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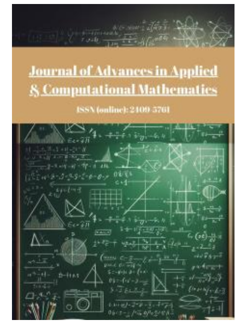


# Certain Fixed Point Results via Contraction Mappings in Neutrosophic Semi-Metric Spaces

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## ABSTRACT

In this work, the authors introduce the concept of neutrosophic semi-metric spaces and prove several common fixed-point theorems for countable and uncountable family of mappings via an implicit relation of contractive and integral type by utilizing locally integrable functions. These results improve and generalize the several results in the existing literature. Further, the authors present some non-trivial examples to support our main results.

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# 1. Introduction and Preliminaries

A metric space is a set-in mathematics and computer science that has a distance function (metric) that fulfils specific criteria, like symmetry, non-negativity, and the triangle inequality. Some of these properties are relaxed in a semi-metric space, usually the symmetry requirement. The distance between point $a$ and point $b$ may not equal the distance between point $b$ and point $a$ in a semi-metric space. In today's world, uncertainty and fuzziness are common in many applications. To capture the fuzziness and ambiguity of information, fuzzy sets (FSs) were first presented by Zadeh [1]. In 1975, Kramosil and Michalek [2] established fuzzy metric spaces (FMSs) by utilizing the concept of FSs. Then, George and Veeramani [3] modified the notion of FMSs by defining a Hausdorff topology on a FMSs presented by [2]. Grabiec [4] defined the notions of convergence, Cauchyness and completeness on FMSs. Further, he proved Banach and Edelstein contraction principal versions in the framework of FMSs. Hu [5] produced some excellent work for various contraction types in FMSs. Deng [6] used the idea of point wise R-weak commutativity to obtained common FP theorems in fuzzy pseudo-metric spaces for a pair of self-maps without making any assumptions about the completeness of space or the continuity of the underlying mappings. Cho *et al.* [7] provided definitions of compatible mappings of types (I) and (II) in FMSs, and proved some common FP theorems for four mappings under the assumption of compatible mappings of types (I) and (II) in complete FMSs. Javed *et al.* [8] introduced $\mathfrak{R}$ -controlled FMSs and proved several fixed point results.

An IFS was established by Atanassove [9] as a generalization of FSs. In 2004, Park [10] developed the concept of intuitionistic fuzzy metric space (IFMS) using continuous t-norms (CTNs) and continuous t-conorms (CTCNs). Alaca *et al.* [11] proposed IFMS with CTNs and CTCNs and demonstrated a number of FP theorems for contraction mappings. Sharma *et al.* [12] proved several FP results for weakly compatible mappings in IFMSs structure. Davvaz *et al.* [13] used IFSs and accomplished a nice work. Kumar *et al*. [14] developed the ideas of weak compatibility and the E. A. property for mixed g-monotone mappings in the context of IFMS, and then used these ideas to derived a coupled FP theorem for these nonlinear contractive mappings. Saadati and Park [15] defined pre compact sets and demonstrated that any subset is only compact if and only if it is pre-compact and complete in IFMSs. In addition, he established the definition of topologically complete fuzzy metrizable spaces that are intuitionistic and demonstrated that any $G\delta$ set in a complete IFMSs is an intuitionistic fuzzy metrizable space that is topologically complete, and vice versa. Furthermore, to relaxing the symmetric condition, Wilson *et al.* [16] on a set $X$ with the function $F$ and without using the triangle inequality or relaxing the symmetric condition, he obtained some common FP theorems for the recently introduced concept of PD-operators.

The notion of neutrosophic sets established by Smarandache [17] as a generalization of IFSs. Based on the idea of NSs, Kirişci and Simşek [18] presented the notion of neutrosophic metric spaces (NMSs) and provided topological structure of NMSs. Simsek and Kirişci [19] proved a number of FP theorems using the idea of NMS. The concept of orthogonal NMSs was coined by Ishtiaq *et al.* [20] and demonstrated some FP results in the context of complete orthogonal NMSs. Sowndrara *et al.* [21] derived several FP results for generalized contractions in NMSs. Uddin *et al.* [22] developed the notion of controlled neutrosophic metric-like spaces, which generalized the concept of NMSs and provided several fixed point results. Ali *et al.* [23] proved several new FP results for weakly compatible and contractive mappings in the context of NMSs.

Aliouche [24] established a common FP theorem for weakly compatible mappings in symmetric spaces that satisfied an integral contractive condition and property (E.A). Merghadi and Godet-Thobie [25] provided common FP results for any family of maps that is not necessarily countable in the context of IFMSs. Sastry and Murthy [26] established a common FP of two partially commuting tangential self-maps on a metric space that extended to symmetric spaces. Aamri and El Moutawakil [27] used the idea of T-weakly and S-weakly commuting mappings satisfying generalized contractive conditions to prove common FP theorems in symmetric spaces for two pairs of hybrid mappings. Al-Thagafi, and Shahzad [28] introduced a new class of self-maps that fulfill the (E.A.) property with respect to some $q \in M$, where $M$ is a q-star-shaped subset of a convex metric space, and established common FP results for this new class of self-maps. Using the concept of a pair of mappings satisfying property (E.A), it was discussed a general common FP theorem of integral $\Phi$- type for two pairs of weakly compatible mappings satisfying specific integral type implicit relations in symmetric spaces by Pathak *et al*. [29]. The idea of weak compatibility was used to demonstrate a common FP theorem of Gregus type for four mappings fulfilled an integral type contractive condition established by Djodi and Aliouche [30]. Alioche and Popa [31] provided the idea of occasionally weakly compatible mappings was used to prove common FP theorems for four mappings that satisfy implicit relations in symmetric spaces. Godet-Thobie and Merghadi [32] proved common fixed point results in the context of intuitionistic fuzzy semi-metric spaces (IFSMSs). Authors in [33-43] worked on different applications of fixed point theory. Younas *et al*. [44-54] worked on different applications by using fixed point results including the vibrations of a vertical heavy hanging cable, damped spring-mass system and deformation of an elastic beam. Authors in [55-63] presented several fixed point and best proximity point results in a number of generalized spaces. Some interesting fixed point results presented in the setting of Gauge spaces, Hausdorff Gauge spaces and some other generalized spaces by [64-69]. By using dislocated b-metric spaces, FMSs and generalized metric spaces, authors [70-77] proved fixed point theorems on closed ball, by using multivalued mappings and single valued mappings. Authors in [78-84] presented fixed point results for interpolative contractions, multivalued contractions and single valued contraction mappings in the settings of dislocated metric spaces and generalized FMSs. Authors in [85-92] provided several fixed point results via probabilistic type contractions, Suzuki type contractions and non-linear contractions in the framework of probabilistic metric spaces and generalized metric spaces. Several properties of k-FMSs, FMSs are discussed in [93-96] with some applications. In [97-100] authors used C*-algebra valued metric spaces and some other spaces to find out the fixed point via different single valued contraction mappings. Authors in [101-109] presented several applications of fixed point theory.

In this paper, the authors defined the concept of neutrosophic semi-metric spaces (NSMSs) as a generalization of IFSMSs and fuzzy semi-metric spaces (FSMSs). In FSMSs, just the membership function is employed, whereas in IFSMSs, both membership and non-membership functions are used. The authors used membership, non-membership, and neutral functions in NSMSs. Several common fixed-point theorems for countable and uncountable families of mappings employing contractive and integral type implicit relations are presented in this new framework, along with non-trivial examples.

**Definition 1.1:** [10] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a CTN if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

**Definition 1.2:** [10] A binary operation $\lozenge: [0,1] \times [0,1] \rightarrow [0,1]$ is a CTCN if it satisfies the following conditions:

(1) $\lozenge$ is associative and commutative,

(2) $\lozenge$ is continuous,

(3) $a \lozenge 0 = a$ for all $a \in [0,1]$,

(4) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

**Definition 1.3:** [18] A 6-tuple $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ is a NMS if $\Omega$ is an arbitrary (non-empty) set, $*$ is a CTN $\lozenge$ a CTCN, $\mathfrak{B}, \mathfrak{D}$ and $\mathfrak{L}$ are NSs on $\Omega^2 \times ]0, +\infty[$, fulfill the following conditions for each $\zeta, \varsigma, \aleph \in \Omega$ and $\wedge, \tau > 0$,

(NMS1) $\mathfrak{B}(\zeta, \varsigma, \wedge) + \mathfrak{D}(\zeta, \varsigma, \wedge) + \mathfrak{L}(\zeta, \varsigma, \wedge) \leq 3$,

(NMS2) $\mathfrak{B}(\zeta, \varsigma, \wedge) > 0$,

(NMS3) $\mathfrak{B}(\zeta, \varsigma, \wedge) = 1$ if and only if $\zeta = \varsigma$,

(NMS4) $\mathfrak{B}(\zeta, \varsigma, \wedge) = \mathfrak{B}(\varsigma, \zeta, \wedge)$, (NMS5) $\mathfrak{B}(\zeta,\varsigma,\lambda) * \mathfrak{B}(\varsigma,\aleph,\tau) \leq \mathfrak{B}(\zeta,\aleph,\lambda+\tau)$,

(NMS6) $\mathfrak{B}(\zeta,\varsigma,\cdot)$: $]0,+\infty] \to ]0.1]$ is continuous,

(NMS7) $\mathfrak{D}(\zeta,\varsigma,\lambda) = 0$ if and only if $\zeta = \varsigma$,

(NMS8) $\mathfrak{D}(\zeta,\varsigma,\lambda) = \mathfrak{D}(\zeta,\varsigma,\lambda)$,

(NMS9) $\mathfrak{D}(\zeta,\varsigma,\lambda) \lozenge \mathfrak{D}(\varsigma,\aleph,\lambda) \geq \mathfrak{D}(\zeta,\aleph,\lambda+\tau)$,

(NMS10) $\mathfrak{D}(\zeta,\varsigma,\cdot)$ : $]0,+\infty[\to ]0,1]$ is continuous,

(NMS11) $\mathfrak{L}(\zeta,\varsigma,\lambda) = 0$ if and only if $\zeta = \varsigma$,

(NMS12) $\mathfrak{L}(\zeta,\varsigma,\lambda) = \mathfrak{L}(\zeta,\varsigma,\lambda)$,

(NMS13) $\mathfrak{L}(\zeta,\varsigma,\lambda) \lozenge \mathfrak{L}(\varsigma,\aleph,\lambda) \geq \mathfrak{L}(\zeta,\aleph,\lambda+\tau)$,

(NMS14) $\mathfrak{L}(\zeta,\varsigma,\cdot)$ : $]0,+\infty[\to ]0,1]$ is continuous.

Then, $(\mathfrak{B},\mathfrak{D},\mathfrak{L})$ is called a neutrosophic metric on $\Omega$.

**Definition 1.4:** [32] A 6-tuple $(\Omega,\mathfrak{B},\mathfrak{D},*,\lozenge)$ is an IFSMS if $\Omega$ is an arbitrary (non-empty) set, $*$ is a CTN, $\lozenge$ a CTCN, $\mathfrak{B}$ and $\mathfrak{D}$ are FSs on $\Omega^2 \times ]0,+\infty[$, satisfying the following conditions for each $\zeta,\varsigma,\aleph \in \Omega$ and $\lambda,\tau > 0$,

(IFSM1) $\mathfrak{B}(\zeta,\varsigma,\lambda) + \mathfrak{D}(\zeta,\varsigma,\lambda) \leq 1$,

(IFSM2) $\mathfrak{B}(\zeta,\varsigma,\lambda) > 0$,

(IFSM3) $\mathfrak{B}(\zeta,\varsigma,\lambda) = 1$ if and only if $\zeta = \varsigma$,

(IFSM4) $\mathfrak{B}(\zeta,\varsigma,\lambda) = \mathfrak{B}(\varsigma,\zeta,\lambda)$,

(IFSM5) $\mathfrak{B}(\zeta,\varsigma,\cdot)$: $]0,+\infty[\to ]0,1]$ is continuous,

(IFSM6) $\mathfrak{D}(\zeta,\varsigma,\lambda) = 0$ if and only if $\zeta = \varsigma$,

(IFSM7) $\mathfrak{D}(\zeta,\varsigma,\lambda) = \mathfrak{D}(\varsigma,\zeta,\lambda)$,

(IFSM8) $\mathfrak{D}(\zeta,\varsigma,\cdot)$: $]0,+\infty[\to ]0,1]$ is continuous.

Then, $(\mathfrak{B},\mathfrak{D})$ is called an intuitionistic fuzzy semi metric on $\Omega$.

# 2. Neutrosophic Semi-Metric Spaces

In this section, we define the notion of NSMSs. We study some topological properties and then we extend, some classical properties which are usually used in generalized metric spaces to prove common FP theorems.

**Definition 2.1:** A 6-tuple $(\Omega,\mathfrak{B},\mathfrak{D},\mathfrak{L},*,\lozenge)$, is a NSMS if $*$ is a CTN, $\lozenge$ a CTCN, $\mathfrak{B},\mathfrak{D}$ and $\mathfrak{L}$ are NSs on $\Omega^2 \times ]0,+\infty[$, satisfying the following conditions for each $\zeta,\varsigma,\aleph \in \Omega$ and $\lambda,\tau > 0$,

(NSMS1) $\mathfrak{B}(\zeta,\varsigma,\lambda) + \mathfrak{D}(\zeta,\varsigma,\lambda) + \mathfrak{L}(\zeta,\varsigma,\lambda) \leq 3$,

(NSMS2) $\mathfrak{B}(\zeta,\varsigma,\lambda) > 0$,

(NSMS3) $\mathfrak{B}(\zeta,\varsigma,\lambda) = 1$ if and only if $\zeta = \varsigma$,

(NSMS4) $\mathfrak{B}(\zeta,\varsigma,\lambda) = \mathfrak{B}(\varsigma,\zeta,\lambda)$,

(NSMS5) $\mathfrak{B}(\zeta,\varsigma,\cdot)$: $]0,+\infty[\to ]0,1]$ is continuous,

(NSMS6) $\mathfrak{D}(\zeta,\varsigma,\lambda) = 0$ if and only if $\zeta = \varsigma$,

(NSMS7) $\mathfrak{D}(\zeta,\varsigma,\lambda) = \mathfrak{D}(\varsigma,\zeta,\lambda)$, (NSMS8) $\mathfrak{D}(\zeta, \varsigma, \cdot)$: $]0, +\infty[ \to ]0, 1]$ is continuous.

(NSMS9) $\mathfrak{L}(\zeta, \varsigma, \curlywedge) = 0$ if and only if $\zeta = \varsigma$,

(NSMS10) $\mathfrak{L}(\zeta, \varsigma, \curlywedge) = \mathfrak{L}(\varsigma, \zeta, \curlywedge)$,

(NSMS11) $\mathfrak{L}(\zeta, \varsigma, \cdot)$: $]0, +\infty[ \to ]0, 1]$ is continuous.

Then, $(\mathfrak{B}, \mathfrak{D}, \mathfrak{L})$ is called a neutrosophic semi-metric (NSM) on $\Omega$.

**Example 2.1:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, *, \lozenge)$ be as following: $\Omega = [0, 1]$, $a * b = ab$, $a \lozenge b = \max\{a, b\}$,

$$\mathfrak{B}(\zeta, \varsigma, \curlywedge) = \begin{cases} 1 & \text{if } \zeta = \varsigma, \\ \max\left\{\frac{\zeta}{2}, \frac{\varsigma}{2}\right\} & \text{if } \zeta \neq \varsigma, \end{cases}$$

$$\mathfrak{D}(\zeta, \varsigma, \curlywedge) = \begin{cases} 0 & \text{if } \zeta \neq \varsigma, \\ \left|\frac{\zeta}{2} - \frac{\varsigma}{2}\right| & \text{if } \zeta \neq \varsigma \end{cases}$$

and

$$\mathfrak{L}(\zeta, \varsigma, \curlywedge) = \begin{cases} 0 & \text{if } \zeta \neq \varsigma, \\ \min\left\{\frac{\zeta}{2}, \frac{\varsigma}{2}\right\} & \text{if } \zeta \neq \varsigma. \end{cases}$$

Then, $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ is a NSMS.

We can easily check that all conditions of Definition 2.1 are satisfied and that conditions (NMS5) and (NMS9) of Definition 1.3 are not satisfied. So $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ is a NSMS.

