\right)|f''(a)|^q+\left(\frac{1}{2}-\frac{1}{2(3-2\alpha)}+\frac{1}{4(4-2\alpha)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{1}{2}-\frac{1}{2(3-2\alpha)}+\frac{1}{4(4-2\alpha)}\right)|f''(a)|^q+\left(\frac{1}{6}-\frac{1}{4(4-2\alpha)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right].$$

Thus, the proof ends.

Now, we show the effectiveness of our theorem with an illustrative example.

Example 1: Let us consider a function $f:[0,2]\to \mathbb{R}$ given by $f(x) = x^3$. Then, for $q=1$, we can calculate the right-hand side of the inequality (3) as follows:

$$5\alpha^2+6(1-\alpha)\left(\frac{2}{3}-\frac{1}{2(3-2\alpha)}\right):=\Psi_1.$$

On the other hand, we obtain that

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right.$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|$$

$$=3\alpha^2+\frac{7}{2}(1-\alpha)-\frac{3(1-\alpha)^2}{4-2\alpha}-\frac{6(1-\alpha)^2}{2-2\alpha}+\frac{6(1-\alpha)^2}{3-2\alpha}:=\Psi_2.$$

As one can see in Fig. (**1**), the left-hand side of the inequality (3) is always below the right-hand side of this inequality for all values of $\alpha\in(0,1)$ and $q=1$.

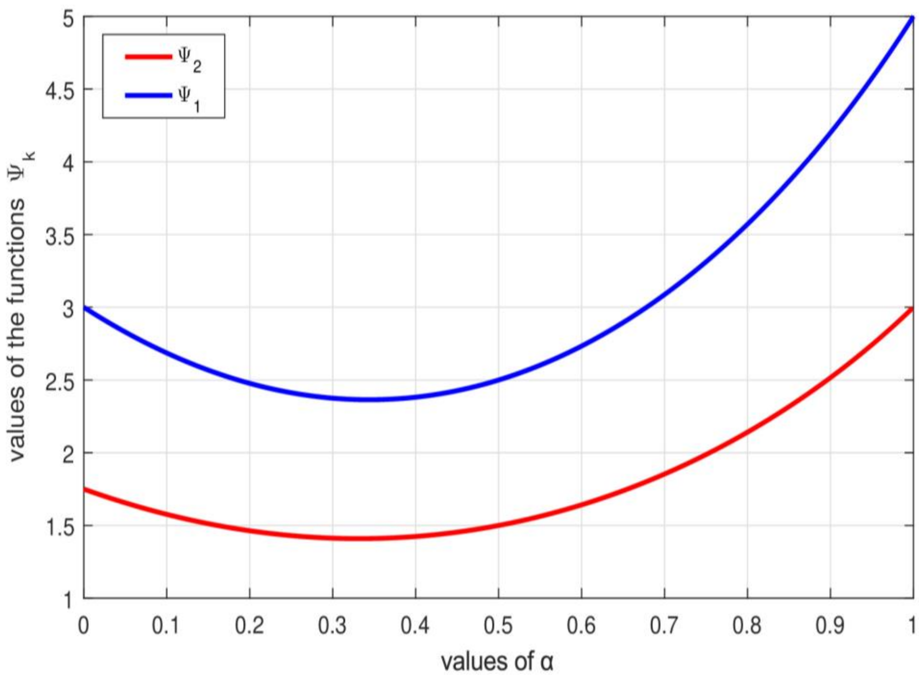

Figure 1: The graph of both sides of the inequality (3) according to Example 1, which is computed and drawn in MATLAB program, depending on $\alpha\in(0,1)$ and $q=1$.

Remark 2: Letting the limit as $\alpha\to 1$ and putting $q=1$ in Theorem 3, it follows that

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{5(b-a)}{24}\left(|f'(a)|+|f'(b)|\right),$$

which was proved by Budak et al. in [16]. Moreover, as α converges to 1 and $q\ge 1$, the inequality in Theorem 3 is given by

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{5(b-a)}{24}\left[\left(\frac{\left|f'(a)\right|^q+4\left|f'(b)\right|^q}{5}\right)^{1/q}+\left(\frac{4\left|f'(a)\right|^q+\left|f'(b)\right|^q}{5}\right)^{1/q}\right],$$

which was proved by Budak et al. in [16].

