Generalized Rational Type Contraction and Fixed Point Theorems in Partially Ordered Metric Spaces

Muhammed Raji\(^1\); Laxmi Rathour\(^2,^*\), Lakshmi N. Mishra\(^3\) and Vishnu N. Mishra\(^4,^*\)

\(^1\)Department of Mathematics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria
\(^2\)Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India
\(^3\)Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
\(^4\)Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

ARTICLE INFO

Article Type: Research Article
Academic Editor: Nurettin Bağirmaz\(^\dagger\)

Keywords:
Fixed point
Metric spaces
Well ordered set
Rational contractions
Partially ordered metric space

Timeline:
Received: November 10, 2023
Accepted: December 13, 2023
Published: December 30, 2023


DOI: https://doi.org/10.15377/2409-5761.2023.10.13

ABSTRACT

In this article, we establish the existence and uniqueness of fixed points for rational type contraction mappings in a metric space that is equipped with a partial order. Our results are shown to improve upon previous results in the literature, and we provide illustrative examples to demonstrate the effectiveness of our approach.

Mathematics Subject Classification: 47H10; 54H25.
1. Introduction

M. Frechet’s introduction of metric spaces in 1906 was a groundbreaking development in the field of functional analysis. This new type of space provided a natural setting for studying functions and operators, and has since been generalized and expanded upon in a variety of ways. Several researchers have developed generalized metric spaces, such as complex valued metric spaces, rectangular metric spaces, semi metric spaces, quasimetric spaces, and more. These spaces have found applications in a wide range of areas, including probability, computer science, and functional analysis.

There are numerous results in the literature that extend or improve upon the existence and uniqueness of fixed points. One such result is that of Ran and Reurings [1], who studied fixed points for certain mappings in partially ordered metric spaces. Nieto and Lopez [2] later generalized this result for non-decreasing mappings and demonstrated its applications to partial differential equations.

The mixed monotone property of contractive operators was introduced by Bhaskar and Lakshmikantham [3], who considered the case of partially ordered metric spaces. They then derived several coupled fixed point theorems using this property. Chatterji [4] later considered contractive conditions for self-mappings in metric spaces, and Dass and Gupta [5] studied rational type contractions in complete metric spaces.

Recently, Seshagiri Rao and Kalyani [6-8] have explored some results on fixed point, coupled fixed point and coincidence point for the mappings in partially ordered metric spaces satisfying rational type contraction.

Motivated by the research of Seshagiri Rao and Kalyani [9], we introduce a class of mappings known as generalized rational type contractive mappings and explore the properties of these mappings and derive fixed point results for them in partially ordered metric spaces. We also provide illustrative examples to demonstrate the improvements offered by our approach.

2. Preliminaries

In this section, we start with the following definitions and theorems that motivate our study as follows:

**Definition 2.1.** [10] The triple \((X, d, \preceq)\) is called partially ordered metric spaces, if \((X, \preceq)\) is a partially ordered set and \((X, d)\) is a metric space.

**Definition 2.2.** [11] Let \((X, d)\) be a complete metric space, then the triple \((X, d, \preceq)\) is called complete partially ordered metric spaces.

**Definition 2.3.** [12] A partially ordered metric space \((X, d, \preceq)\) is called ordered complete if for each convergent sequence \(\{x_n\}_{n=0}^{\infty} \subset X\), the following condition holds: either

i. if \(x_n\) is a non-increasing sequence in \(X\) such that \(x_n \to x\) implies \(x \leq x_n\), for all \(n \in \mathbb{N}\) that is, \(x = \inf\{x_n\}\), or

ii. if \(x_n\) is a non-decreasing sequence in \(X\) such that \(x_n \to x\) implies \(x_n \leq x\), for all \(n \in \mathbb{N}\) that is, \(x = \sup\{x_n\}\).