**Example 2.2:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, *, \lozenge)$ be as following: $\Omega = [0, 1]$, $a * b = ab, a \lozenge b = \min\{1, a + b\}$,

$$\mathfrak{B}(\zeta, \varsigma, \curlywedge) = \begin{cases} 1 & \text{if } \zeta = \varsigma, \\ \max\left\{\frac{\zeta}{2}, \frac{\varsigma}{2}\right\} & \text{if } \zeta \neq \varsigma, \end{cases}$$

$$\mathfrak{D}(\zeta, \varsigma, \curlywedge) = \begin{cases} 0 & \text{if } \zeta = \varsigma, \\ \min\left\{\frac{\zeta}{2}, \frac{\varsigma}{2}\right\} & \text{if } \zeta \neq \varsigma, \end{cases}$$

and

$$\mathfrak{L}(\zeta, \varsigma, \curlywedge) = \begin{cases} 0 & \text{if } \zeta = \varsigma, \\ \left|\frac{\zeta}{2} - \frac{\varsigma}{2}\right| & \text{if } \zeta \neq \varsigma. \end{cases}$$

Then, $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ is a NSMS.

Following George and Veeramani [3], we can define a topology on $\Omega$ by the family of open sets as follows.

**Definition 2.2:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ ,be a NSMS. For $\curlywedge > 0$, the open ball $B(\zeta, r, \curlywedge)$ with center $\zeta \in \Omega$ and radius $0 < r < 1$ is defined by

$$B(\zeta, r, \curlywedge) = \{\varsigma \in \Omega: \mathfrak{B}(\zeta, \varsigma, \curlywedge) > 1 - r, \quad \mathfrak{D}(\zeta, \varsigma, \curlywedge) < r \text{ and } \mathfrak{L}(\zeta, \varsigma, \curlywedge) < r\}$$

A subset $A \subset \Omega$ is called an open set if, for each $\zeta \in A$, there exist $\curlywedge > 0$ and $0 < r < 1$ such that $B(\zeta, r, \curlywedge) \subset A$. Let $\mathcal{T}(m, n, o)$ denote the family of all open subsets of $\Omega$. Then $\mathcal{T}(m, n, o)$ is called the topology on $\Omega$ induced by the NSM $(\mathfrak{B}, \mathfrak{D}, \mathfrak{L})$. It is easy to show that $\mathcal{T}(m, n, o)$ is a topology on $\Omega$. This topology is generally not Hausdorff in context to NSMSs.

**Definition 2.3:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond)$ be a NSMS.

(I) A sequence $(\zeta_n)$ in $\Omega$ converges to $\zeta$ if and only if $\mathfrak{B}(\zeta_{n,}, \zeta, \wedge) \to 1$ , $\mathfrak{D}(\zeta_n, \zeta, \wedge) \to 0$ and $\mathfrak{L}(\zeta_n, \zeta, \wedge) \to 0$ as $n \to \infty$, for each $\wedge > 0$.

(II) A sequence $(\zeta_n)$ in $\Omega$ is called a Cauchy sequence if, for every $0 < \varepsilon < 1$ and every $\wedge > 0$, there exists $n_0 \in \mathfrak{D}$ such that, for all $n \geq n_0$ and $m \geq n_0, \mathfrak{B}(\zeta_n, \zeta_m, \wedge) > 1 - \varepsilon$, $\mathfrak{D}(\zeta_n, \zeta_m, \wedge) < \varepsilon$ and $\mathfrak{L}(\zeta_n, \zeta_m, \wedge) < \varepsilon$.

**Proposition 2.1:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, * ,\Diamond )$ be a NSMS and $\mathcal{T}(m, n, o)$ is topology on $\Omega$ induced by the NSM $(\mathfrak{B}, \mathfrak{D}, \mathfrak{L})$. Then the convergence in the topological space $(\Omega, \mathcal{T}_{(m,n,o)})$ coincides with that one of Definition 2.3.

**Proof:** The proof similar to that one of Theorem 3.11 (Park [10]) of given in the context of NSMS. If, according to Definition 2.3

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \zeta, \wedge) = 0,$$

for every $\wedge > 0$, then for $r \in [0,1]$ there exist $n_0$ such that

$$\mathfrak{B}(\zeta_n, \zeta, \wedge) > 1 - r,\ \mathfrak{D}(\zeta_n, \zeta, \wedge) < r \text{ and } (\zeta_n, \zeta, \wedge) < r\ ,$$

for all $n \geq n_0$. So, $\zeta_n \in B(\zeta, r, \wedge)$ for all $n \geq n_0$, that is, $(\zeta_n)$ converges to $\zeta$ in $(\Omega, \mathcal{T}_{(\mathfrak{B},\mathfrak{D},\mathfrak{L})})$. Now let

$\wedge > 0$ and $\zeta_n \to \zeta$. For $r \in ]0,1[$ there exist $n_0$ such that for every $n \geq n_0, \zeta_n \in B(\zeta, r, \wedge)$. That is

$\mathfrak{B}(\zeta_n, \zeta, \wedge) > 1 - r$ and $\mathfrak{D}(\zeta_n, \zeta, \wedge) < r$ and $\mathfrak{L}(\zeta_n, \zeta, \wedge) < r$ for all $n \geq n_0$ and

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \zeta, \wedge) = 0.$$

**Proposition 2.2:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond )$ be a NSMS. The topology $\mathcal{T}_{(m,n,o)}$ on $\Omega$ is independent of $\mathfrak{D}$ and identical to that one of the fuzzy semi-metric spaces (FSMS) $(\Omega, \mathfrak{B}, *)$.

**Proof**: It is sufficient to prove that open balls of two topologies are same.

Let

$$B_{(\mathfrak{B},\mathfrak{D})}(\zeta, r, \wedge) = \{\varsigma \in \Omega : \mathfrak{B}(\zeta, \varsigma, \wedge) > 1 - r\ \}, \mathfrak{D}(\zeta, \varsigma, \wedge) < r \text{ and } \mathfrak{L}(\zeta, \varsigma, \wedge) < r\ \},$$

the ball in the NSMS and

$$B(\zeta, r, \wedge) = \{\varsigma \in \Omega : \mathfrak{B}(\zeta, \varsigma, \wedge) > 1 - r\},$$

that one in the FSMS $(\Omega, \mathfrak{B}, *)$. It is clear that $B_{(\mathfrak{B},\mathfrak{D},\mathfrak{L})}(\zeta, r, \wedge) \subset B(\zeta, r, \wedge)$. For opposite inclusion, if

$$\varsigma \in B(\zeta, \varsigma, \wedge), \mathfrak{B}(\zeta, \varsigma, \wedge) > 1 - r,$$

and from (NSMS1) of Definition 2.1,

$$\mathfrak{D}(\zeta, \varsigma, \wedge) < r, \ \mathfrak{L}(\zeta, \varsigma, \wedge) < r \text{ and } \varsigma \in B_{(\mathfrak{B},\mathfrak{D},\mathfrak{L})}(\zeta, r, \wedge).$$

# 3. Some Properties of the NMS and NSMS

In the first part, we show that every NSMS satisfies classical properties which are usually used in generalized metric spaces to prove common FP theorems. They are properties (W4), (W3) in Wilson [16], (H.E) in (Aamri and El Moutawakil, [27]), (CE.1) and (CE.2) in (Pathak *et al.* [29]). In the second part, we extend in the context of NSMSs, some properties of compatibility.

Now, we recall their definitions. The two next properties were introduced by Wilson (Wilson [16]). We give an extension of them which is adapted to NSMSs.

**Definition 3.1:** A NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ satisfies the property (W3) if and only if, given $(\zeta_n)_{n\in\mathfrak{D}}$, $\zeta$ and $\varsigma$ in $\Omega$. If for each $\wedge > 0$, on the one hand, $\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1$ and $\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \varsigma, \wedge) = 1$ and on the other hand $\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0$ and $\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \varsigma, \wedge) = 0$, $\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0$ and $\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0$ then $\zeta = \varsigma$.

**Definition 3.2:** A NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ satisfies the property (W4) if and only if given $\zeta$, $(\zeta_n)_{n\in\mathfrak{D}}$, and $(\varsigma_n)_{n\in\mathfrak{D}}$ in $\Omega$, if

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \varsigma_n, \wedge) = 1,$$

for each $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \varsigma_n, \wedge) = 0.$$

Similarly,

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0.$$

For each $\wedge > 0$, we have

$$\lim_{n\to\infty} \mathfrak{B}(\varsigma_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \zeta, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0.$$

It is clear that $(W4)$ implies $(W3)$.

The three following definitions were introduced in (Aamri & El Moutawakil [27]) in the context of symmetric spaces.

**Definition 3.3:** A NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ satisfies the property $(\mathrm{H.E})$ if and only if given $(\zeta_n)_{n\in\mathfrak{D}}$ ,$(\varsigma_n)_{n\in\mathfrak{D}}$ and $\zeta$ in $\Omega$, if,

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{B}(\varsigma_n, \zeta, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{D}(\varsigma_n, \zeta, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \zeta, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(\varsigma_n, \zeta, \wedge) = 0.$$

For every $\wedge > 0$, then for every $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \varsigma_n, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}\big(\zeta_n, \varsigma_\zeta, \wedge\big) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = 0.$$

**Definition 3.4:** A NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$,satisfies the property (CE.1) if and only if given, $\zeta$ and $\varsigma$ in $\Omega$ for each $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \zeta, \wedge) = 1,$$

implies that

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \varsigma, \wedge) = \mathfrak{B}(\zeta, \varsigma, \wedge),$$

and

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \zeta, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \varsigma, \wedge) = \mathfrak{D}(\zeta, \varsigma, \wedge),$$

similarly,

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \zeta, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma, \wedge) = \mathfrak{L}(\zeta, \varsigma, \wedge).$$

**Definition 3.5:** A NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ satisfies the property (CE.2) if and only if given$(\zeta_n)_{n\in\mathfrak{D}}$,$(\varsigma_n)_{n\in\mathfrak{D}}$ and $(\aleph_n)_{n\in\mathfrak{D}}$ in $\Omega$ if, for each $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{B}(\zeta_n, \varsigma_n, \wedge) = 1,$$

Implies

$$\lim_{n\to\infty} \inf \mathfrak{V}(\aleph_n, \varsigma_n, \wedge) = \lim_{n\to\infty} \inf \mathfrak{V}(\aleph_n, \zeta_n, \wedge).$$

and

$$\lim_{n\to\infty} \mathfrak{D}(\zeta_n, \varsigma_n, \wedge) = 0,$$

implies

$$\lim_{n\to\infty} \sup \mathfrak{D}(\aleph_n, \varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{D}(\aleph_n, \zeta_n, \wedge).$$

In the same manner, we have

$$\lim_{n\to\infty} \mathfrak{L}(\zeta_n, \varsigma_n, \wedge) = 0,$$

implies

$$\lim_{n\to\infty} \sup \mathfrak{L}(\aleph_n, \varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{L}(\aleph_n, \zeta_n, \wedge).$$

For NMS and NSMS, we give an extension of property (E.A) which was defined in (Sastry and Murphy [26]) as the Tangential Property and in (Aamri and El Moutawakil [27]) as Property (E.A).

**Definition 3.6:** Let $S, T: \Omega \to Y$. The pair $(S, T)$ satisfies property (E.A) if there exist sequence $(\zeta_n)$ in $\Omega$ such that

$$\lim_{n\to\infty} S\zeta_n = \lim_{\zeta\to\infty} T\zeta_n = u \in \Omega,$$

that is for every $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{V}(S\zeta_n, u, \wedge) = \lim_{n\to\infty} \mathfrak{V}(T\zeta_n, u, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(S\zeta_n, u, \wedge) = \lim_{n\to\infty} \mathfrak{D}(T\zeta_n, u, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(S\zeta_n, u, \wedge) = \lim_{n\to\infty} \mathfrak{L}(T\zeta_n, u, \wedge) = 0.$$

**Proposition 3.7:** Every NSMS satisfies (W4), (W3), (H.E), (CE.1) and (CE.2).

**Proof**: It is sufficient to prove (W4), (H.E), (CE.1) and (CE.2). Since (W4) implies (W3). We remark that (W3) is satisfied because the topology $T_{(\mathfrak{V},\mathfrak{D})}$ of every NSMS is Hausdorff (see Theorem 3.5 of (Park [10]). Let $(\Omega, \mathfrak{V}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ be a NSMS. To show that (W4) and $(\mathrm{H.E})$ are satisfied, we use properties (5) and (9) of Definition 1.4 as following:

(I) Since $\mathfrak{V}(\zeta_n, \zeta, u) \to 1$ and $\mathfrak{V}(\zeta_n, \varsigma_n, u) \to 1$ for each $u$, by (2) and (3) of Definition 1.1,

$$\mathfrak{V}\left(\zeta_n, \zeta, \frac{\wedge}{2}\right) * \mathfrak{V}\left(\zeta_n, \varsigma_n, \frac{\wedge}{2}\right) \leq \mathfrak{V}(\zeta, \varsigma_n, \wedge).$$

We obtain that $\mathfrak{V}(\varsigma_n, \zeta, \wedge) \to 1$, for every $\wedge \in ]0, \infty[$. By similar manner, utilizing (2) and (3) of Definition 1.2, we obtain $\mathfrak{D}(\varsigma_n, \zeta, \wedge) \to 0$ and $\mathfrak{L}(\varsigma_n, \zeta, \wedge) \to 0$. Then (W4) is satisfied.

(II) If we suppose that, for every $u$, $\mathfrak{V}(\zeta_n, \zeta, u) \to 1$ and $\mathfrak{V}(\varsigma_n, \zeta, u) \to 1$ from the inequality

$$\mathfrak{V}\left(\zeta_n, \zeta, \frac{\wedge}{2}\right) * \mathfrak{V}\left(\zeta, \varsigma_n, \frac{\wedge}{2}\right) \leq \mathfrak{V}(\zeta_n, \varsigma_n, \wedge),$$

we obtain $\mathfrak{V}(\zeta_n, \varsigma_n, \wedge) \to 1$. For every $u$, $\mathfrak{D}(\zeta_n, \zeta, u) \to 0$ and $\mathfrak{D}(\varsigma_n, \zeta, u) \to 0$ from the inequality

$$\mathfrak{D}\left(\zeta_n,\zeta,\frac{\Lambda}{2}\right) \lozenge \mathfrak{D}\left(\zeta,\varsigma_n,\frac{\Lambda}{2}\right) \le \mathfrak{D}(\zeta_n,\varsigma_n,\Lambda),$$

we obtain $\mathfrak{D}(\zeta_n,\varsigma_n,\Lambda) \to 0$. Similarly, for every $u$, $\mathfrak{L}(\zeta_n,\zeta,u) \to 0$ and $\mathfrak{L}(\varsigma_n,\zeta,u) \to 0$ from the inequality

$$\mathfrak{L}\left(\zeta_n,\zeta,\frac{\Lambda}{2}\right) \lozenge \mathfrak{L}\left(\zeta,\varsigma_n,\frac{\Lambda}{2}\right) \le \mathfrak{L}(\zeta_n,\varsigma_n,\Lambda),$$

we obtain $\mathfrak{L}(\zeta_n,\varsigma_n,\Lambda) \to 0$. So, (H.E) is satisfied.