Corollary 2: In the limiting case $\alpha = 0$ in Theorem 3, we obtain

$$\left|\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_a^{\frac{a+b}{2}} f(x)dx\right)\right|$$

$$\leq \frac{(b-a)}{4}\left[\left(\frac{5}{24}\left|f''(a)\right|^q+\frac{19}{24}\left|f''(b)\right|^q\right)^{1/q}+\left(\frac{19}{24}\left|f''(a)\right|^q+\frac{5}{24}\left|f''(b)\right|^q\right)^{1/q}\right].$$

Theorem 4: Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $a,b \in I^o$ satisfying $a<b$ and let $f, f', f'' \in L_1[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q>1$, then the following inequality holds:

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right. \tag{4}$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)$$

$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha} f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha} f(a)\right]\right|$$

$$\leq \frac{\alpha^2(b-a)^{1+\alpha}2^{-\alpha-1}}{6}\left(\frac{2^{2p+2}-1}{3p+3}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\frac{2^{2p}\left((2-2\alpha)p+1\right)-3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\alpha^2 (b-a)^{1+\alpha} 2^{-\alpha-1}}{6}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)$$

$$+(1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-3}\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}\left(\left|f^{''}(a)\right|+\left|f^{''}(b)\right|\right)$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. By employing the well-known Hölder's inequality and the convexity of $|f^{'}|^q$ and $|f^{''}|^q$, based on Lemma 1, we have

$$\left|\frac{\alpha^2(b-a)^{\alpha}2^{-\alpha}}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)\right. \tag{5}$$

$$+\frac{(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}}{3}\left(2f^{'}(a)-f^{'}\left(\frac{a+b}{2}\right)+2f^{'}(b)\right)$$

$$\left.-\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{\left(\frac{a+b}{2}\right)^+}D_b^{\alpha}f(b)+{}^{PC}_{\left(\frac{a+b}{2}\right)^-}D_a^{\alpha}f(a)\right]\right|$$

$$\leq \alpha^2(b-a)^{1+\alpha}2^{-\alpha-1}\left(\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left|f^{'}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f^{'}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left|f^{''}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f^{''}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

When we calculate the integrals in the above inequality, we get

$$\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)^p dt=\frac{2^{2p+3}-2}{6^{p+1}\left(p+1\right)},$$

$$\int_0^1\left[\frac{2-t}{2}\left|f'(a)\right|^q+\frac{t}{2}\left|f'(b)\right|^q\right]dt=\frac{3|f'(a)|^q+|f'(b)|^q}{4},$$

$$\int_0^1\left[\frac{t}{2}\left|f'(a)\right|^q+\frac{2-t}{2}\left|f'(b)\right|^q\right]dt=\frac{|f'(a)|^q+3|f'(b)|^q}{4},$$

$$\int_0^1\left[\frac{2-t}{2}\left|f''(a)\right|^q+\frac{t}{2}\left|f''(b)\right|^q\right]dt=\frac{3|f''(a)|^q+|f''(b)|^q}{4},$$

$$\int_0^1\left[\frac{t}{2}\left|f''(a)\right|^q+\frac{2-t}{2}\left|f''(b)\right|^q\right]dt=\frac{|f''(a)|^q+3|f''(b)|^q}{4}.$$

Also, using the property $(A-B)^p \le A^p - B^p$ for $A > B \ge 0$ and $p \ge 1$, we obtain

$$\int_0^1\left(\frac{2}{3}-\frac{t^{2-2\alpha}}{2}\right)^p dt \le \int_0^1\left(\frac{2}{3}\right)^p-\frac{t^{2p-2\alpha p}}{2^p}dt=\left(\frac{2}{3}\right)^p-\frac{1}{2^p\left((2-2\alpha)p+1\right)}.$$

Thus, the first part of (4) can be obtained by replacing the calculated integral results into inequality (5) . Furthermore, it is known that we have the property

$$\sum_{k=1}^n\left(a_k+b_k\right)^s \le \sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s$$

for $0 \le s < 1$ and $a_k, b_k \ge 0$ with $k \in \{1,2,\ldots,n\}$. Thus, for the proof of the second part of (4), if we take $a_1=\left|f'(a)\right|^q$, $b_1=3\left|f'(b)\right|^q$, $a_2=3\left|f'(a)\right|^q$, $b_2=\left|f'(b)\right|^q$, $a_1'=\left|f''(a)\right|^q$, $b_1'=3\left|f''(b)\right|^q$, $a_2'=3\left|f''(a)\right|^q$, $b_2'=\left|f''(b)\right|^q$, by $1+3^{\frac{1}{q}} \le 4$, the intended result can be attained in an easy way.

Example 2: Take a function f as defined in Example 1. Then, we obtain that the left-hand side and right-hand side of (4) are

$$3\alpha^2+\frac{7}{2}(1-\alpha)-\frac{3(1-\alpha)^2}{4-2\alpha}-\frac{6(1-\alpha)^2}{2-2\alpha}+\frac{6(1-\alpha)^2}{3-2\alpha}:=\Omega_1$$

and

$$2\alpha^2\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}+6(1-\alpha)\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}:=\Omega_2.$$

Furthermore, we have

$$\frac{\alpha^2(b-a)^{1+\alpha}2^{-\alpha-1}}{6}\left(\frac{2^{2p+2}-1}{3p+3}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\left(\frac{2^{2p}\left((2-2\alpha)p+1\right)-3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$=\frac{\alpha^2}{2}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(1+3^{\frac{1}{q}}\right)+\frac{3(1-\alpha)}{2}\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}\left(1+3^{\frac{1}{q}}\right):=\Omega_3$$