**Definition 2.4.** [13] Let \((X, \preceq)\) be a partially ordered set and let \(T: X \to X\) be a mapping. Then

i. elements \(x, y \in X\) are comparable, if \(x \preceq y\) or \(y \preceq x\) holds;

ii. a non-empty set \(X\) is called well ordered set, if every two elements of it are comparable;

iii. \(T\) is said to be monotone non-decreasing with respect to \(\preceq\), if for all \(x, y \in X\),

\[x \leq y \text{ implies } Tx \leq Ty.\]

iv. \(T\) is said to be monotone non-increasing with respect to \(\preceq\), if for all \(x, y \in X\),

\[x \leq y \text{ implies } Tx \geq Ty.\]
we have the following two cases:

**Theorem 2.4.** [7] Let $T:X \to X$ be a complete metric space $(X, d)$ satisfying the following condition

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)[1 + d(y, Ty)]}{1 + d(x, y)} + \beta [d(x, Tx) + d(y, Ty)] + \gamma [d(x, Ty) + d(y, Tx)] + \delta d(x, y)$$

(2.1)

for all distinct $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + 2(\beta + \gamma) + \delta < 1$. Then $T$ has a unique fixed point in $X$.

For more details on generalized metric spaces, one can see [14-24].

### 3. Main Results

**Theorem 3.1.** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that $T:X \to X$ is a non-decreasing, continuous self mapping satisfying

$$d(Tx, Ty) \leq \begin{cases} 
\varepsilon d(x, y) + \delta [d(x, Ty) + d(y, Tx)] + \gamma \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(x, Ty)} & \text{if } K \neq 0 \\
0 & \text{if } K = 0
\end{cases}$$

(3.1)

for all distinct $x, y \in X$ with $y \preceq x$, where $K = d(y, Tx) + d(x, Ty)$ and $\alpha, \beta, \gamma, \varepsilon, \delta \geq 0$ with $\varepsilon + 2\delta + \gamma + \beta + \alpha < 1$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point in $X$.

**Proof.** Suppose $x_0 = Tx_0$, then we have the result complete. Let that $x_0 < Tx_0$. Since $T$ is a non-decreasing mapping, then by induction we obtain that

$$x_0 < Tx_0 \preceq T^2x_0 \preceq \ldots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq$$

(3.2)

Now, we construct a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Since $T$ is non-decreasing mapping, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq$$

Suppose there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $x_{n+1} = Tx_n = x_n$, implies $x_n$ is a fixed point and the proof is finished. Again, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since the points $x_n$ and $x_{n-1}$ are comparable for $n \in \mathbb{N}$ with (3.2), we have the following two cases:

**Case 1:** If $K = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) \neq 0$, then, with (3.1), we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$$

$$\leq \varepsilon d(x_n, x_{n-1}) + \delta [d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] + \gamma \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} + \beta \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1})}$$

implies that

$$d(x_{n+1}, x_n) \leq \varepsilon d(x_n, x_{n-1}) + \delta [d(x_n, x_{n-1}) + d(x_n, x_{n+1})] + \gamma \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1})} + \beta \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_n, x_{n-1}) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}$$

By triangular inequality $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})$ and $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$, we have
\[ d(x_{n+1}, x_n) \leq \left( \frac{\varepsilon + \delta + \beta + \alpha}{1 - \delta - \gamma} \right) d(x_{n-1}, x_n) \]

Inductively, we have

\[ d(x_{n+1}, x_n) \leq r^n d(x_0, x_1) \]

where \( r = \frac{\varepsilon + \delta + \beta + \alpha}{1 - \delta - \gamma} < 1 \). Now, we prove that \( \{x_n\} \) is a Cauchy sequence. For \( m \geq n \) and by triangular inequality, we have

\[ d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \leq \frac{r^n}{1-r} d(x_1, x_0) \]

as \( m, n \to +\infty, d(x_m, x_n) \to 0 \). Thus, \( \{x_n\} \) is a Cauchy sequence in a complete metric space \( X \). Hence, there exists \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \). Then, by continuity of \( T \) we have

\[
Tu = T \left( \lim_{n \to \infty} x_n \right) \\
= \lim_{n \to \infty} Tx_n \\
= \lim_{n \to \infty} x_{n+1} \\
= u.
\]

Thus, \( u \) is a fixed point of \( T \).