(III) Now we suppose $\mathfrak{V}(\zeta_n,\zeta,u) \to 1$ for each $u \in ]0,\infty[$. For each $\Lambda \in ]0,\infty[$ and every $0 < \varepsilon < \Lambda$, from $\mathfrak{V}(\zeta_n,\varsigma,\Lambda) \ge \mathfrak{V}(\zeta_n,\zeta,\varepsilon) * \mathfrak{V}(\zeta,\varsigma,\Lambda-\varepsilon)$, we obtain

$\lim_{n\to\infty} \inf \mathfrak{V}(\zeta_n,\varsigma,\Lambda) \ge \mathfrak{V}(\zeta,\varsigma,\Lambda-\varepsilon)$, and by property (6) of Definition 2.1,

$$\lim_{n\to\infty} \inf \mathfrak{V}(\zeta_n,\varsigma,\Lambda) \ge \mathfrak{V}(\zeta,\varsigma,\Lambda). \qquad (1)$$

From

$$\mathfrak{V}(\zeta,\varsigma,\Lambda+\varepsilon) \ge \mathfrak{V}(\zeta_n,\zeta,\varepsilon) * \mathfrak{V}(\zeta_n,\varsigma,\Lambda),$$

for every $\Lambda > 0$ and $\varepsilon > 0$, we obtain

$$\mathfrak{V}(\zeta,\varsigma,\Lambda+\varepsilon) \ge \lim_{n\to\infty} \sup \mathfrak{V}(\zeta_n,\varsigma,\Lambda),$$

and

$$\mathfrak{V}(\zeta,\varsigma,\Lambda) \ge \lim_{n\to\infty} \mathrm{Sup}\, \mathfrak{V}(\zeta_n,\varsigma,\Lambda). \qquad (2)$$

By using (1) and (2), we have

$$\mathfrak{V}(\zeta,\varsigma,\Lambda) = \lim_{n\to\infty} \mathrm{Sup}\, \mathfrak{V}(\zeta_n,\varsigma,\Lambda),$$

and suppose $\mathfrak{D}(\zeta_n,\zeta,u) \to 0$ for each $u \in ]0,\infty[$. For each $\Lambda \in ]0,\infty[$ and every $0 < \varepsilon < \Lambda$, from

$$\mathfrak{D}(\zeta_n,\varsigma,\Lambda) \le \mathfrak{D}(\zeta_n,\zeta,\varepsilon) \lozenge \mathfrak{D}(\zeta,\varsigma,\Lambda-\varepsilon),$$

we obtain,

$$\lim_{n\to\infty} \sup \mathfrak{D}(\zeta_n,\varsigma,\Lambda) \le \mathfrak{D}(\zeta,\varsigma,\Lambda-\varepsilon),$$

and by property (6) of Definition 2.1, we have

$$\lim_{n\to\infty} \sup \mathfrak{D}(\zeta_n,\varsigma,\Lambda) \le \mathfrak{D}(\zeta,\varsigma,\Lambda). \qquad (3)$$

From

$$\mathfrak{D}(\zeta,\varsigma,\Lambda+\varepsilon) \le \mathfrak{D}(\zeta_n,\zeta,\varepsilon) \lozenge \mathfrak{D}(\zeta_n,\varsigma,\Lambda),$$

for every $\Lambda > 0$ and $\varepsilon > 0$, we obtain

$$\mathfrak{D}(\zeta,\varsigma,\Lambda+\varepsilon) \le \lim_{n\to\infty} \inf \mathfrak{D}(\zeta_n,\varsigma,\Lambda),$$

and

$$\mathfrak{D}(\zeta,\varsigma,\Lambda) \le \lim_{n\to\infty} \inf \mathfrak{D}(\zeta_n,\varsigma,\Lambda). \qquad (4)$$

By applying (3) and (4), we have

$$\mathfrak{D}(\zeta, \varsigma, \Lambda) = \lim_{n\to\infty} \inf \mathfrak{D}(\zeta_n, \varsigma, \Lambda).$$

Similarly, suppose $\mathfrak{L}(\zeta_n, \zeta, u) \to 0$ for each $u \in ]0, \infty[$. For each $\Lambda \in ]0, \infty[$ and every $0 < \varepsilon < \Lambda$, from

$$\mathfrak{L}(\zeta_n, \varsigma, \Lambda) \leq \mathfrak{L}(\zeta_n, \zeta, \varepsilon) \lozenge \mathfrak{L}(\zeta, \varsigma, \Lambda - \varepsilon),$$

we obtain

$$\lim_{n\to\infty} \sup \mathfrak{L}(\zeta_n, \varsigma, \Lambda) \leq \mathfrak{L}(\zeta, \varsigma, \Lambda - \varepsilon),$$

and by property (6) of Definition 2.1,

$$\lim_{n\to\infty} \sup \mathfrak{L}(\zeta_n, \varsigma, \Lambda) \leq \mathfrak{L}(\zeta, \varsigma, \Lambda). \tag{5}$$

From

$$\mathfrak{L}(\zeta, \varsigma, \Lambda + \varepsilon) \leq \mathfrak{L}(\zeta_n, \zeta, \varepsilon) \lozenge \mathfrak{L}(\zeta_n, \varsigma, \Lambda),$$

for every $\Lambda > 0$ and $\varepsilon > 0$, we obtain

$$\mathfrak{L}(\zeta, \varsigma, \Lambda + \varepsilon) \leq \lim_{n\to\infty} \inf \mathfrak{L}(\zeta_n, \varsigma, \Lambda),$$

and

$$\mathfrak{L}(\zeta, \varsigma, \Lambda) \leq \lim_{n\to\infty} \inf \mathfrak{L}(\zeta_n, \varsigma, \Lambda). \tag{6}$$

By utilizing (5) and (6), we have

$$\mathfrak{L}(\zeta, \varsigma, \Lambda) = \lim_{n\to\infty} \inf \mathfrak{L}(\zeta_n, \varsigma, \Lambda).$$

Hence, property (CE.1) is satisfied.

(IV) If we suppose $\mathfrak{B}(\zeta_n, \varsigma_n, u) \to 1$ for each $u > 0$, by (6) of Definition 1.4, (2) of Definition 1.1, and from

$$\mathfrak{B}(\aleph_n, \varsigma_n, \Lambda + \varepsilon) \geq \mathfrak{B}(\aleph_n, \zeta_n, \Lambda) * \mathfrak{B}(\zeta_n, \varsigma_n, \varepsilon),$$

we obtain for every $\Lambda$ and $\varepsilon$,

$$\lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \varsigma_n, \Lambda + \varepsilon) \geq \lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \zeta_n, \Lambda),$$

then,

$$\lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \varsigma_n, \Lambda) \geq \lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \zeta_n, \Lambda),$$

for every $\Lambda$. We get

$$\mathfrak{B}(\aleph_n, \zeta_n, \Lambda + \varepsilon) \geq \mathfrak{B}(\aleph_n, \varsigma_n, \Lambda) * \mathfrak{B}(\zeta_n, \varsigma_n, \varepsilon),$$

then,

$$\lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \varsigma_n, \Lambda) = \lim_{n\to\infty} \inf \mathfrak{B}(\aleph_n, \zeta_n, \Lambda).$$

In the same manner, we obtain

$$\lim_{n\to\infty} \sup \mathfrak{D}(\aleph_n, \varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{D}(\aleph_n, \zeta_n, \wedge),$$

and

$$\lim_{n\to\infty} \sup \mathfrak{L}(\aleph_n, \varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{L}(\aleph_n, \zeta_n, \wedge).$$

Hence, Property $(\mathrm{CE}.2)$ is satisfied.

We extend some classical compatibility properties to the context of NSMSs. The study of common FP theorems calls for various compatibility properties. Recently, several authors (Aliouche *et al.* [24]; Cho [7]; Grabiec [4]; Kumar *et al.* [14]; Sharma and Deshpande [12]) proved common FP theorems by extending them to FMS. For the reader's convenience, we recall the following definitions.

Two mappings $f$ and $g$ of a FMS $(\Omega, \mathfrak{B}, *)$ into itself are said to be

(I) **Weekly commuting:** if $\mathfrak{B}(fg\zeta, gf\zeta, \wedge) \geq \mathfrak{B}(f\zeta, g\zeta, \wedge), \forall\, \zeta \in \Omega$.

(II) **Compatible:** if for each $\wedge > 0$,

$$\lim_{n\to\infty} \mathfrak{B}(fg\zeta_n, gf\zeta_n, \wedge) = 1,$$

whenever $(\zeta_n)$ is a sequence in $\Omega$ such that $\lim f\zeta_n = \lim g\zeta_n = p$ for some $p$ in $\Omega$.

(III) $R$ −**weekly commuting:** if there exist $R > 0$ such that, for every $\zeta \in \Omega$,

$$\mathfrak{B}(fg\zeta, gf\zeta, \wedge) \geq \mathfrak{B}\left(f\zeta, g\zeta, \frac{\wedge}{R}\right).$$

In following definitions, we extend these notions to NSMSs.

**Definition 3.8:** Two mappings $S$ and $T$ of a NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond)$ into itself are said to be $R$ -weakly commuting if there exists an $R > 0$ such that, for every $\zeta \in \Omega$,

$$\mathfrak{B}(ST\zeta, TS\zeta, \wedge) \geq \mathfrak{B}\left(T\zeta, S\zeta, \frac{\wedge}{R}\right),$$

$$\mathfrak{D}(ST\zeta, TS\zeta, \wedge) \leq \mathfrak{D}\left(T\zeta, S\zeta, \frac{\wedge}{R}\right),$$

and

$$\mathfrak{L}(ST\zeta, TS\zeta, \wedge) \leq \mathfrak{L}\left(T\zeta, S\zeta, \frac{\wedge}{R}\right).$$

**Definition 3.9:** Two mappings $S$ and $T$ are said to be compatible if

$$\lim_{n\to\infty} \mathfrak{B}(ST\zeta_n, TS\zeta_n, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(ST\zeta_n, TS\zeta_n, \wedge) = 0,$$

and

$$\lim_{n\to\infty} \mathfrak{L}(ST\zeta_n, TS\zeta_n, \wedge) = 0.$$

Whenever $(\zeta_n)$ is a sequence in $\Omega$ such that $\lim S\zeta_n = \lim T\zeta_n = p$ for some $p$ in $\Omega$.

**Definition 3.10:** Two mappings $S$ and $T$ are said weakly compatible if they commute at their coincidence points i.e.

$$\{\zeta \in \Omega: S\zeta = T\zeta\} \subseteq \{\zeta \in \Omega: ST\zeta = TS\zeta\}.$$

**Definition 3.11:** Let $S$ and $T$ be self maps of a metric space. Then $S$ and $T$ are said to be occasionally weakly compatible if

$$\{\zeta \in \Omega: S\zeta = T\zeta\} \cap \{\zeta \in \Omega: ST\zeta = TS\zeta\} \neq \emptyset.$$

**Remarks 3.12**

i) With the help of numerous examples from the literature, it is simple to demonstrate that weakly commutativity implies compatibility and that compatibility implies weak compatibility.

ii) It is proved that $R$-weakly commutativity is equivalent to commutativity at coincidence points; i.e., $S$ and $T$ are point-wise $R$ -weakly commuting if and only if they are weakly compatible.

iii) The set of all occasionally weakly compatible self-maps, which is known to be a proper subclass of the set of all nontrivially weakly compatible self-maps (See Al-Thagafi and Shahzad [33]).

iv) Property (E.A) and properties of weak compatibility or compatibility are independent as it is shown by Examples 1 and 2 of (Merghadi and Thobie [17]).

# 4. Main Results

We start by defining implicit relations, which will be used in the following result. In the following, $\Theta: R_+ \to R_+$ is a locally integrable function which fulfills $\int_\delta^\varepsilon \Theta(\Lambda)d\Lambda > 0$, for every $0 < \delta < \varepsilon$. We denote by $\Phi$ (resp. $\Psi$) the set of all continuous functions $\phi$ ( resp. $\psi$): $R_+^6 \to R$ such that, if

$$(\Theta_1)\ \ \phi\left(\int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda \int_0^v \Theta(\Lambda) d\Lambda\right) \geq 0,$$

or

$$(\Theta_2)\ \ \phi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda\right) \geq 0,$$

or

$$(\Theta_3)\ \ \phi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) \geq 0.$$

Then,

$$\int_0^u \Theta(\Lambda) d\Lambda \geq \int_0^v \Theta(\Lambda) d\Lambda.$$

$$(\psi_1)\ \ \psi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda\right) \leq 0,$$

or

$$(\psi_2)\ \ \psi\left(\int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) \leq 0,$$

or

$$(\psi_3) \quad \psi\left(\int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) \leq 0.$$

Then,

$$\int_0^u \Theta(\Lambda) d\Lambda \leq \int_0^v \Theta(\Lambda) d\Lambda.$$

Respectively, if

$$(\xi_1) \quad \xi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda\right) \leq 0,$$

or

$$(\xi_2) \quad \xi\left(\int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) \leq 0,$$

or

$$(\xi_3) \quad \xi\left(\int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^u \Theta(\Lambda)\, d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) \leq 0.$$

Then,

$$\int_0^u \Theta(\Lambda) d\Lambda \leq \int_0^v \Theta(\Lambda) d\Lambda.$$

We give some examples of $\phi \in \Phi$ and $\psi, \xi \in \Psi$.

**Example 4.1:** Let

$$\phi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \psi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \xi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \frac{1}{5}(\Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5 + \Lambda_6).$$

We, have to prove $(\phi_i)$ for $i = 1$ to 3. For $i = 1$,

$$\phi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda\right) = \frac{3}{5}\left(\int_0^u \Theta(\Lambda) d\Lambda - \int_0^v \Theta(\Lambda) d\Lambda\right)$$

which is $\geq 0$ if and only if $\int_0^u \Theta(\Lambda) d\Lambda \geq \int_0^v \Theta(\Lambda) d\Lambda$. So $(\phi_1)$ is satisfied for each $\Theta$. For $i = 2$ ,

$$\phi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) = \frac{3}{5}\left(\int_0^u \Theta(\Lambda) d\Lambda - \int_0^v \Theta(\Lambda) d\Lambda\right)$$

which is $\geq 0$ if and only if $\int_0^u \Theta(\Lambda) d\Lambda \geq \int_0^v \Theta(\Lambda) d\Lambda$. So $(\phi_2)$ is satisfied for each $\Theta$. For $i = 3$,

$$\phi\left(\int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^v \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda, \int_0^u \Theta(\Lambda) d\Lambda\right) = \frac{2}{5}\left(\int_0^u \Theta(\Lambda) d\Lambda - \int_0^v \Theta(\Lambda) d\Lambda\right)$$

which is $\geq 0$ if and only if $\int_0^u \Theta(\Lambda) d\Lambda \geq \int_0^v \Theta(\Lambda) d\Lambda$. So $(\phi_3)$ is satisfied for each $\Theta$ and $\phi \in \Phi$.

Similarly, we can show that $\psi$ and $\xi \in \Psi$.

**Example 4.2:** If

$$\phi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \min\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\},$$

$$\psi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \max\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\},$$

and

$$\xi(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \max\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\}.$$

We have to prove $(\phi_i)$ for $i = 1$ to 3. For $i = 1$,

$$\phi\left(\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right) = \int_0^u \Theta(\Lambda)d\Lambda - \min\left\{\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right\},$$

which is $\geq 0$ if and only if $\int_0^u \Theta(\Lambda)d\Lambda \geq \int_0^v \Theta(\Lambda)d\Lambda$. It is easy to verify $(\phi_2), (\phi_3)$ are satisfied. So $\phi \in \Phi$. We have to prove $(\psi_i)$ for $i = 1$ to 3. For $i = 1$,

$$\psi\left(\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right) = \int_0^u \Theta(\Lambda)d\Lambda - \max\left\{\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right\},$$

which is$\leq 0$ if and only if $\int_0^u \Theta(\Lambda)d\Lambda \leq \int_0^v \Theta(\Lambda)d\Lambda$. It is easy to verify $(\psi_2)$ and $(\psi_3)$ are satisfied. So $\psi \in \Psi$. For $(\xi_i)$ and $i = 1$ to 3. For $i = 1$,

$$\xi\left(\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda, \int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right) = \int_0^u \Theta(\Lambda)d\Lambda - \max\left\{\int_0^u \Theta(\Lambda)d\Lambda, \int_0^v \Theta(\Lambda)d\Lambda\right\},$$

which is $\leq 0$ if and only if $\int_0^u \Theta(\Lambda)d\Lambda \leq \int_0^v \Theta(\Lambda)d\Lambda$. It is easy to verify $(\xi_2)$ and $(\xi_3)$ are satisfied. So $\psi, \xi \in \Psi$.

Now firstly, we present two theorems for infinity (not necessarily countable) of mappings in NSMS and NMS with hypothesis of Property (E.A) and weak compatibility.

**Theorem 4.3:** Let $(\Omega, \mathfrak{V}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ be a NSMS for $\Omega$ which satisfies (W4), (H.E), (CE.1) and (CE.2) and $A, (A_i)_{i\in I}$, $S$ and $T$ be self mappings of $\Omega$ satisfying $A\Omega \subset T\Omega$ and $A_i\Omega \subset S\Omega$ for all $i \in I$ and

$$\phi\begin{pmatrix} \int_0^{\mathfrak{V}(A\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{V}(S\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{V}(A\zeta, S\zeta, \Lambda)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{V}(A_i\varsigma, T\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{V}(A\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{V}(S\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)d\tau, \end{pmatrix} \geq 0, \tag{7}$$

$$\psi\begin{pmatrix} \int_0^{\mathfrak{D}(A\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{D}(S\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{D}(A\zeta, S\zeta, \Lambda)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{D}(A_i\varsigma, T\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{D}(A\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{D}(S\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)d\tau, \end{pmatrix} \leq 0, \tag{8}$$

$$\xi\begin{pmatrix} \int_0^{\mathfrak{L}(A\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{L}(S\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{L}(A\zeta, S\zeta, \Lambda)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{L}(A_i\varsigma, T\varsigma, \Lambda)} \Theta(\tau)\,d\tau, \int_0^{\mathfrak{L}(A\zeta, T\varsigma, \Lambda)} \Theta(\tau), \int_0^{\mathfrak{L}(S\zeta, A_i\varsigma, \Lambda)} \Theta(\tau)d\tau, \end{pmatrix} \leq 0. \tag{9}$$

For every $\zeta$ and every $\varsigma \in \Omega$, for every $i \in I$, where $\phi \in \Phi$, and $\psi, \xi \in \Psi$. $\varphi: R_+ \to R_+$ is locally integrable function which satisfies $\int_\delta^\varepsilon \Theta(\Lambda)d\Lambda > 0$, for every $0 < \delta < \varepsilon$. Suppose that:

(I) $(A, S)$ satisfies property (E.A);

(II) $(A, S)$ and $(A_k, T)$ are weekly compatible for some $k$.