Therefore, in view of inequality (4), we get the inequality

$$3\alpha^2+\frac{7}{2}(1-\alpha)-\frac{3(1-\alpha)^2}{4-2\alpha}-\frac{6(1-\alpha)^2}{2-2\alpha}+\frac{6(1-\alpha)^2}{3-2\alpha} \tag{6}$$

$$\leq\frac{\alpha^2}{2}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(1+3^{\frac{1}{q}}\right)+\frac{3(1-\alpha)}{2}\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}\left(1+3^{\frac{1}{q}}\right)$$

$$\leq 2\alpha^2\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}+6(1-\alpha)\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}.$$

Thus, the validity of inequality (6) is demonstrated in Fig. (**2**).

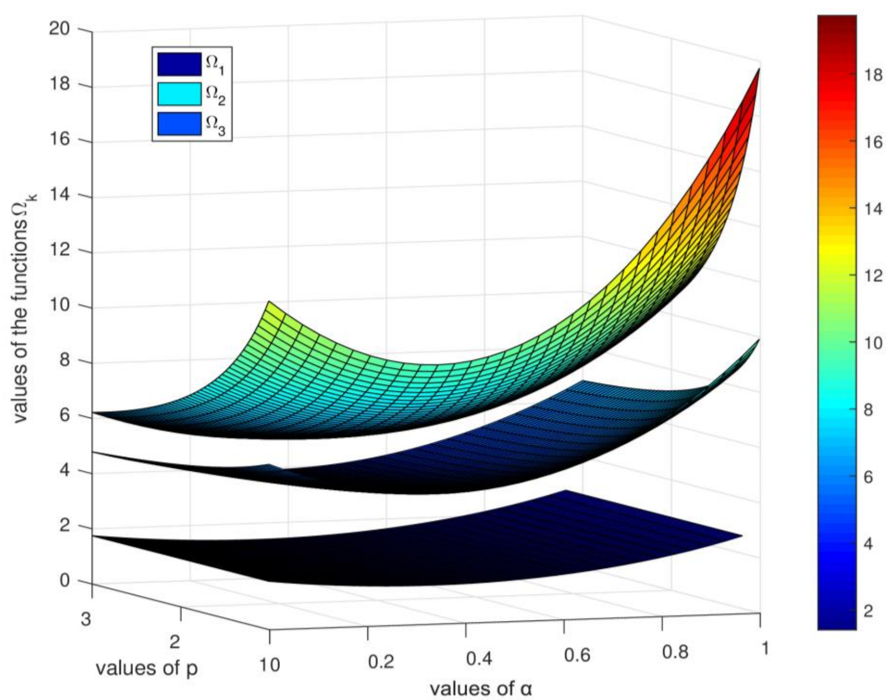

Figure 2: The graph of three parts of the inequality (6) in Example 2, which is computed and drawn in MATLAB program, depending on $\alpha\in(0,1)$ and $p\in(1,3]$.

Remark 3: In the special case when α tends to 1 in Theorem 4, we reach

$$\left|\frac{1}{3}\left(2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{12}\left(\frac{2^{2p+2}-1}{3p+3}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{12}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),$$

which was proved by Budak et al. in [16].

Corollary 3: The following specific situation occurs as α approaches 0, according to Theorem 4:

$$\left|\frac{1}{3}\left(2f'(a)-f'\left(\frac{a+b}{2}\right)+2f'(b)\right)-\frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b} f(x)dx-\int_a^{\frac{a+b}{2}} f(x)dx\right)\right|$$

$$\leq \frac{(b-a)}{2}\left(\frac{2^{2p}\left((2-2\alpha)p+1\right)-3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{1/q}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{1/q}\right]$$

$$\leq \frac{(b-a)}{2}\left(\frac{2^{2p+2}\left((2-2\alpha)p+1\right)-4.3^p}{6^p\left((2-2\alpha)p+1\right)}\right)^{\frac{1}{p}}\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

4. Conclusion

The purpose of this work is to develop new Milne-type integral inequalities for twice-differentiable convex mappings by using a proportional Caputo hybrid operator. We start by demonstrating a new integral identity of the Milne-type associated with proportional Caputo-hybrid operator in order to accomplish this purpose. Next, utilizing convexity, the Hölder inequality, and the power mean inequality, we present many Milne-type inequalities. Since our results for $\alpha \to 1$ represent the specific case of previously established bounds, they are more useful in this study than in traditional calculus. Therefore, we hope that our methods and results will inspire readers to investigate this subject further. In future work, one can explore similar inequalities for distinct fractional integrals and obtain new Milne-type inequalities through the use of various forms of convexity.

Conflict of Interest

The author has no competing interests to declare that are relevant to the content of this article.

Funding

The study received no financial support.

Data Availability

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

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