**Case 2:** If \( K = d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) = 0 \), then \( d(x_{n+1}, x_n) = 0 \). Which implies that \( x_{n+1} = x_n \), a contradiction as the sequence points is comparable. Thus, there exists a fixed point \( u \) of \( T \).

**Example 3.2.** Suppose \( X = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2 \), and define the usual order

\[
U: (u, v) \preceq (z, t) \iff u \leq z \text{ and } v \leq t. \tag{3.3}
\]

Let \( T: X \to X \) define by \( T(x, y) = (x, y) \). Then \( T \) has a fixed point in \( X \).

**Proof.** Let \( (X, \preceq) \) be a partially ordered set, whose different elements are not comparable. Besides, \( (X, d_2) \) is a complete metric space considering \( d_2 \), the Euclidean distance. The identity map \( T(x, y) = (x, y) \) is trivially continuous and nondecreasing and condition

\[
d(T(u, v), T(z, t)) \\
\leq \varepsilon d((u, v), (z, t)) + \delta d((u, v), T(z, t)) + d((z, t), T(u, v)) + \gamma \frac{d((u, v), T(u, v))d((z, t), T(z, t))}{d((u, v), (z, t))} \\
+ \beta \frac{d((u, v), (z, t)) + d((u, v), T(z, t)) + d((z, t), T(u, v))}{d((z, t), T(z, t))} \\
+ \alpha \frac{d((u, v), T(u, v))d((z, t), T(z, t)) + d((z, t), (u, v))d((z, t), T(z, t))}{d((u, v), T(u, v)) + d((u, v), T(z, t))}
\]

for all \( \alpha, \beta, \gamma, \delta, \varepsilon \in [0,1] \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). Since elements in \( X \) are only comparable to themselves. Furthermore, \((1,0) \preceq T(1,0) = (1,0) \). Thus, there are two fixed points in \( X \), the assertions in Theorem 3.1 holds and \( T \) has two fixed points, \((1,0)\) and \((0,1)\).

Now, we prove that Theorem 3.1 is still valid for \( T \), not necessarily continuous, by assuming the following hypothesis in \( X \).
If \(\{x_n\}\) is a non-decreasing sequence in \(X\) such that \(x_n \to x\), then
\[
x = \sup\{x_n\}.
\]

**Theorem 3.3.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Assume that \(T:X \to X\) is a monotone non-decreasing self mapping satisfying
\[
d(Tx, Ty) \leq \frac{\epsilon d(x, y) + \delta [d(x, Ty) + d(y, Tx)] + \gamma \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta \frac{d(x, Ty)d(y, Tx)}{d(x, y)} + \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \delta \frac{d(x, Ty)d(y, Tx)}{d(x, y)}}{0},
\]
if \(K \neq 0\)
\[
+ \frac{\alpha}{\beta} \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \frac{\alpha}{\gamma} \frac{d(x, Ty)d(y, Tx)}{d(x, y)}\]
if \(K = 0\)
for all distinct \(x, y \in X\) with \(y \preceq x\), where \(K = d(y, Tx) + d(x, Ty)\) and \(\alpha, \beta, \gamma, \epsilon, \delta \geq 0\) with \(\epsilon + 2\delta + \gamma + \beta + \alpha < 1\). If there exists \(x_0 \in X\) with \(x_0 \preceq Tx_0\), then \(T\) has a fixed point in \(X\).