If one of the subspaces $AS, S\Omega, A_i\Omega$ and $T\Omega$ of $\Omega$ is closed , then $A, A_{i,}$ for every $i \in I, S$ and $T$ have a unique common FP in $\Omega$.

**Proof:** Since $(A, S)$ satisfies Property (E.A), there exists a sequence $(\zeta_n)_{n\in\mathfrak{D}}$ in $\Omega$ such that, for some $\aleph$ in $\Omega$.

$$\lim_{n\to\infty} \mathfrak{B}(A\zeta_n, \aleph, \wedge) = \lim_{n\to\infty} \mathfrak{B}(S\zeta_n, \aleph, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(A\zeta_n, \aleph, \wedge) = \lim_{n\to\infty} \mathfrak{D}(S\zeta_n, \aleph, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(A\zeta_n, \aleph, \wedge) = \lim_{n\to\infty} \mathfrak{L}(S\zeta_n, \aleph, \wedge) = 0.$$

By property (H.E), we have

$$\lim_{n\to\infty} \mathfrak{B}(A\zeta_n, S\zeta_n, \wedge) = 1\ , \tag{10}$$

$$\lim_{n\to\infty} \mathfrak{D}(A\zeta_n, S\zeta_n, \wedge) = 0\ , \tag{11}$$

$$\lim_{n\to\infty} \mathfrak{L}(A\zeta_n, S\zeta_n, \wedge) = 0. \tag{12}$$

Since $A\Omega \subset T\Omega$, there exists a sequence $(\varsigma_n)_{n\in\mathfrak{D}}$ in $\Omega$ such that $A\zeta_n = T\varsigma_n$ for all $n \in \mathfrak{D}$ and for all $i$. Furthermore, we get

$$\lim_{n\to\infty} \mathfrak{B}(A\zeta_n, T\varsigma_n, \wedge) = 1, \tag{13}$$

$$\lim_{n\to\infty} \mathfrak{D}(A\zeta_n, T\varsigma_n, \wedge) = 0, \tag{14}$$

$$\lim_{n\to\infty} \mathfrak{L}(A\zeta_n, T\varsigma_n, \wedge) = 0. \tag{15}$$

By (CE.2), (10, ) (11), and (12), we obtain

$$\lim_{n\to\infty} \inf \mathfrak{B}(A\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \inf \mathfrak{B}(S\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \inf \mathfrak{B}(T\zeta_n, A_i\varsigma_n, \wedge),$$

$$\lim_{n\to\infty} \sup \mathfrak{D}(A\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{D}(S\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{D}(T\zeta_n, A_i\varsigma_n, \wedge),$$

$$\lim_{n\to\infty} \sup\ \mathfrak{L}(A\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{L}(S\zeta_n, A_i\varsigma_n, \wedge) = \lim_{n\to\infty} \sup \mathfrak{L}(T\zeta_n, A_i\varsigma_n, \wedge)\,.$$

Now we have to show that $\lim_{n\to\infty} \inf A_i\, \varsigma_n = \aleph$. Let be $a_i = \lim_{n\to\infty} \inf \mathfrak{B}(A\zeta_n, A_i\varsigma_n, \wedge)$ and $\wedge_I = \lim_{n\to\infty} \mathrm{Sup}\, \mathfrak{D}(A\zeta_n, A_i\varsigma_n, \wedge)$. From (7), (8) and (9) with $\zeta = \zeta_n$ and $\varsigma = \varsigma_n$, we get

$$\phi\begin{pmatrix} \int_0^{\mathfrak{B}(A\zeta_n, A_i\varsigma_n, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(S\zeta_n, T\varsigma_n, \wedge)} \Theta(\tau), \int_0^{\mathfrak{B}(A\zeta_n, S\zeta_n, \wedge)} \Theta(\tau) d\tau\,, \\ \int_0^{\mathfrak{B}(A_i\varsigma_n, T\varsigma_n, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(A\zeta_n, T\varsigma_n, \wedge)} \Theta(\tau), \int_0^{\mathfrak{B}(S\zeta_n, A_i\varsigma_n, \wedge)} \Theta(\tau) d\tau\,, \end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix} \int_0^{\mathfrak{D}(A\zeta_n, A_i\varsigma_n, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(S\zeta_n, T\varsigma_n, \wedge)} \Theta(\tau), \int_0^{\mathfrak{D}(A\zeta_n, S\zeta_n, \wedge)} \Theta(\tau) d\tau\,, \\ \int_0^{\mathfrak{D}(A_i\varsigma_n, T\varsigma_n, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(A\zeta_n, T\varsigma_n, \wedge)} \Theta(\tau), \int_0^{\mathfrak{D}(S\zeta_n, A_i\varsigma_n, \wedge)} \Theta(\tau) d\tau\,, \end{pmatrix} \leq 0,$$

and

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\zeta_n,A_i\varsigma_n,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\zeta_n,T\varsigma_n,\wedge)}\Theta(\tau),\int_0^{\mathfrak{L}(A\zeta_n,S\zeta_n,\wedge)}\Theta(\tau)d\tau\,,\\ \int_0^{\mathfrak{L}(A_i\varsigma_n,T\varsigma_n,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\zeta_n,T\varsigma_n,\wedge)}\Theta(\tau),\int_0^{\mathfrak{L}(S\zeta_n,A_i\varsigma_n,\wedge)}\Theta(\tau)d\tau\,,\end{pmatrix}\leq 0.$$

Letting $n\to\infty$, we find

$$\phi\left(\int_0^{\alpha_i}\Theta(\tau)\,d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^{\alpha_i}\Theta(\tau)d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^{\alpha_i}\Theta(\tau)d\tau\right)\geq 0,$$

$$\psi\left(\int_0^{\wedge_i}\Theta(\tau)\,d\tau,0,0,\int_0^{\wedge_i}\Theta(\tau)\,d\tau,0,\int_0^{\wedge_i}\Theta(\tau)\,d\tau\right)\leq 0,$$

and

$$\xi\left(\int_0^{\wedge_i}\Theta(\tau)\,d\tau,0,0,\int_0^{\wedge_i}\Theta(\tau)\,d\tau,0,\int_0^{\wedge_i}\Theta(\tau)\,d\tau\right)\leq 0.$$

Then

$$a_i=\lim_{n\to\infty}\inf\mathfrak{V}(A\zeta_n,A_i\varsigma_n,\wedge)=1,$$
$$\wedge_I=\lim_{n\to\infty}\operatorname{Sup}\mathfrak{D}(A\zeta_n,A_i\varsigma_n,\wedge)=0,$$
$$\mathfrak{D}_I=\lim_{n\to\infty}\operatorname{Sup}\mathfrak{L}(A\zeta_n,A_i\varsigma_n,\wedge)=0,$$

by $(\phi_2)$, $(\psi_2)$ and $\xi_2$.Then, we have

$$\lim_{n\to\infty}\mathfrak{V}(A\zeta_n,A_i\varsigma_n,\wedge)=1,\tag{16}$$

$$\lim_{n\to\infty}\mathfrak{D}(A\zeta_n,A_i\varsigma_n,\wedge)=0,\tag{17}$$

$$\lim_{n\to\infty}\mathfrak{L}(A\zeta_n,A_i\varsigma_n,\wedge)=0.\tag{18}$$

By (W4), $(16)$, $(17)$, and $(18)$, we deduce

$$\lim_{n\to\infty}\mathfrak{V}(A_i\varsigma_n,\aleph,\wedge)=1,$$

$$\lim_{n\to\infty}\mathfrak{D}(A_i\varsigma_n,\aleph,\wedge)=0,$$

$$\lim_{n\to\infty}\mathfrak{L}(A_i\varsigma_n,\aleph,\wedge)=0.$$

$$\lim_{n\to\infty}A_i\,\varsigma_n=\lim_{n\to\infty}A\zeta_n=\lim_{n\to\infty}S\zeta_n=\lim_{n\to\infty}T\varsigma_n=\aleph,\forall i.$$

If, we suppose that the $T(\Omega)$ is closed, $\aleph\in T(\Omega)$ and there exist $u\in\Omega$ such that $\aleph=Tu$. By (7), (8 ), and (9 ) with $\zeta=\zeta_n$ and $\varsigma=u$, we get

$$\phi\begin{pmatrix}\int_0^{\mathfrak{V}(A\zeta_n,A_iu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{V}(S\zeta_n,Tu,\wedge)}\Theta(\tau),\int_0^{\mathfrak{V}(A\zeta_n,S\zeta_n,\wedge)}\Theta(\tau)d\tau\,,\\ \int_0^{\mathfrak{V}(A_iu,Tu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{V}(A\zeta_n,Tu,\wedge)}\Theta(\tau),\int_0^{\mathfrak{V}(S\zeta_n,A_iu,\wedge)}\Theta(\tau)d\tau\,,\end{pmatrix}\geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\zeta_n,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\zeta_n,Tu,\Lambda)}\Theta(\tau),\int_0^{\mathfrak{D}(A\zeta_n,S\zeta_n,\Lambda)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{D}(A_iu,Tu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\zeta_n,Tu,\Lambda)}\Theta(\tau),\int_0^{\mathfrak{D}(S\zeta_n,A_iu,\Lambda)}\Theta(\tau)d\tau,\end{pmatrix}\le 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\zeta_n,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\zeta_n,Tu,\Lambda)}\Theta(\tau),\int_0^{\mathfrak{L}(A\zeta_n,S\zeta_n,\Lambda)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{L}(A_iu,Tu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\zeta_n,Tu,\Lambda)}\Theta(\tau),\int_0^{\mathfrak{L}(S\zeta_n,A_iu,\Lambda)}\Theta(\tau)d\tau,\end{pmatrix}\le 0.$$

When $n\to\infty$, using (CE.1), we obtain

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(\aleph,,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^1\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{B}(,\aleph,A_iu,,\Lambda)}\Theta(\tau)d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(\aleph,A_iu,\Lambda)}\Theta(\tau)d\tau\end{pmatrix}\ge 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,0,0\int_0^{\mathfrak{D}(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\\ 0\int_0^{\mathfrak{D}(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\end{pmatrix}\le 0,$$

$$\xi\left(\int_0^{\mathfrak{L}(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,0,0\int_0^{\mathfrak{L}(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,0\int_0^{\mathfrak{L}.(\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\right)\le 0.$$

Which implies by (CE.1) by $(\phi_2)$, $(\psi_2)$ and $(\xi_2)$

$$(A_iu)=Tu=\aleph,\quad \forall i. \tag{19}$$

Since $\aleph\in A_i\Omega\subset S\Omega$, there exists $v\in\ \Omega$ such that $\aleph=\ A_Iu\ =\ Sv\ =\ Tu$ and applying (7), (8), and (9) with $\zeta=v$ and $\varsigma=u$, we get

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(Av,\aleph,\Lambda)}\Theta(\tau)\,d\tau,\int_0^1\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(Av,\aleph,\Lambda)}\Theta(\tau)\\ \int_0^1\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(Av,\aleph,\Lambda)}\Theta(\tau)d\tau,\int_0^1\Theta(\tau)\end{pmatrix}\ge 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(Av,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0,\int_0^{\mathfrak{D}(Sv,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0,\\ \int_0^{\mathfrak{D}(Av,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0\end{pmatrix}\le 0,$$

$$\xi\left(\int_0^{\mathfrak{L}(Av,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0,\int_0^{\mathfrak{L}(Sv,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0,\int_0^{\mathfrak{L}(Av,\aleph,\Lambda)}\Theta(\tau)\,d\tau,0\right)\le 0.$$

So, by $(\phi_1)$, $(\psi_1)$ and $(\xi_1)$, $\aleph=Av=Sv$. Since the couple $(A,S)$ is weekly compatible, $ASv=SAv\ \ i.e.$

$A\aleph=S\aleph$. Now (7), (8), and (9) with $\zeta=\aleph$ and $\varsigma=u$ gives

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\aleph,A_iu,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(S\aleph,Tu,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\aleph,S\aleph,\Lambda)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{B}(A_iu,Tu,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\aleph,Tu,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\aleph,A_iu,\Lambda)}\Theta(\tau)d\tau\end{pmatrix}\ge 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\aleph,A_iu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\aleph,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\aleph,S\aleph,\wedge)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{D}(A_iu,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\aleph,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\aleph,A_iu,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\aleph,A_iu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\aleph,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\aleph,S\aleph,\wedge)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{L}(A_iu,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\aleph,Tu,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\aleph,A_iu,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0.$$

That is

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{1}\Theta(\tau)\,d\tau,\\ \int_0^{1}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0,\\ \int_0^{\mathfrak{D}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\aleph,A_iu,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0,\\ \int_0^{\mathfrak{L}(A\aleph,\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\aleph,A_iu,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\leq 0.$$

So by $(\phi_3),(\psi_3),\text{and }(\xi_3), A\aleph=\aleph$. By weak compatibility of $A_k$ and $T$, we have

$$A_kTu=A_k\aleph=TA_ku=T\aleph,$$

and applying $(\phi_3),(\psi_3),\text{and }(\xi_3)$ with $\zeta=\varsigma=\aleph$, we obtain

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{1}\Theta(\tau)\,d\tau,\\ \int_0^{1}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0\\ ,\int_0^{\mathfrak{D}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0\\ ,\int_0^{\mathfrak{L}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(\aleph,A_k\aleph,\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\leq 0.$$

From $(\phi_3),(\psi_3),(\xi_3)$, (16, (17) and (18), it follows $A_k\aleph=T\aleph=S\aleph=\aleph$. So, $\aleph$ is a common FP of $A,S,T$ and $A_K$. But, for every $i$, we have

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(S\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\aleph,S\aleph,\wedge)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{B}(A_i\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\aleph,A_i\aleph,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\aleph,S\aleph,\wedge)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{D}(A_i\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\aleph,A_i\aleph,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\le 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\aleph,S\aleph,\wedge)}\Theta(\tau)d\tau\\ \int_0^{\mathfrak{L}(A_i\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\aleph,T\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\aleph,A_i\aleph,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\le 0.$$

That is

$$\phi\left(\int_0^{\mathfrak{B}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,\int_0^{1}\Theta(\tau)d\tau,\int_0^{1}\Theta(\tau)d\tau\int_0^{\mathfrak{B}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)d\tau,\int_0^{1}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)d\tau\right)\ge 0,$$

$$\psi\left(\int_0^{\mathfrak{D}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0,\int_0^{\mathfrak{D}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,0,\int_0^{\mathfrak{D}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau\right)\le 0,$$

$$\xi\left(\int_0^{\mathfrak{L}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,0,0,\int_0^{\mathfrak{L}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau,0,\int_0^{\mathfrak{L}(\aleph,A_i\aleph,\wedge)}\Theta(\tau)\,d\tau\right)\le 0.$$

Then from $(\phi_2),(\psi_2)$ and $(\xi_3)$, $\aleph=A_i\aleph$ for each $i$ and $\aleph$ is a common FP of $A,S,T$ and $A_i$ for every $i$. Now we show the uniqueness of the common FP. If $\aleph$ is another common FP, from (7), (8) and (9) with $\zeta=\aleph$ and $y=\overline{\aleph}$, we get

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{1}\Theta(\tau)\,d\tau,\\ \int_0^{1}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\ge 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,0,0,\\ \int_0^{\mathfrak{D}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\le 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,0,0,\\ \int_0^{\mathfrak{L}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(\aleph,\overline{\aleph},\wedge)}\Theta(\tau)\,d\tau\end{pmatrix}\le 0.$$

So, by $(\phi_3)$ and $(\psi_3)$, we obtain $\aleph=\overline{\aleph}$. Then, $\aleph$ is unique common FP of $A,S,T$ and $A_I$ for every $i$.