**Proof.** In view of the proof of Theorem 3.1, we have \(\{x_n\}\) is Cauchy sequence. Now, we need to check that \(Tu = u\). Since \(\{x_n\}\) is a non-decreasing sequence in \(X\) such that \(x_n \to u\), then \(u = \sup\{x_n\}\) for all \(n \in \mathbb{N}\) by (3.4). Also since \(T\) is a non-decreasing mapping, then \(Tx_n \preceq Tu\) for all \(n \in \mathbb{N}\) or, equivalently, \(x_{n+1} \preceq Tu\) for all \(n \in \mathbb{N}\). Moreover, as \(x_0 < x_1 \preceq Tu\) and \(u = \sup\{x_n\}\), we get \(u \preceq Tu\). Suppose that \(u < Tu\). Using the same argument as that in the proof of Theorem 3.1 for \(x_0 < x_0\), we get a non-decreasing sequence \(\{Tu\}\) in \(X\) such that \(\lim_{n \to \infty} Tu = y\) for certain \(y \in X\). Again, with (3.4), we obtain that \(y = \sup\{Tu\}\). Moreover, from \(x_0 < u\), we get \(x_n < Tu < x_0 < Tu\) for \(n \geq 1\) and \(x_n < Tu < x_0 < Tu\) for \(n \geq 1\) because \(x_n < u < Tu < x_0 < Tu\) for all \(n \geq 1\).

Since \(x_n\) and \(Tu\) are comparable and distinct for \(n \geq 1\), we consider the following cases:

**Case 1:** If \(d(Tu, Tu) + d(x_n, Tu) \neq 0\), then, by using (3.1), we have
\[
d(x_{n+1}, Tu) = d(Tx_n, Tu) \\
\leq \epsilon d(x_n, Tu) + \delta [d(x_n, Tu) + d(Tu, x_{n+1})] + \gamma \frac{d(x_n, x_{n+1})d(Tu, Tu)}{d(x_n, Tu)} + \beta \frac{d(x_n, Tu)d(Tu, Tu)}{d(x_n, Tu)} + \alpha \frac{d(x_n, Tu)d(Tu, Tu)}{d(x_n, Tu)}
\]
On letting \(n \to +\infty\), we have
\[
d(u, y) \leq \epsilon d(u, y) + \delta [d(u, y) + d(y, u)] + \gamma \frac{d(u, y)d(y, u)}{d(u, y)} + \beta \frac{d(u, y)d(u, y)}{d(u, y)} + \alpha \frac{d(u, y)d(y, u)}{d(u, y)}
\]
\[
d(u, y) \leq (\epsilon + 2\delta)d(u, y)
\]
as \(\epsilon + 2\delta < 1\), \(d(u, y) = 0\), that is, \(u = y\). Hence, \(u = y = \sup\{Tu\}\) and consequently, \(Tu \preceq u\), a contradiction. Thus, \(u\) is a fixed point of \(T\).

**Case 2:** If \(d(Tu, Tu) + d(x_n, Tu) = 0\), then \(d(x_{n+1}, Tu) = 0\). Taking the limit as \(n \to +\infty\), we get \(d(u, y) = 0\). That is, \(u = y = \sup\{Tu\}\), which implies that \(Tu \preceq u\), a contradiction. Thus, \(Tu = u\).

To show uniqueness of fixed point that exists in Theorem 3.1 and Theorem 3.3, we give sufficient condition as follows
\[
every\ pair\ of\ elements\ has\ a\ lower\ bound\ or\ an\ upper\ bound.
\]
In [12], the above condition is equivalent to
for every $y, z \in X$, there exists $x \in X$ that is comparable to $y$ and $z$.

**Theorem 3.4.** In addition to the hypothesis of Theorem 3.1 (or Theorem 3.3), assume that for every $y, z \in X$, there exists $x \in X$ that is comparable to $y$ and $z$, then $T$ has a unique fixed point.

**Proof.** In view of Theorem 3.1 (or Theorem 3.3), the set of fixed points of $T$ is non-empty. Let $y, z \in X$ be two fixed points of $T$. Then, we distinguish two cases:

**Case 1:** If $y$ and $z$ are comparable and $y \neq z$. Then, we obtain the following two subcases:

(i). If $d(z, Ty) + d(y, Tz) \neq 0$, then using (3.1), we have

$$d(y, z) = d(Ty, Tz)$$

$$\leq \epsilon d(y, z) + \delta [d(y, Ty) + d(z, Tz)] + \gamma \frac{d(y, Ty) d(z, Tz)}{d(y, z)} + \beta \frac{d(y, Ty) d(z, Tz)}{d(y, z) + d(y, Tz) + d(z, Tz)}$$

$$+ \alpha \frac{d(y, Ty) d(z, Tz) + d(z, Tz) d(z, Ty) + d(z, Ty) d(z, Ty)}{d(z, Ty) + d(y, Tz) + d(z, Ty) + d(y, Tz) + d(z, Ty)}$$

implies

$$d(y, z) \leq (\epsilon + 2\delta) d(y, z)$$

$$< d(y, z) \text{ as } \epsilon + 2\delta < 1$$

a contradiction. Thus, $y = z$.