**Theorem 4.4:** Let $(\Omega,\mathfrak{B},\mathfrak{D},\mathfrak{L},*,\lozenge)$ be a NSMS for $\Omega$ which satisfies (W4), (H.E), (CE.1), (CE.2) and $A,(A_i)_{i\in I}$, $S$ and $T$ be self mappings satisfying $A\Omega\subset T\Omega$ and $A_i\Omega\subset S\Omega$ for all $i\in I$ and

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{B}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\ge 0,\tag{20}$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{D}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\le 0,\tag{21}$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{L}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\le 0.\tag{22}$$

For every $\zeta$ and every $\varsigma \in \Omega$, for every $i \in I$,where $\phi \in \Phi, \psi \in \Psi$ and $\phi: R_+ \to R_+$ is locally integrable function which satisfies $\int_\delta^\varepsilon \Theta(\wedge) d\wedge > 0$, for every $0 < \delta < \varepsilon$. Suppose that:

(I) $(A_i, T)$ satisfies property (E.A) for every $i \in I$;

(II) $(A, S)$ and $(A_K, T)$ are weekly compatible for some $k$. If one of the subspaces $A\Omega, S\Omega, A_i\Omega$ and $T\Omega$ of $\Omega$ is closed then $A, A_i$, for every $i \in I, S$ and $T$ have a unique common FP in $\Omega$.

**Proof:** Since $(A_i, T)$ satisfies property (E.A),

$$\lim_{n\to\infty} \mathfrak{B}(A_i\zeta_n^i, \aleph_i, \wedge) = \lim_{n\to\infty} \mathfrak{B}(T\zeta_n^i, \aleph_i, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(A_i\zeta_n^i, \aleph_i, \wedge) = \lim_{n\to\infty} \mathfrak{D}(T\zeta_n^i, \aleph_i, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(A_i\zeta_n^i, \aleph_i, \wedge) = \lim_{n\to\infty} (T\zeta_n^i, \aleph_i, \wedge).$$

for some $\aleph_i$ in $\Omega$. By property (H.E), we have

$$\lim_{n\to\infty} \mathfrak{B}(A_i\zeta_n^i, T\zeta_n^i, \wedge) = 1, \tag{23}$$

$$\lim_{n\to\infty} \mathfrak{D}(A_i\zeta_n^i, T\zeta_n^i, \wedge) = 0, \tag{24}$$

$$\lim_{n\to\infty} \mathfrak{L}(A_i\zeta_n^i, T\zeta_n^i, \wedge) = 0. \tag{25}$$

Since $A_i\Omega \subset S\Omega$ , there exist a sequence $(\varsigma_n^i)_n$ in $\Omega$ such that $A_i\zeta_n^i = S\varsigma_n^i$ for all $n \in \mathfrak{D}$ and for all $i$.

$$\lim_{n\to\infty} \mathfrak{B}(A_i\zeta_n^i, S\varsigma_n^i, \wedge) = 1, \tag{26}$$

$$\lim_{n\to\infty} \mathfrak{D}(A_i\zeta_n^i, , S\varsigma_n^i \wedge) = 0, \tag{27}$$

$$\lim_{n\to\infty} \mathfrak{L}(A_i\zeta_n^i, , S\varsigma_n^i \wedge) = 0. \tag{28}$$

By (CE.2),(23), (24), and (25), we obtain

$$\lim_{n\to\infty} \inf \mathfrak{B}(A\varsigma_n^i, A_I\zeta_n^i, \wedge) = \lim_{n\to\infty} \inf \mathfrak{B}(A\varsigma_n^i, T\zeta_n^i, \wedge),$$

$$\lim_{n\to\infty} \sup \mathfrak{D}(A\varsigma_n^i, A_I\zeta_n^i, \wedge) = \lim_{n\to\infty} \sup \mathfrak{D}(A\varsigma_n^i, T\zeta_n^i, \wedge),$$

$$\lim_{n\to\infty} \sup \mathfrak{L}(A\varsigma_n^i, A_I\zeta_n^i, \wedge) = \lim_{n\to\infty} \sup \mathfrak{L}(A\varsigma_n^i, T\zeta_n^i, \wedge).$$

Now, we show that $\lim_{n\to\infty} A\varsigma_n^i = \aleph_{i.}$ Let be

$$\propto_i = \lim_{n\to\infty} \inf \mathfrak{B}(A\varsigma_n^i, A_i\zeta_n^i, \wedge),$$

and

$$\wedge_i = \lim_{n\to\infty} \sup \mathfrak{D}(A\varsigma_n^i, A_i\zeta_n^i, \wedge).$$

Using (20), (21), and (22) with $\zeta = \varsigma_n^i$ and $\varsigma = \zeta_n^i$. We get

$$\phi\left(\begin{array}{l}\int_0^{\mathfrak{B}(A\varsigma_n^i, A_i\zeta_n^i, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{B}(A\varsigma_n^i, T\zeta_n^i, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{B}(A\varsigma_n^i, S\zeta_n^i, \wedge)} \Theta(\tau) d\tau, \\ \int_0^{\mathfrak{B}(A_i\zeta_n^i, T\zeta_n^i, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(A\varsigma_n^i, T\zeta_n^i, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(S\varsigma_n^i, A_i\zeta_n^i, \wedge)} \Theta(\tau) d\tau\end{array}\right) \geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\varsigma_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\varsigma_n^i,S\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{D}(A_i\zeta_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\varsigma_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(S\varsigma_n^i,A_i\zeta_n^i,\Lambda)}\Theta(\tau)d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\varsigma_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\varsigma_n^i,S\zeta_n^i,\Lambda)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{L}(A_i\zeta_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\varsigma_n^i,T\zeta_n^i,\Lambda)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(S\varsigma_n^i,A_i\zeta_n^i,\Lambda)}\Theta(\tau)d\tau\end{pmatrix}\leq 0.$$

Letting $n \to \infty$, we deduce

$$\phi\begin{pmatrix}\int_0^{\propto_i}\Theta(\tau)d\tau,\int_0^{1}\Theta(\tau)d\tau,\int_0^{\propto_i}\Theta(\tau)d\tau,\\ \int_0^{1}\Theta(\tau)d\tau,\int_0^{\propto_i}\Theta(\tau)d\tau,\int_0^{1}\Theta(\tau)d\tau,\end{pmatrix}\geq 0,$$

$$\psi\left(\int_0^{\Lambda_i}\Theta(\tau)d\tau,0,\int_0^{\Lambda_i}\Theta(\tau)d\tau,0,\int_0^{\Lambda_i}\Theta(\tau)d\tau,0\right)\leq 0,$$

$$\xi\left(\int_0^{\Lambda_i}\Theta(\tau)d\tau,0,\int_0^{\Lambda_i}\Theta(\tau)d\tau,0,\int_0^{\Lambda_i}\Theta(\tau)d\tau,0\right)\leq 0.$$

By $(\phi_1)$, $(\psi_1)$, and $(\xi_1)$, $\propto_i = 1$ and $\Lambda_i = 0$; i.e.,

$$\lim_{n\to\infty}\inf\mathfrak{B}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=1,$$

$$\lim_{n\to\infty}\sup\mathfrak{D}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0,$$

$$\lim_{n\to\infty}\sup\mathfrak{L}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0.$$

Thus, we have

$$\lim_{n\to\infty}\mathfrak{B}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=1,$$

$$\lim_{n\to\infty}\mathfrak{D}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0,$$

and

$$\lim_{n\to\infty}\mathfrak{L}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0.$$

By (W4), since

$$\lim_{n\to\infty}\mathfrak{B}(A_I\zeta_n^i,\aleph_i,\Lambda)=1,$$

$$\lim_{n\to\infty}\mathfrak{D}(A_i\zeta_n^i,\aleph_i,\Lambda)=0,$$

$$\lim_{n\to\infty}\mathfrak{L}(A_i\zeta_n^i,\aleph_i,\Lambda)=0.$$

and

$$\lim_{n\to\infty}\mathfrak{B}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=1,$$

$$\lim_{n\to\infty}\mathfrak{D}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0,$$

$$\lim_{n\to\infty}\mathfrak{L}(A\varsigma_n^i,A_i\zeta_n^i,\Lambda)=0.$$

we have

$$\lim_{n\to\infty} \mathfrak{B}(A\varsigma_n^i, \aleph_i, \wedge) = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(A\varsigma_n^i, \aleph_i, \wedge) = 0,$$

$$\lim_{n\to\infty} \mathfrak{L}(A\varsigma_n^i, \aleph_i, \wedge) = 0.$$

That is

$$\lim_{n\to\infty} A_i\zeta_n^i = \lim_{n\to\infty} T\zeta_n^i = \lim_{n\to\infty} A\varsigma_n^i = \lim_{n\to\infty} S\varsigma_n^i = \aleph_i, \qquad \forall i.$$

Suppose that $S(\Omega)$ is closed. Then, $\aleph_i \in S(\Omega)$ and there exist $u_i \in \Omega$ such that $\aleph_i = Su_i$. By (20), (21), and (22) with $\zeta = u_i$ and $\varsigma = \zeta_n^i$, we get

$$\phi\begin{pmatrix} \int_0^{\mathfrak{B}(Au_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(Su_i,T\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(Au_i,Su_i,\wedge)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{B}(A_i\zeta_n^i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Au_i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Su_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau \end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix} \int_0^{\mathfrak{D}(Au_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{D}(Su_i,T\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{D}(Au_i,Su_i,\wedge)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{D}(A_i\zeta_n^i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(Au_i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(Su_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau \end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix} \int_0^{\mathfrak{L}(Au_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{L}(Su_i,T\zeta_n^i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{L}(Au_i,Su_i,\wedge)} \Theta(\tau)d\tau, \\ \int_0^{\mathfrak{L}(A_i\zeta_n^i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{L}(Au_i,T\zeta_n^i,\wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{L}(Su_i,A_i\zeta_n^i,\wedge)} \Theta(\tau)d\tau \end{pmatrix} \leq 0.$$

When $n \to \infty$ using (CE. 1), we get

$$\phi\begin{pmatrix} \int_0^{\mathfrak{B}(Au_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^1 \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(Au_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \\ \int_0^1 \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Au_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^1 \Theta(\tau)d\tau, \end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix} \int_0^{\mathfrak{D}(Au_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, 0 \int_0^{\mathfrak{D}(Su_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \\ 0, \int_0^{\mathfrak{D}(Au_i,\aleph_i,\wedge)} \Theta(\tau)\, d\tau, 0 \end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix} \int_0^{\mathfrak{L}(Au_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, 0 \int_0^{\mathfrak{L}(Su_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \\ 0, \int_0^{\mathfrak{L}(Au_i,\aleph_i,\wedge)} \Theta(\tau)\, d\tau, 0 \end{pmatrix} \leq 0.$$

Which implies, by (CE.1), $(\phi_1)$, $(\psi_1)$, and $(\xi_1)$, $Au_i = Su_i = \aleph_i$, $\forall i$. As $A\Omega \subset T\Omega$, $\exists\, v_i \in \Omega$ such that

$\aleph_i = Au_i = Tv_i$. Applying again (20), (21), and (22) with $\zeta = u_i$ and $\varsigma = v_i$, we have

$$\phi\left(\int_0^{\mathfrak{B}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 1,1, \int_0^{\mathfrak{B}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 1, \int_0^{\mathfrak{B}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau\right) \geq 0,$$

$$\psi\left(\int_0^{\mathfrak{D}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 0,0, \int_0^{\mathfrak{D}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 0, \int_0^{\mathfrak{D}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau\right) \leq 0,$$

$$\xi\left(\int_0^{\mathfrak{L}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 0,0, \int_0^{\mathfrak{L}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau, 0, \int_0^{\mathfrak{L}(\aleph_i,A_iv_i\wedge)} \Theta(\tau)d\tau\right) \leq 0.$$

Which implies $A_i v_i = T v_i = \aleph_i$ by using the conditions $(\phi_2), (\psi_2), \text{and } (\xi_2)$. Since the pair $(A, S)$ is weakly compatible, $A\aleph_i = S\aleph_i$. Using (20), (21), and (22) with $\zeta = \aleph_i$ and $\varsigma = v_i$, we get

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^1 \Theta(\tau)d\tau,\\ \int_0^1 \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{B}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{D}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{D}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{D}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{L}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{L}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{L}(A\aleph_i,\aleph_i,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0.$$

So, by $(\phi_3), (\psi_3), \text{and } (\xi_3)$, we have

$$\aleph_i = A\aleph_i = S\aleph_i \tag{29}$$

for every $i \in I$. By weak compatibility of $A_k$ and $T$, we have $A_k\aleph_k = T\aleph_k$. And with $\zeta = \varsigma = \aleph_k$, we obtain

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^1 \Theta(\tau)d\tau,\\ \int_0^1 \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{B}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{D}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{D}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{D}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{L}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{L}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{L}(\aleph_k,A_k\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0.$$

From $(\phi_3), (\psi_3), (\xi_3)$ and (4.3), it follows $A_k\aleph_k = T\aleph_k = S\aleph_k = \aleph_k$ . So $\aleph_k$ is a common FP of $A, S, T$ and $A_k$. But, for every $i$, we have

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{B}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^1 \Theta(\tau)d\tau,\\ \int_0^1 \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{B}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{D}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{D}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{D}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, \int_0^{\mathfrak{L}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau, 0,\\ 0, \int_0^{\mathfrak{L}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\, , \int_0^{\mathfrak{L}(\aleph_k,A_i\aleph_k,\wedge)} \Theta(\tau)d\tau\end{pmatrix} \leq 0.$$

Then, from $(\phi_3)$, $(\psi_3)$, and $(\xi_3)$, $A_i\aleph_k = \aleph_k$ for each $i$ and $\aleph_k$ is a common FP of $A, S, T$ and $A_i$, for every $i$. The unicity of common FP is shown as in previous theorem .Then, $\aleph_k$ is the unique common FP of $A, S, T$ and $A_k$, for every $i$.

**Remark 4.5:** When $T\Omega$ is assumed to be a closed subspace of $\Omega$, then proof is similar. On the other hand the case in which $A\Omega$ or $A_i\Omega$ is a closed subspace of $\Omega$ are similar to the case in which $T\Omega$ or $S\Omega$ is closed.

**Theorem 4.6:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ be a NSMS and $A$, $(A_i)_{i\in I}$, $S$ and $T$ be self mappings of $\Omega$ satisfying $A\Omega \subset T\Omega$ and $A_i\Omega \subset S\Omega$ for all $i \in I$ and (20), (21), and (22) for all $\zeta, \varsigma \in \Omega$ and every $\wedge > 0$, where $\phi \in \Phi, \psi \in \Psi$ and $\Theta: R_+ \to R_+$ is a locally integrable function which satisfies $\int_\delta^\varepsilon \Theta(\wedge)d\wedge > 0$, for every $0 < \delta < \varepsilon$. Suppose that:

(I) $(A, S)$ or $(A_i, T)$ for every $i \in I$ satisfies property (E.A);

(II) $(A, S)$ and $(A_k, T)$ are weakly compatible for some $k$. If one of the subspaces $A\Omega, S\Omega, A_i\Omega$ and $T\Omega$ of $\Omega$ is closed, then $A, A_i$ for every $i \in I, S$ and $T$ have a unique common FP in $\Omega$.

**Proof:** By the Proposition 3.7, the result follows immediately from the previous Theorems 4.3 or 4.4. In the following results, the condition of compatibility is a slightly enhanced occasionally weak compatibility. And hypotheses on the ranges are removed.

**Theorem 4.7:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond)$ be a NSMS and $A, (A_i)_{i\in I}, S$ and $T$ be self mappings of $\Omega$ satisfying the following conditions:

i) the pair $(A, S)$ is occasionally weakly compatible,

ii) there exists $v \in \bigcap_{i\in I} C(A_i, T)$ such that $A_i T v = T A_i v$ for all $i \in I$, where $C(A_i, T)$ is the set coincidence point of $A_i$ and $T$,

iii) $A, A_i$ for every $i \in I, S$ and $T$ satisfies (20), (21) and (22) for every $\zeta$ and $\varsigma \in \Omega$ where

$\phi, \psi: R_+^6 \to R$ Satisfies the equation (20), (21) and (22) and $\Theta: R_+ \to R_+$ is a locally integrable function which satisfies $\int_\delta^\varepsilon \Theta(\wedge) d\wedge > 0$ for every $0 < \delta < \varepsilon$.Then $A, A_i$ for every

$i \in I, S$ and $T$ have a unique common FP in $\Omega$.