(ii). If $d(z, Ty) + d(y, Tz) = 0$, then $d(y, z) = 0$, a contradiction again. Hence, $y = z$.

**Case 2:** If $y$ and $z$ are not comparable, then, by (3.1), there exists $x \in X$ comparable to $y$ and $z$. By monotonicity it implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0, 1, 2, \ldots$.

If there exists $n_0 \geq 1$ such that $T^{n_0} x = y$, then as $y$ is a fixed point, the sequence $\{T^n x : n \geq n_0\}$ is constant and consequently $\lim_{n \to \infty} T^n x = y$. On the other hand, if $T^n x \neq y$ for all $n \geq 1$.

Again, we have two subcases:

(i). If $d(T^{n-1} y, T^n x) + d(T^{n-1} x, T^n y) \neq 0$, then with (3.1) for $n \geq 2$, we have

$$d(T^n x, y) = d(T^n x, T^n y)$$

$$\leq \epsilon d(T^{n-1} x, y) + \delta [d(T^{n-1} x, y) + d(y, T^{n-1} x)] + \gamma \frac{d(T^{n-1} x, T^n y)}{d(T^{n-1} x, y)}$$

$$+ \beta \frac{d(T^{n-1} x, T^n y) d(y, y)}{d(T^{n-1} x, y) + d(T^{n-1} x, y) + d(y, T^{n-1} x) + d(y, T^{n-1} y)}$$

By triangular inequality $d(T^n x, T^{n-1} x) \leq d(T^n x, y) + d(y, T^{n-1} x)$, we have

$$d(T^n x, y) \leq \epsilon d(T^{n-1} x, y) + \delta [d(T^{n-1} x, y) + d(y, T^{n-1} x)] + \alpha d(T^{n-1} x, y)$$

implies that

$$d(T^n x, y) \leq \left(\frac{\epsilon + \delta + \alpha}{1 - \delta}\right) d(T^{n-1} x, y).$$
Then, by induction, we have
\[ d(T^n x, y) \leq \left( \frac{\varepsilon + \delta + \alpha}{1 - \delta} \right)^n d(x, y). \]

Using \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \) and taking limit as \( n \to +\infty \) in the above inequality, we get
\[ \lim_{n \to +\infty} T^n x = y. \]

Similarly, we get
\[ \lim_{n \to +\infty} T^n x = z. \]

Now, the uniqueness of the limit gives that \( y = z \).

(ii) If \( d(T^{n-1} x, T^n x) + d(T^{n-1} y, T^n y) = 0 \), then using (3.1), we have \( d(T^n x, y) = 0 \). Therefore,
\[ \lim_{n \to +\infty} T^n x = y. \]

Similarly, we get
\[ \lim_{n \to +\infty} T^n x = z. \]

Again, the uniqueness of the limit gives that \( y = z \). Thus, \( T \) has a unique fixed point in \( X \).

**Example 3.5.** Suppose \( X = \{(0,0), (\frac{1}{2}, 0), (1,1)\} \) is a subset of \( \mathbb{R}^2 \) with the order \( \preceq \) define as: for \( (x_1, y_1), (x_2, y_2) \in X \) with \( (x_1, y_1) \preceq (x_2, y_2) \) if and only if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). Let the distance function \( d: X \times X \to \mathbb{R} \) be defined by
\[ d\left((x_1,y_1),(x_2,y_2)\right) = \max\{|x_1-x_2|,|y_1-y_2|\}. \]  

Again, let \( d: X \to X \) be defined by \( T(0,0) = (0,0), T(0,1) = (\frac{1}{2}, 0) \) and \( T(\frac{1}{2}, 0) = (0,0) \). Observe that all the conditions of Theorem 3.1 and 3.3 are satisfied and \( (0,0) \) is the unique fixed point of \( T \).