**Proof:** By (i) and (ii), there exists $u$ and $v \in \Omega$ such that for every $i \in I$

$$Au = Su \qquad ASu = SAu \qquad A_i v = Tv \qquad A_i Tv = TA_i v. \tag{30}$$

Using (20), (21), and (22) with $\zeta = u$ and $\varsigma = v$, we have

$$\phi\begin{pmatrix} \int_0^{\mathfrak{B}(Au, A_i v, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{B}(Su,, Tv, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{B}(Au, Su, \wedge)} \Theta(\tau) d\tau, \\ \int_0^{\mathfrak{B}(A_i v,, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Au, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Su, A_i v, \wedge)} \Theta(\tau) d\tau \end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix} \int_0^{\mathfrak{D}(Au, A_i v, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{D}(Su,, Tv, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{D}(Au, Su, \wedge)} \Theta(\tau) d\tau, \\ \int_0^{\mathfrak{D}(A_i v,, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(Au, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{D}(Su, A_i v, \wedge)} \Theta(\tau) d\tau \end{pmatrix} \leq 0,$$

$$\xi\begin{pmatrix} \int_0^{\mathfrak{L}(Au, A_i v, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{L}(Su,, Tv, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{L}(Au, Su, \wedge)} \Theta(\tau) d\tau, \\ \int_0^{\mathfrak{L}(A_i v,, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{L}(Au, Tv, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{L}(Su, A_i v, \wedge)} \Theta(\tau) d\tau \end{pmatrix} \leq 0.$$

Then by $(\phi_3), (\psi_3)$ and $(\xi_3), Au = Tv$ we have for every $i$,

$$Au = Su = A_i v = Tv. \tag{31}$$

From (30), we can write for every $i$

$$A_i Au = A_i Tv = TA_i v = TAu \tag{32}$$

Using (20), (21) and (22) again with $\zeta = u$ and $\varsigma = Tu$, we obtain with (32)

$$\phi\begin{pmatrix} \int_0^{\mathfrak{B}(Au, TAv, \wedge)} \Theta(\tau) d\tau, \int_0^{\mathfrak{B}(Au, TAu, \wedge)} \Theta(\tau) d\tau, \int_0^{1} \Theta(\tau) d\tau, \\ \int_0^{1} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Au, TAu, \wedge)} \Theta(\tau)\, d\tau, \int_0^{\mathfrak{B}(Au, TAu, \wedge)} \Theta(\tau) d\tau \end{pmatrix} \geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(Au,TAv,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(Au,TAu,\wedge)}\Theta(\tau)d\tau,0,\\ 0,\int_0^{\mathfrak{D}(Au,TAu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(Au,TAu,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(Au,TAv,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(Au,TAu,\wedge)}\Theta(\tau)d\tau,0,\\ 0,\int_0^{\mathfrak{L}(Au,TAu,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(Au,TAu,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0.$$

So, by $(\phi_3),(\psi_3),\text{and }(\xi_3), Au=TAu$ . Therefore, we have by $(30),(31),\text{and }(32)$, for every $i\in I$.

$$A_i(Au)=T(Au)=Au, \tag{33}$$

and

$$A(Au)=A(Su)=S(Au). \tag{34}$$

Using $(20),(21),\text{and }(22)$ again with $\zeta=\varsigma=Au$, we get

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(AAu,Au,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(AAu,Au,\wedge)}\Theta(\tau)d\tau,\int_0^{1}\Theta(\tau)d\tau,\\ \int_0^{1}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(AAu,Au,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(AAu,Au,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\geq 0,$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(AAu,Au,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(AAu,Au,\wedge)}\Theta(\tau)d\tau,0,\\ 0,d\tau,\int_0^{\mathfrak{D}(AAu,Au,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(AAu,Au,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0,$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(AAu,Au,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(AAu,Au,\wedge)}\Theta(\tau)d\tau,0,\\ 0,d\tau,\int_0^{\mathfrak{L}(AAu,Au,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(AAu,Au,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0.$$

Hence by $(\phi_3),(\psi_3),(\xi_3)$, (33), and (34) for all $i$, we have

$$A(Au)=S(Au)=A_i(Au)=T(Au)=Au.$$

So, $Au$ is a FP of $A,S,T,\text{and }A_i$ for every $i$. The unicity of the common FP is shown as in the previous theorem. And the proof is finished.

As a particular case, we get the following theorem which generalizes Theorem 4.1 of (Pathak *et al.* [29]), Theorem 2.3 of (Merghadi and Godet-Thobie [25]) and Theorem 3.1 of (Aliouche and Popa [31]) among others.

**Theorem 4.8:** Let $A,B,S$ and $T$ be self-mappings of a NSMS $(\Omega,\mathfrak{B},\mathfrak{D},\mathfrak{L},*,\lozenge)$, which satisfies

$$\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{B}(B\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\geq 0, \tag{35}$$

$$\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{D}(B\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0, \tag{36}$$

$$\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{L}(B\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\zeta,B\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}\leq 0. \tag{37}$$

for every $\zeta$ and every $\varsigma\in\Omega$, where $\phi,\psi: R_+^6\to R$ satisfy the conditions $(\phi_3)$ and $(\psi_3)$ and

$\Theta: R_+\to R_+$ is a locally integrable function which satisfies $\int_\delta^\varepsilon\Theta(\wedge)d\wedge>0$, for every $0<\delta<\varepsilon$. If the pairs $(A,S)$ and $(B,T)$ are occasionally weakly compatible, then $A,B,S$ and $T$ have a unique common FP.

## 5. Examples and Applications

We provide some examples that illustrate our theorems before explaining a number of previously published results that can be obtained as special cases of our earlier theorems.

**Example 5.1:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, *, \lozenge)$ be as following: $\Omega = [0, 4]$,

$$\mathfrak{B}(\zeta, \varsigma, \wedge) = \frac{\wedge}{\wedge + |\zeta - \varsigma|},$$

$$\mathfrak{D}(\zeta, \varsigma, \wedge) = \frac{|\zeta - \varsigma|}{\wedge + |\zeta - \varsigma|},$$

$$\mathfrak{L}(\zeta, \varsigma, \wedge) = \frac{|\zeta - \varsigma|}{\wedge}.$$

$a * b = ab$ and $a \lozenge b = \min\{1, a + b\}$. Let

$$\phi(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - \min\{\tau_2, \tau_3, \tau_4, \tau_5, \tau_6\},$$

$$\psi(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - \max\{\tau_2, \tau_3, \tau_4, \tau_5, \tau_6\},$$

$$\xi(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - \max\{\tau_2, \tau_3, \tau_4, \tau_5, \tau_6\},$$

and $A, A_{i\in I}, S$, and $T$ be self-mapping of $\Omega$ such that

$$A\zeta = \begin{cases} \frac{7}{3} \text{ if } \zeta \in [0,2[ \\ 2 \text{ if } \zeta \in [2,3[, \\ \frac{3}{2} \text{ if } \zeta \in [3,4[ \end{cases} \qquad S\zeta = \begin{cases} 3 \text{ if } \zeta \in [0,2[ \\ \zeta \text{ if } \zeta \in [2,3[ \\ \frac{7}{2} \text{ if } \zeta \in [3,4[ \end{cases},$$

for all $i \in \mathfrak{D}$,

$$A_i\varsigma = \begin{cases} 2 \text{ if } \varsigma \in [0,3[ \\ 2 + \frac{1}{i} \text{ if } \varsigma \in [3,4]' \end{cases} \qquad T\varsigma = \begin{cases} 4 - \varsigma \text{ if } \varsigma \in [0,3[ \\ \frac{3}{4} \text{ if } \varsigma \in [3,4]. \end{cases}$$

Note that there exists $\aleph + 2$ in $\Omega$ such that $A2 = A_i 2 = S\,2 = T2 = 2, \forall i.$ It is clear with the sequence $\zeta_n = 2 + \frac{1}{n}$ that $(A, S)$ satisfies the property (E.A). $C(A, S) = \{2\}, AS2 = SA2$; $(A, S)$ are weakly compatible. $C(A_2, T) = \{2\}$ and $A_2T2 = TA_2 2 = 2,$ then hypotheses (I) and (II) of Theorem 4.3 are satisfied.

$$A\Omega = \left\{\frac{3}{2}, 2, \frac{7}{3}\right\} \subset T\Omega = \left\{\frac{3}{4}\right\} \cup ]1,4],$$

and

$$A_i\Omega = \left\{2, 2 + \frac{1}{i}\right\} \subset S\Omega = [2,3] \cup \left\{\frac{7}{2}\right\}.$$

Now, we verify the condition (7), (8) and (9) of Theorem 4.3. Let $\Theta(\tau) = 1$

$$\int_0^{\mathfrak{B}(\zeta,\varsigma,\wedge)} \Theta(\tau) d\tau = \frac{|\zeta - \varsigma|}{\wedge + |\zeta - \varsigma|)},$$

$$\int_0^{\mathfrak{D}(\zeta,\varsigma,\wedge)} \Theta(\tau) d\tau = \frac{|\zeta - \varsigma|}{\wedge + |\zeta - \varsigma|)},$$

$$\int_0^{\mathfrak{L}(\zeta,\varsigma,\wedge)} \Theta(\tau) d\tau = \frac{|\zeta - \varsigma|}{\wedge}.$$

If we define $R, L, J, P\ Q$ and $U$ by

$$R(\zeta,\varsigma,\wedge)=\phi\begin{pmatrix}\int_0^{\mathfrak{B}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{B}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau\,,\int_0^{\mathfrak{B}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix},$$

$$L(\zeta,\varsigma,\wedge)=\psi\begin{pmatrix}\int_0^{\mathfrak{D}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{D}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau\,,\int_0^{\mathfrak{D}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix},$$

$$J(\zeta,\varsigma,\wedge)=\xi\begin{pmatrix}\int_0^{\mathfrak{L}(A\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau,\int_0^{\mathfrak{L}(A\zeta,S\zeta,\wedge)}\Theta(\tau)d\tau,\\ \int_0^{\mathfrak{L}(A_i\varsigma,T\varsigma,\wedge)}\Theta(\tau)\,d\tau,\int_0^{\mathfrak{L}(A\zeta,T\varsigma,\wedge)}\Theta(\tau)d\tau\,,\int_0^{\mathfrak{L}(S\zeta,A_i\varsigma,\wedge)}\Theta(\tau)d\tau\end{pmatrix}.$$

$$P(\zeta,\varsigma,\wedge)=\min\left\{\begin{matrix}\frac{\wedge}{\wedge+|S\zeta-T\varsigma|},\frac{\wedge}{\wedge+|A\zeta-S\zeta|},\frac{\wedge}{\wedge+|A_i\varsigma-T\varsigma|},\\ \frac{\wedge}{\wedge+|A\zeta-T\varsigma|},\frac{\wedge}{\wedge+|S\zeta-A_i\varsigma|}\end{matrix}\right\},$$

$$Q(\zeta,\varsigma,\wedge)=\max\left\{\begin{matrix}\frac{|S\zeta-T\varsigma|}{\wedge+|S\zeta-T\varsigma|},\frac{|A\zeta-S\zeta|}{\wedge+|A\zeta-S\zeta|},\frac{|A_i\varsigma-T\varsigma|}{\wedge+|A_i\varsigma-T\varsigma|},\\ \frac{|A\zeta-T\varsigma|}{\wedge+|A\zeta-T\varsigma|},\frac{|S\zeta-A_i\varsigma|}{\wedge+|S\zeta-A_i\varsigma|}\end{matrix}\right\},$$

$$U(\zeta,\varsigma,\wedge)=\max\left\{\begin{matrix}\frac{|S\zeta-T\varsigma|}{\wedge},\frac{|A\zeta-S\zeta|}{\wedge},\frac{|A_i\varsigma-T\varsigma|}{\wedge},\\ \frac{|A\zeta-T\varsigma|}{\wedge},\frac{|S\zeta-A_i\varsigma|}{\wedge}\end{matrix}\right\}.$$

Then

$$R(\zeta,\varsigma,\wedge)=\frac{\wedge}{\wedge+|A\zeta-A_i\varsigma|}-P(\zeta,\varsigma,\wedge),$$

and

$$L(\zeta,\varsigma,\wedge)=\frac{|A\zeta-A_i\varsigma|}{\wedge+|A\zeta-A_i\varsigma|}-Q(\zeta,\varsigma,\wedge),$$

$$J(\zeta,\varsigma,\wedge)=\frac{|A\zeta-A_i\varsigma|}{\wedge}-U(\zeta,\varsigma,\wedge).$$

We have to prove that $R(\zeta,\varsigma,\wedge)\geq 0, L(\zeta,\varsigma,\wedge)\leq 0$ and $J(\zeta,\varsigma,\wedge)\leq 0$, for every $\zeta$ and every $\varsigma$. It is clear that $R(\zeta,\varsigma,\wedge)\geq 0, L(\zeta,\varsigma,\wedge)\leq 0$ and $J(\zeta,\varsigma,\wedge)\leq 0$ when $\zeta\in[2,3[$ and $\varsigma\in[0,3[$. Since, in this case,

$$|A\zeta-A_i\varsigma|=0.$$

We have to study the other cases. For this, it is easy to see that $R(\zeta,\varsigma,\wedge)\geq 0, L(\zeta,\varsigma,\wedge)\leq 0$ and $J(\zeta,\varsigma,\wedge)\leq 0$, if

$$V(\zeta,\varsigma)=\left\{\begin{matrix}|S\zeta-T\varsigma|,|A\zeta-S\zeta|,|A_i\varsigma-T\varsigma|,\\ |A\zeta-T\varsigma|,|S\zeta-A_i\varsigma|\end{matrix}\right\}\geq|A\zeta-A_i\varsigma|\}.$$

See Tables (**1**-**8**) 5.1 and 5.2 for required values of this example.

**Table 1: Some values connected with Example 5.1.**

| | $\zeta \in [0,2[$ $\varsigma \in [0,1[$ | $\zeta \in [0,2[$ $\varsigma \in [1,2[$ | $\zeta \in [0,2[$ $\varsigma \in [2,3[$ | $\zeta \in [0,2[$ $\varsigma \in [3,4[$ | $\zeta \in \left[2,2+\frac{1}{i}\right[$ $\varsigma \in [3,4[$ |
|---|---|---|---|---|---|
| $\lvert A\zeta - A_i\varsigma\rvert$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}-\frac{1}{i}$ | $\frac{1}{i}$ |
| $\lvert S\zeta - T\varsigma\rvert$ | $1-\varsigma$ | $\varsigma-1$ | $\varsigma-1$ | $\frac{9}{4}$ | $\zeta-\frac{3}{4}$ |
| $\lvert A\zeta - S\zeta\rvert$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\zeta-2$ |
| $\lvert A_i - T\varsigma\rvert$ | $2-\varsigma$ | $2-\varsigma$ | $\varsigma-2$ | $\frac{5}{4}+\frac{1}{i}$ | $\frac{5}{4}+\frac{1}{i}$ |
| $\lvert A\zeta - T\varsigma\rvert$ | $\frac{5}{3}-\varsigma$ | $\left\lvert\varsigma-\frac{5}{3}\right\rvert$ | $\varsigma-\frac{5}{3}$ | $\frac{19}{12}$ | $\frac{5}{4}$ |
| $\lvert S\zeta - A_i\varsigma\rvert$ | $1$ | $1$ | $1$ | $1-\frac{1}{i}$ | $2+\frac{1}{i}-\zeta$ |
| $V(\zeta,\varsigma)$ | $2-\varsigma$ | $1$ | $\varsigma-1$ | $\frac{9}{4}$ | $\frac{5}{4}+\frac{1}{i}$ |
| | $\geq\frac{1}{3}$ | $\geq\frac{1}{3}$ | $\geq\frac{1}{3}$ | $\geq\frac{1}{3}-\frac{1}{i}$ | $\geq\frac{1}{i}$ |

**Table 2: Some values connected with Example 5.1.**

| | $\zeta \in \left[2+\frac{1}{i},3\right[$ $\varsigma \in [3,4[$ | $\zeta \in [3,4[$ $\varsigma \in \left[0,\frac{1}{2}\right[$ | $\zeta \in [3,4[$ $\varsigma \in \left[\frac{1}{2},\frac{5}{2}\right[$ | $\zeta \in [3,4[$ $\varsigma \in \left[\frac{5}{2},3\right[$ | $\zeta \in [3,4[$ $\varsigma \in [3,4[$ |
|---|---|---|---|---|---|
| $\lvert A\zeta - A_i\varsigma\rvert$ | $\frac{1}{i}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\left\lvert\frac{1}{2}-\frac{1}{i}\right\rvert$ |
| $\lvert S\zeta - T\varsigma\rvert$ | $\zeta-\frac{3}{4}$ | $\frac{1}{2}-\varsigma$ | $\varsigma-\frac{1}{2}$ | $\varsigma-\frac{1}{2}$ | $\frac{11}{4}$ |
| $\lvert A\zeta - S\zeta\rvert$ | $\zeta-2$ | $2$ | $2$ | $2$ | $2$ |
| $\lvert A_i\varsigma - T\varsigma\rvert$ | $\frac{5}{4}+\frac{1}{i}$ | $2-\varsigma$ | $\lvert\varsigma-2\rvert$ | $\varsigma-2$ | $\frac{5}{4}+\frac{1}{i}$ |
| $\lvert A\zeta - T\varsigma\rvert$ | $\frac{5}{4}$ | $\frac{5}{2}-\varsigma$ | $\frac{5}{2}-\varsigma$ | $\varsigma-\frac{5}{2}$ | $\frac{3}{2}$ |
| $\lvert S\zeta - A_i\varsigma\rvert$ | $2+\frac{1}{i}-\zeta$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}-\frac{1}{i}$ |
| $V(\zeta,\varsigma)$ | $\frac{5}{4}+\frac{1}{i}$ | $\frac{5}{2}-\varsigma$ | $2$ | $\varsigma-\frac{1}{2}$ | $\frac{11}{4}$ |
| | $\geq\frac{1}{i}$ | $\geq\frac{1}{2}$ | $\geq\frac{1}{2}$ | $\geq\frac{1}{2}$ | $\geq\left\lvert\frac{1}{2}-\frac{1}{i}\right\rvert$ |

So, for every $\zeta$, every $\varsigma$ and every $\wedge$, $R(\zeta,\varsigma,\wedge) \geq 0$, $L(\zeta,\varsigma,\wedge) \leq 0$ and $J(\zeta,\varsigma,\wedge) \leq 0$. Then, all conditions of Theorem 4.3 are satisfied. We give now an example which illustrates Theorem 4.7.