**Corollary 3.6.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing, continuous self mapping satisfying
\[ d(Tx, Ty) \leq \varepsilon d(x, y) + \delta[d(x, Ty) + d(y, Tx)] + \gamma \frac{d(Tx,d(y,Ty))}{d(x,y)} + \beta \frac{d(Tx,d(y,Ty)+d(y,Tx))}{d(x,y)+d(x,Tx)+d(y,Tx)} + \alpha \frac{d(Tx,d(y,Ty)+d(y,Tx))}{d(y,Tx)+d(x,Ty)} \]  

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).

Corollary 3.6 is still valid for \( T \), not necessarily continuous, by assuming the following hypothesis in \( X \).

If \( \{x_n\} \) is a non-decreasing sequence in \( X \) such that \( x_n \to x \), then
\[ x = \operatorname{sup}\{x_n\}. \]  

**Corollary 3.7.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Assume that \( T: X \to X \) is a monotone non-decreasing self mapping satisfying
\[ d(Tx, Ty) \leq \varepsilon d(x, y) + \delta[d(x, Ty) + d(y, Tx)] + \gamma \frac{d(Tx,d(y,Ty))}{d(x,y)} + \beta \frac{d(Tx,d(y,Ty)+d(y,Tx))}{d(x,y)+d(x,Tx)+d(y,Tx)} + \alpha \frac{d(Tx,d(y,Ty)+d(y,Tx))}{d(y,Tx)+d(x,Ty)} \]  

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).
for all distinct \( x, y \in X \) with \( y \preceq x \), where \( K = d(y, Tx) + d(x, Ty) \) and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).

**Corollary 3.8.** Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing self mapping and that for some positive integer \( m \), self mapping \( T \) satisfying

\[
d(T^m x, T^m y) \leq \varepsilon d(x, y) + \delta [d(x, T^m y) + d(y, T^m x)] + \gamma \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} + \beta \frac{d(x, T^m x) d(y, T^m y)}{d(x, y) + d(T^m x, T^m y) + d(y, T^m y)} + \alpha \frac{d(x, T^m x) d(y, T^m y)}{d(y, T^m x) + d(x, T^m y)}
\]

(3.11)

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). Suppose there exists \( x_0 \in X \) with \( x_0 \preceq T^m x_0 \). If \( T^m \) is continuous, then \( T \) has a fixed point in \( X \).

### 4. Application

In this section, we consider some of the application of our results for the self mapping involving integral type contractions areas follows:

**Theorem 4.1.** Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing, continuous self mapping satisfying

\[
\int_0^{d(Tx, Ty)} ds \leq \varepsilon \int_0^{d(x, y)} ds + \delta \int_0^{d(x, Ty) + d(y, Tx)} ds + \gamma \frac{d(x, Ty) d(y, Tx)}{d(x, y)} ds + \beta \frac{d(x, Ty) d(y, Tx)}{d(y, Ty) + d(x, Ty) + d(y, Tx)} ds + \alpha \frac{d(x, Ty) d(y, Tx)}{d(x, Ty) + d(y, Tx)} ds
\]

(4.1)

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta + \alpha < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).

**Corollary 4.2.** Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing, continuous self mapping satisfying

\[
\int_0^{d(Tx, Ty)} ds \leq \varepsilon \int_0^{d(x, y)} ds + \delta \int_0^{d(x, Ty) + d(y, Tx)} ds + \gamma \frac{d(x, Ty) d(y, Tx)}{d(x, y)} ds + \beta \frac{d(x, Ty) d(y, Tx)}{d(y, Ty) + d(x, Ty) + d(y, Tx)} ds + \alpha \frac{d(x, Ty) d(y, Tx)}{d(x, Ty) + d(y, Tx)} ds
\]

(4.2)

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \beta < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).

**Corollary 4.3.** Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing, continuous self mapping satisfying

\[
\int_0^{d(Tx, Ty)} ds \leq \varepsilon \int_0^{d(x, y)} ds + \delta \int_0^{d(x, Ty) + d(y, Tx)} ds + \gamma \frac{d(x, Ty) d(y, Tx)}{d(x, y)} ds + \beta \frac{d(x, Ty) d(y, Tx)}{d(y, Ty) + d(x, Ty) + d(y, Tx)} ds + \alpha \frac{d(x, Ty) d(y, Tx)}{d(x, Ty) + d(y, Tx)} ds
\]

(4.3)

for all distinct \( x, y \in X \) with \( y \preceq x \), and \( \alpha, \beta, \gamma, \delta, \varepsilon \geq 0 \) with \( \varepsilon + 2\delta + \gamma + \alpha < 1 \). If there exists \( x_0 \in X \) with \( x_0 \preceq Tx_0 \), then \( T \) has a fixed point in \( X \).

**Corollary 4.4.** Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Assume that \( T: X \to X \) is a non-decreasing, continuous self mapping satisfying
\begin{equation}
\int_{0}^{d(Tx,Ty)} ds \leq \varepsilon \int_{0}^{d(x,y)} ds + \delta \int_{0}^{[d(x,y)+d(y,Tx)]} ds + \beta \int_{0}^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,Tx)+d(y,Tx)}} ds + \alpha \int_{0}^{\frac{d(x,Tx)d(y,Ty)}{d(y,Tx)+d(y,Tx)}} ds
\end{equation}

for all distinct \(x, y \in X\) with \(y \leq x\), and \(\alpha, \beta, \delta, \varepsilon \geq 0\) with \(\varepsilon + 2\delta + \beta + \alpha < 1\). If there exists \(x_0 \in X\) with \(x_0 \leq Tx_0\), then \(T\) has a fixed point in \(X\).

**Corollary 4.5.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Assume that \(T: X \to X\) is a non-decreasing, continuous self mapping satisfying

\begin{equation}
\int_{0}^{d(Tx,Ty)} ds \leq \delta \int_{0}^{[d(x,y)+d(y,Tx)]} ds + \gamma \int_{0}^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,Tx)+d(y,Tx)}} ds + \beta \int_{0}^{\frac{d(x,Tx)d(y,Ty)}{d(y,Tx)+d(y,Tx)}} ds + \alpha \int_{0}^{\frac{d(x,Tx)d(y,Ty)}{d(y,Tx)}} ds
\end{equation}

for all distinct \(x, y \in X\) with \(y \leq x\), and \(\alpha, \beta, \gamma, \delta \geq 0\) with \(2\delta + \gamma + \beta + \alpha < 1\). If there exists \(x_0 \in X\) with \(x_0 \leq Tx_0\), then \(T\) has a fixed point in \(X\).

5. **Remark:** If \(\gamma = 0, \beta = 0\) in Theorem 3.1, 3.3 and 3.4, then we obtain Theorem 31, 32 and 35 of [5].

6. **Conclusion**

In this paper, we considered rational type contraction mappings in metric spaces that are partially ordered. We demonstrated that such mappings have a unique fixed point, and we presented several results that support this conclusion. We also presented examples to highlight the improvements made over previous work on this topic.

**Conflict of Interest**

The authors declare that there are no conflict of interest.

**Funding**

The study received no financial support.

**Acknowledgment**

We would like to express our deepest gratitude to everyone who has contributed to this work in any way, whether directly or indirectly.

**Availability of Data and Materials**

Not applicable.

**References**


Raji M, Rajpoot AK, Hussain A, Rathour L, Mishra LN, Mishra VN. Results in fixed point theorems for relational-theoretic contraction mappings in metric spaces. Tujin Jishu. 2024; 45(1): 4356-68.


Raji M, Ibrahim MA. Results in cone metric spaces and related fixed point theorems for contractive type mappings with application. Qeios. 2024; 1-17. https://doi.org/10.32388/4ON167.2