**Example 5.2:** Let $(\Omega,\mathfrak{B},\mathfrak{D},\mathfrak{L},*,\lozenge)$, be the NSMS in Example 2.5, $\Theta(\wedge) = 1$,

$$\phi(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6) = \tau_1 - \min\{\tau_2,\tau_3,\tau_4,\tau_5,\tau_6\},$$

$$\psi(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6) = \tau_1 - \max\{\tau_2,\tau_3,\tau_4,\tau_5,\tau_6\},$$

$$\xi(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6) = \tau_1 - \max\{\tau_2,\tau_3,\tau_4,\tau_5,\tau_6\}.$$

Let $A, A_{i\in\mathfrak{D}^*}, S$, and $T$ be self mapping of $\Omega$ such that

$$A\zeta=\begin{cases}\frac{5}{6} & \text{if } \zeta\in\left[0,\frac{1}{2}\right[\\ \frac{1}{2} & \text{if } \zeta\in\left[\frac{1}{2},1\right]\end{cases},\qquad S\zeta=\begin{cases}\frac{1}{2} & \text{if } \zeta\in\left[0,\frac{1}{2}\right[\\ \frac{1}{2} & \text{if } \zeta=\frac{1}{2},\\ \frac{1}{4} & \text{if } \zeta\in\left[\frac{1}{2},1\right[\end{cases}$$

for all $i\in\mathfrak{D}^*$,

$$A_i\varsigma=\begin{cases}\frac{3}{4} & \text{if } \varsigma\in\left[0,\frac{1}{2}\right[\\ \frac{1}{2} & \text{if } \varsigma=\frac{1}{2}\\ \frac{2}{7} & \text{if } \varsigma\in\left[\frac{1}{2},1\right[\end{cases},\qquad T\varsigma=\begin{cases}\frac{4}{5} & \text{if } \varsigma\in\left[0,\frac{1}{2}\right[\\ \frac{1}{2} & \text{if } \varsigma\in\left[\frac{1}{2},1\right]\end{cases}.$$

If, we define $R,L,P,J,U$ and $Q$ as in the previous example, we have, except when the values are equal in which case, as indicated by $*$, $\mathfrak{B}$ takes the value 1 and $\mathfrak{D},\mathfrak{L}$ are 0,

$$2P(\zeta,\varsigma,\wedge)=\min\left\{\begin{matrix}\max\{S\zeta,T\varsigma\},\max\{A\zeta,S\zeta\},\\ \max\{A_i\varsigma,T\varsigma\},\max\{A\zeta,T\varsigma\},\max\{S\zeta,A_i\varsigma\}\end{matrix}\right\},$$

$$2Q(\zeta,\varsigma,\wedge)=\max\left\{\begin{matrix}\min\{S\zeta,T\varsigma\},\min\{A\zeta,S\zeta\},\min\{A_i\varsigma,T\varsigma\},\\ \min\{A\zeta,T\varsigma\},\min\{S\zeta,A_i\varsigma\}\end{matrix}\right\},$$

$$2U(\zeta,\varsigma,\wedge)=\max\left\{\begin{matrix}\min\{S\zeta,T\varsigma\},\min\{A\zeta,S\zeta\},\min\{A_i\varsigma,T\varsigma\},\\ \min\{A\zeta,T\varsigma\},\min\{S\zeta,A_i\varsigma\}\end{matrix}\right\}.$$

$$R(\zeta,\varsigma,\wedge)=\frac{1}{2}\max\{A\zeta,A_i\varsigma\}-P(\zeta,\varsigma,\wedge),$$

$$L(\zeta,\varsigma,\wedge)=\frac{1}{2}\min\{A\zeta,A_i\varsigma\}-Q(\zeta,\varsigma,\wedge),$$

$$J(\zeta,\varsigma,\wedge)=\frac{1}{2}\min\{A\zeta,A_i\varsigma\}-U(\zeta,\varsigma,\wedge).$$

We have to prove that $R(\zeta,\varsigma,\wedge)\geq 0, L(\zeta,\varsigma,\wedge)\leq 0$ and $J(\zeta,\varsigma,\wedge)\leq 0$, for every $\zeta$ and every $\varsigma$. It is evident that $R(\zeta,\varsigma,\wedge)\geq 0$ $L(\zeta,\varsigma,\wedge)\leq 0$ and $J(\zeta,\varsigma,\wedge)\leq 0$ when $\zeta=\varsigma=\frac{1}{2}$ since,

$$A\zeta=A_i\varsigma=S\zeta=T\varsigma=\frac{1}{2}.$$

We represent the other cases in following Tables **3**-**8**.

**Table 3: Some values connected with Example 5.2.**

| | $\zeta\in\left[0,\frac{1}{2}\right[$ $\varsigma\in\left[0,\frac{1}{2}\right[$ | $\zeta\in\left[0,\frac{1}{2}\right[$ $\varsigma=\frac{1}{2}$ | $\zeta\in\left[0,\frac{1}{2}\right[$ $\varsigma\in\left]\frac{1}{2},1\right]$ | $\zeta=\frac{1}{2}$ $\varsigma\in\left[0,\frac{1}{2}\right[$ |
|---|---|---|---|---|
| $\max\{A\zeta,A_i\varsigma\}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{3}{4}$ |
| $\max\{S\zeta,T\varsigma\}$ | $\frac{4}{5}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{4}{5}$ |
| $\max\{A\zeta,S\zeta\}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $*$ |
| $\max\{A_i\varsigma,T\varsigma\}$ | $\frac{4}{5}$ | $*$ | $\frac{1}{2}$ | $\frac{4}{5}$ |
| $\max\{A\zeta,T\varsigma\}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{4}{5}$ |
| $\max\{S\zeta,A_i\varsigma\}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{2}{7}+\frac{1}{2i}$ | $\frac{3}{4}$ |

**Table 3 (contd….)**

| | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left[0, \frac{1}{2}\right[$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma = \frac{1}{2}$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ | $\zeta = \frac{1}{2}$ $\varsigma \in \left[0, \frac{1}{2}\right[$ |
|---|---|---|---|---|
| $P(\zeta, \varsigma, \wedge)$ | $\frac{3}{8}$ | $\frac{1}{4}$ | $\frac{1}{7} + \frac{1}{4i}$ | $\frac{3}{8}$ |
| $R(\zeta, \varsigma, \wedge)$ | $\frac{5}{12} - \frac{3}{8} \geq 0$ | $\frac{5}{12} - \frac{1}{4} \geq 0$ | $\frac{5}{12} - \frac{1}{7} - \frac{1}{4i} \geq 0$ | $\frac{3}{8} - \frac{3}{8} \geq 0$ |

### Table 4: Some values connected with Example 5.2.

| | $\zeta = \frac{1}{2}$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma \in \left[0, \frac{1}{2}\right[$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma = \frac{1}{2}$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma = \frac{1}{2}$ |
|---|---|---|---|---|
| $\max\{A\zeta, A_i\varsigma\}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | $*$ | $\frac{1}{2}$ |
| $\max\{S\zeta, T\varsigma\}$ | $*$ | $\frac{4}{5}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\max\{A\zeta, S\zeta\}$ | $*$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\max\{A_i\varsigma, T\varsigma\}$ | $\frac{1}{2}$ | $\frac{4}{5}$ | $*$ | $\frac{1}{2}$ |
| $\max\{A\zeta, T\varsigma\}$ | $*$ | $\frac{4}{5}$ | $*$ | $*$ |
| $\max\{S\zeta, A_i\varsigma\}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{2}{7} + \frac{1}{2i}$ |
| $P(\zeta, \varsigma, \wedge)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{7} + \frac{1}{4i}$ |
| $R(\zeta, \varsigma, \wedge)$ | $\frac{1}{4} - \frac{1}{4} \geq 0$ | $\frac{3}{8} - \frac{1}{4} \geq 0$ | $1 - \frac{1}{4} \geq 0$ | $\frac{1}{4} - \frac{1}{7} - \frac{1}{4i} \geq 0$ |

### Table 5: Some values connected with Example 5.2.

| | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left[0, \frac{1}{2}\right[$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma = \frac{1}{2}$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ | $\zeta = \frac{1}{2}$ $\varsigma \in \left[0, \frac{1}{2}\right[$ |
|---|---|---|---|---|
| $\max\{A\zeta, A_i\varsigma\}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{1}{2}$ |
| $\max\{S\zeta, T\varsigma\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{2}$ |
| $\max\{A\zeta, S\zeta\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $*$ |
| $\max\{A_i\varsigma, T\varsigma\}$ | $\frac{3}{4}$ | $*$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{3}{4}$ |
| $\max\{A\zeta, T\varsigma\}$ | $\frac{4}{5}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\max\{S\zeta, A_i\varsigma\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{2}$ |
| $Q(\zeta, \varsigma, \wedge)$ | $\frac{2}{5}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{8}$ |
| $L(\zeta, \varsigma, \wedge)$ | $\frac{3}{8} - \frac{1}{5} \leq 0$ | $\frac{1}{4} \leq 0$ | $\frac{1}{7} + \frac{1}{4i} - \frac{1}{4} \leq 0$ | $\frac{1}{4} - \frac{3}{8} \leq 0$ |

**Table 6: Some values connected with Example 5.2.**

| | $\zeta = \frac{1}{2}$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma \in \left[0, \frac{1}{2}\right[$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma = \frac{1}{2}$ | $\zeta \in \left]\frac{1}{2}, 1\right]$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ |
|---|---|---|---|---|
| $\max\{A\zeta, A_i\varsigma\}$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{1}{2}$ | * | $\frac{2}{7} + \frac{1}{2i}$ |
| $\max\{S\zeta, T\varsigma\}$ | * | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $\max\{A\zeta, S\zeta\}$ | * | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $\max\{A_i\varsigma, T\varsigma\}$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{3}{4}$ | * | $\frac{2}{7} + \frac{1}{2i}$ |
| $\max\{A\zeta, T\varsigma\}$ | * | $\frac{1}{2}$ | * | * |
| $\max\{S\zeta, A_i\varsigma\}$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $Q(\zeta, \varsigma, \wedge)$ | $\frac{1}{7} + \frac{1}{4i}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{1}{7} - \frac{1}{4i}$ |
| $L(\zeta, \varsigma, \wedge)$ | $\frac{1}{7} + \frac{1}{4i} - \frac{1}{7} - \frac{1}{4i} \leq 0$ | $\frac{1}{4} - \frac{3}{8} \leq 0$ | $0 - \frac{1}{4} \leq 0$ | $\frac{1}{7} + \frac{1}{4i} - \frac{1}{7} - \frac{1}{4i} \leq 0$ |

**Table 7: Some values connected with Example 5.2.**

| | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left[0, \frac{1}{2}\right[$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma = \frac{1}{2}$ | $\zeta \in \left[0, \frac{1}{2}\right[$ $\varsigma \in \left]\frac{1}{2}, 1\right]$ | $\zeta = \frac{1}{2}$ $\varsigma \in \left[0, \frac{1}{2}\right[$ |
|---|---|---|---|---|
| $\max\{A\zeta, A_i\varsigma\}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{1}{2}$ |
| $\max\{S\zeta, T\varsigma\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{2}$ |
| $\max\{A\zeta, S\zeta\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | * |
| $\max\{A_i\varsigma, T\varsigma\}$ | $\frac{3}{4}$ | * | $\frac{2}{7} + \frac{1}{2i}$ | $\frac{3}{4}$ |
| $\max\{A\zeta, T\varsigma\}$ | $\frac{4}{5}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\max\{S\zeta, A_i\varsigma\}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{2}$ |
| $U(\zeta, \varsigma, \wedge)$ | $\frac{2}{5}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{8}$ |
| $J(\zeta, \varsigma, \wedge)$ | $\frac{3}{8} - \frac{1}{5} \leq 0$ | $\frac{1}{4} \leq 0$ | $\frac{1}{7} + \frac{1}{4i} - \frac{1}{4} \leq 0$ | $\frac{1}{4} - \frac{3}{8} \leq 0$ |

**Table 8: Some values connected with Example 5.2.**

| | $\zeta=\frac{1}{2}$ $\varsigma\in\left]\frac{1}{2},1\right]$ | $\zeta\in\left]\frac{1}{2},1\right]$ $\varsigma\in\left[0,\frac{1}{2}\right[$ | $\zeta\in\left]\frac{1}{2},1\right]$ $\varsigma=\frac{1}{2}$ | $\zeta\in\left]\frac{1}{2},1\right]$ $\varsigma\in\left]\frac{1}{2},1\right]$ |
|---|---|---|---|---|
| $\max\{A\zeta,A_i\varsigma\}$ | $\frac{2}{7}+\frac{1}{2i}$ | $\frac{1}{2}$ | * | $\frac{2}{7}+\frac{1}{2i}$ |
| $\max\{S\zeta,T\varsigma\}$ | * | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $\max\{A\zeta,S\zeta\}$ | * | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $\max\{A_i\varsigma,T\varsigma\}$ | $\frac{2}{7}+\frac{1}{2i}$ | $\frac{3}{4}$ | * | $\frac{2}{7}+\frac{1}{2i}$ |
| $\max\{A\zeta,T\varsigma\}$ | * | $\frac{1}{2}$ | * | * |
| $\max\{S\zeta,A_i\varsigma\}$ | $\frac{2}{7}+\frac{1}{2i}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $U(\zeta,\varsigma,\wedge)$ | $\frac{1}{7}+\frac{1}{4i}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{1}{7}-\frac{1}{4i}$ |
| $J(\zeta,\varsigma,\wedge)$ | $\frac{1}{7}+\frac{1}{4i}-\frac{1}{7}-\frac{1}{4i}\leq 0$ | $\frac{1}{4}-\frac{3}{8}\leq 0$ | $0-\frac{1}{4}\leq 0$ | $\frac{1}{7}+\frac{1}{4i}-\frac{1}{7}-\frac{1}{4i}\leq 0$ |

Theorem 4.7 is explained by this example because other theorem's assumptions are met. As specific examples of our earlier theorems, we now mention a few results that have already been published. The subsequent theorem enhances Theorem 3 of (Djoudi and Aliouche [30]). Since this result only applies to symmetric spaces, no restrictions on the ranges of $A, B, S$, and $T$ are made.

Additionally, the upper semi continuity and non-decrease of $\psi$ hypotheses are dropped, and the weak compatibility of the pairs $(A,S)$ and $(B,T)$ is replaced by the occasionally weak computability.

**Theorem 5.3:** Suppose $(\Omega,\mathfrak{B},\mathfrak{D},\mathfrak{L},*,\Diamond)$ be a NSMS and $A,B,S$ and $T$ be self- mappings of $\Omega$ which fulfil the following axioms, for every $\zeta$ and every $\varsigma\in\Omega$ and $\wedge>0$

$$\left(\int_0^{\mathfrak{B}(A\zeta,B\varsigma,\wedge)}\phi(\tau)d\tau\right)^p \geq \phi\left[(1-a)\left(\min\left\{\begin{array}{c} a\left(\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p+\left(\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p, \\ \left(\int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)}\phi(\tau)d\tau\right)^p,\left(\int_0^{\mathfrak{B}(B\varsigma,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p, \\ \left(\int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)}\phi(\tau)d\tau\int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^{\frac{p}{2}}, \\ \left(\int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\int_0^{\mathfrak{B}(S\zeta,B\varsigma,\wedge)}\phi(\tau)d\tau\right)^{\frac{p}{2}} \end{array}\right\}\right)\right],$$

and

$$\left(\int_0^{\mathfrak{D}(A\zeta,B\varsigma,\wedge)}\phi(\tau)d\tau\right)^p \leq \psi\left[(1-a)\left(\max\left\{\begin{array}{c} a\left(\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p+\left(\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p, \\ \left(\int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)}\phi(\tau)d\tau\right)^p,\left(\int_0^{\mathfrak{D}(B\varsigma,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^p, \\ \left(\int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)}\phi(\tau)d\tau\int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\right)^{\frac{p}{2}}, \\ \left(\int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)}\phi(\tau)d\tau\int_0^{\mathfrak{D}(S\zeta,B\varsigma,\wedge)}\phi(\tau)d\tau\right)^{\frac{p}{2}} \end{array}\right\}\right)\right],$$

$$\left(\int_0^{\mathfrak{L}(A\zeta,B\varsigma,\Lambda)}\phi(\tau)d\tau\right)^p\le\xi\left[(1-a)\left(\max\left\{\begin{array}{c}a\left(\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\Lambda)}\phi(\tau)d\tau\right)^p+\left(\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\Lambda)}\phi(\tau)d\tau\right)^p,\\ \left(\int_0^{\mathfrak{L}(A\zeta,S\zeta,\Lambda)}\phi(\tau)d\tau\right)^p,\left(\int_0^{\mathfrak{L}(B\varsigma,T\varsigma,\Lambda)}\phi(\tau)d\tau\right)^p,\\ \left(\int_0^{\mathfrak{L}(A\zeta,S\zeta,\Lambda)}\phi(\tau)d\tau\int_0^{\mathfrak{L}(A\zeta,T\varsigma,\Lambda)}\phi(\tau)d\tau\right)^{\frac{p}{2}},\\ \left(\int_0^{\mathfrak{L}(A\zeta,T\varsigma,\Lambda)}\phi(\tau)d\tau\int_0^{\mathfrak{L}(S\zeta,B\varsigma,\Lambda)}\phi(\tau)d\tau\right)^{\frac{p}{2}}\end{array}\right\}\right)\right].$$

Where $\phi,\psi,\xi: R_+\to R_+$ satisfy $\phi(\Lambda)>\Lambda,\psi(\Lambda)<\Lambda$ and $\xi(\Lambda)<\Lambda$ for every $\Lambda>0, a\in[0,1[$ and $\phi: R_+\to R_+$ is integrable function locally which accomplish $\int_\delta^\varepsilon\Theta(\Lambda)d\Lambda>0$, for each $0<\delta<\epsilon$. $A,B,S$ and $T$ have a unique common FP in $\Omega$, if pairs $(A,S)$ and $(B,T)$ are occasionally weakly compatible.

**Proof:** We define $F,G,H: R_+^6\to R_+$ respectively by

$$F(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\Lambda_5,\Lambda_6)=\Lambda_1^p-\phi\left(a\Lambda_1^p+(1-a)\min\left\{\Lambda_2^p,\Lambda_3^p,\Lambda_4^p,(\Lambda_3,\Lambda_5)^{\frac{p}{2}},(\Lambda_5,\Lambda_6)^{\frac{p}{2}}\right\}\right),$$

$$G(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\Lambda_5,\Lambda_6)=\Lambda_1^p-\psi\left(a\Lambda_1^p+(1-a)\max\left\{\Lambda_2^p,\Lambda_3^p,\Lambda_4^p,(\Lambda_3,\Lambda_5)^{\frac{p}{2}},(\Lambda_5,\Lambda_6)^{\frac{p}{2}}\right\}\right),$$

$$H(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\Lambda_5,\Lambda_6)=\Lambda_1^p-\xi\left(a\Lambda_1^p+(1-a)\max\left\{\Lambda_2^p,\Lambda_3^p,\Lambda_4^p,(\Lambda_3,\Lambda_5)^{\frac{p}{2}},(\Lambda_5,\Lambda_6)^{\frac{p}{2}}\right\}\right),$$

$$F\left(\int_0^u\Theta(\Lambda)\,d\Lambda,\int_0^u\Theta(\Lambda)\,d\Lambda,\int_0^v\Theta(\Lambda)\,d\Lambda,\int_0^v\Theta(\Lambda)\,d\Lambda,\int_0^u\Theta(\Lambda)\,d\Lambda,\int_0^u\Theta(\Lambda)\,d\Lambda\right)\ge0,$$

is written

$$\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p-\phi\left[a\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p+(1-a)\min\left\{\begin{array}{c}\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^p\\ \left(\int_0^u\Theta(\Lambda)\,d\Lambda\int_0^v\Theta(\Lambda)\,d\Lambda\right)^{\frac{p}{2}}\end{array}\right\}\right]\ge0,$$

that is

$$\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p\ge\phi\left[a\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p+(1-a)\min\left\{\begin{array}{c}\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^p\\ (\int_0^u\Theta(\Lambda)\,d\Lambda\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^{\frac{p}{2}}\end{array}\right\}\right].$$

Since $\Theta(\Lambda)>\Lambda$ and $a\in[0,1[$, we obtain

$$\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p\ge\ a\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p+(1-a)\min\left\{\begin{array}{c}\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^p\\ \left(\int_0^u\Theta(\Lambda)\,d\Lambda\int_0^v\Theta(\Lambda)\,d\Lambda\right)^{\frac{p}{2}}\end{array}\right\},$$

and

$$\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p\ge\min\left\{\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^u\Theta(\Lambda)\,d\Lambda\right)^p,\left(\int_0^v\Theta(\Lambda)\,d\Lambda\right)^{\frac{p}{2}}\right\}.$$

Which implies

$$\int_0^u \Theta(\wedge)\, d\wedge \geq \int_0^v \Theta(\wedge)\, d\wedge.$$

So, $F$ satisfies $(\phi_3)$ similarly, since $\psi(\wedge) < \wedge$, we show that

$$G\left(\int_0^u \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge, \int_0^v \Theta(\wedge)\, d\wedge, \int_0^v \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge\right) \leq 0$$

this implies that

$$\int_0^u \Theta(\wedge)\, d\wedge \leq \int_0^v \Theta(\wedge)\, d\wedge.$$

Then $G$ satisfies $(\xi_3)$ and since $\psi(\wedge) < \wedge$, we show that

$$H\left(\int_0^u \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge, \int_0^v \Theta(\wedge)\, d\wedge, \int_0^v \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge, \int_0^u \Theta(\wedge)\, d\wedge\right) \leq 0,$$

this implies that

$$\int_0^u \Theta(\wedge)\, d\wedge \leq \int_0^v \Theta(\wedge)\, d\wedge.$$

Then $H$ satisfies $(\xi_3)$ and by Theorem 4.8, proof is finished.

If we put $p = 1$, we can give the suitable theorem which derived Theorems 3.1 and 3.2 of (Kumar *et al.* [14]) Theorem 2 of (Aliouche [24]), Theorem 5 of (Popa, 1999), Theorem 3.1 of (Aliouche and Popa [31]) and is an improved variant of Theorem 1 of (Aliouche [24]) since, in it, about the range of the maps $A, B, S$ and $T$ there is no hypotheses of inclusion and also not required the property (E.A). If in the following theorem, we put $\Theta(\tau) = 1$ for every $\tau$, we obtain an improvement of Theorem 2 of (Aamri & El Moutawakil 2002).

**Theorem 5.4:** Assume the NSMS $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond)$ and $A, B, S$ and $T$ be self- mappings of $\Omega$ which fulfil the following axioms, for every $\zeta$ and every $\varsigma \in \Omega$, and every $\wedge > 0$,

$$\int_0^{\mathfrak{B}(A\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau \geq \phi\left(\min\left\{\begin{array}{c}\int_0^{\mathfrak{B}(S\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{B}(A\zeta,S\zeta,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{B}(B\varsigma,T\varsigma,\wedge)} \phi(\tau)d\tau, \\ \int_0^{\mathfrak{B}(A\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{B}(S\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau\}\end{array}\right.\right),$$

$$\int_0^{\mathfrak{D}(A\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau \leq \psi\left(\max\left\{\begin{array}{c}\int_0^{\mathfrak{D}(S\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{D}(A\zeta,S\zeta,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{D}(B\varsigma,T\varsigma,\wedge)} \phi(\tau)d\tau, \\ \int_0^{\mathfrak{D}(A\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{D}(S\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau\}\end{array}\right.\right),$$

and

$$\int_0^{\mathfrak{L}(A\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau \leq \xi\left(\max\left\{\begin{array}{c}\int_0^{\mathfrak{L}(S\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{L}(A\zeta,S\zeta,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{L}(B\varsigma,T\varsigma,\wedge)} \phi(\tau)d\tau, \\ \int_0^{\mathfrak{L}(A\zeta,T\varsigma,\wedge)} \phi(\tau)d\tau, \int_0^{\mathfrak{L}(S\zeta,B\varsigma,\wedge)} \phi(\tau)d\tau\}\end{array}\right.\right),$$

where $\phi, \psi: R_+ \to R_+$ satisfy $\phi(\wedge) > \wedge$ and $\psi(\wedge) < \wedge$ for every $\wedge > 0$ and $\phi: R_+ \to R_+$ is locally integrable function, which satisfies $\int_\delta^\varepsilon \Theta(\wedge)d\wedge > 0$, for each $0 < \delta < \in$. $A, B, S$ and $T$ have a specific common FP in $\Omega$, if pairs $(A, S)$ and $(B, T)$ are occasionally weakly compatible.

**Proof:** Same as to the preceding theorem by defining mappings $F, G, H : R^6 \to R$ as

$$F(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \phi(\min\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\}),$$

$$G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \psi(\max\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\}),$$

and

$$H(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6) = \Lambda_1 - \xi(\max\{\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6\}).$$

As before, we show that $F$ and $G$ satisfy respectively $(\phi_3), (\psi_3)$ and $(\xi_3)$. And Theorem 4.8 allows us to conclude and to finish the proof.

Now we select $\theta(\tau) = 1$ for every $\tau > 0$ in the preceding theorems and we express by individually $F$ and $G$ are the sets of all continuous functions $F, G: R^6 \to R$ satisfying the following axioms:

$$F(u, v, u, v, u, v) \geq 0, \tag{38}$$

or

$$F(u, v, v, u, v, u) \geq 0, \tag{39}$$

or

$$F(u, u, v, v, u, u) \geq 0. \tag{40}$$

Implies $u \geq v$, and respectively,

$$G(u, v, u, v, u, v) \leq 0, \tag{41}$$

or

$$G(u, v, v, u, v, u) \leq 0, \tag{42}$$

or

$$G(u, u, v, v, u, u) \leq 0. \tag{43}$$

Implies $u \leq v$ and

$$H(u, v, u, v, u, v) \leq 0, \tag{44}$$

or

$$H(u, v, v, u, v, u) \leq 0, \tag{45}$$

or

$$H(u, u, v, v, u, u) \leq 0. \tag{46}$$

Implies $u \leq v$.

Then, we give following theorems.

**Theorem 5.5:** Let $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \Diamond)$ be a NSMS for $\Omega$ which satisfies (W4), (H.E), (CE.1), (CE.2) and $A, (A_i)_{i \in I}, S$ and $T$ be self mappings of $\Omega$ satisfying $A\Omega \subset T\Omega$ and $A_i\Omega \subseteq S\Omega$ for every $i \in I$ and

$$F\begin{pmatrix}\mathfrak{B}(A\zeta, A_i\varsigma, \wedge), \mathfrak{B}(S\zeta, T\varsigma, \wedge), \mathfrak{B}(A\zeta, S\zeta, \wedge)\\ \mathfrak{B}(A_i\varsigma, T\varsigma, \wedge), \mathfrak{B}(A\zeta, T\varsigma, \wedge), \mathfrak{B}(S\zeta, A_i\varsigma, \wedge)\end{pmatrix} \geq 0, \tag{47}$$

$$G\begin{pmatrix}\mathfrak{D}(A\zeta, A_i\varsigma, \wedge), \mathfrak{D}(S\zeta, T\varsigma, \wedge), \mathfrak{D}(A\zeta, S\zeta, \wedge)\\ \mathfrak{D}(A_i\varsigma, T\varsigma, \wedge), \mathfrak{D}(A\zeta, T\varsigma, \wedge), \mathfrak{D}(S\zeta, A_i\varsigma, \wedge)\end{pmatrix} \leq 0, \tag{48}$$

$$H\begin{pmatrix}\mathfrak{L}(A\zeta, A_i\varsigma, \wedge), \mathfrak{L}(S\zeta, T\varsigma, \wedge), \mathfrak{L}(A\zeta, S\zeta, \wedge)\\ \mathfrak{L}(A_i\varsigma, T\varsigma, \wedge), \mathfrak{L}(A\zeta, T\varsigma, \wedge), \mathfrak{L}(S\zeta, A_i\varsigma, \wedge)\end{pmatrix} \leq 0 \tag{49}$$

for every $\zeta$, every $\varsigma$ in $\Omega$ and every $\wedge > 0$, where $F \in \mathcal{F}, G \in \mathfrak{H}$ and $H \in \mathcal{H}$. Suppose:

(I) $(A, S)$ or, for all $i$, $(A_i, T)$ satisfies assumptions (E.A),

(II) $(A, S)$ and for some $k \in I$, $(A_k, T)$ are weakly compatible.

Whenever one of the subspaces $A\Omega, S\Omega, A_i\Omega$ and $T\Omega$ of $\Omega$ is closed, then $A, S, T$ and $A_i$, for all $i \in I$, have a unique common FP in $\Omega$.

**Theorem 5.6:** Suppose $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ be a NSMS. Let $F, G, H: R_+^6 \to R$ satisfying $(40)$, $(43)$, and $(46)$ respectively and $(A_i)_{i\in I}$, the self-mappings of $\Omega$ is $S$ and $T$ which satisfy $(47), (48)$, and $(49)$ respectively and fulfill the conditions which are given below:

(I) for some $k \in I$, $(A_k, S)$ is pair which is sometimes weakly compatible,

(II) there exists $v \in \cap i \in I\ C(A_i, T)$ such that $A_iTv = TA_iv$ for all $i \neq k$,

where $C(A_i, T)$ is the set of points of coincidence of $A_i$ and $T$. Then $A, S, T$ and $A_i$, for all $i \in I$, have a unique common FP in $\Omega$.

For four applications with its specific case, the consecutive theorems, we have improved Theorems 2.1 and 2.2 of (Aamri and El Moutawakil [27]), Theorem 2.8 of (Merghadi and Godet-Thobie [25]) and Theorem 4.1 of (Pathak *et al.* [29]) among others.

**Theorem 5.7:** Suppose $(\Omega, \mathfrak{B}, \mathfrak{D}, \mathfrak{L}, *, \lozenge)$ be a NSMS and $A, B, S$ and $T$ be sel-mappings of $\Omega$ satisfying, for all $\zeta$ and $\varsigma$ in $\Omega$ and every $\wedge > 0$,

$$F\begin{pmatrix}\mathfrak{B}(A\zeta, B\varsigma, \wedge), \mathfrak{B}(S\zeta, T\varsigma, \wedge), \mathfrak{B}(A\zeta, S\zeta, \wedge)\\ \mathfrak{B}(B\varsigma, T\varsigma, \wedge), \mathfrak{B}(A\zeta, T\varsigma, \wedge), \mathfrak{B}(S\zeta, B\varsigma, \wedge)\end{pmatrix} \geq 0,$$

$$G\begin{pmatrix}\mathfrak{D}(A\zeta, B\varsigma, \wedge), \mathfrak{D}(S\zeta, T\varsigma, \wedge), \mathfrak{D}(A\zeta, S\zeta, \wedge)\\ \mathfrak{D}(B\varsigma, T\varsigma, \wedge), \mathfrak{D}(A\zeta, T\varsigma, \wedge), \mathfrak{D}(S\zeta, B\varsigma, \wedge)\end{pmatrix} \leq 0,$$

$$H\begin{pmatrix}\mathfrak{L}(A\zeta, B\varsigma, \wedge), \mathfrak{L}(S\zeta, T\varsigma, \wedge), \mathfrak{L}(A\zeta, S\zeta, \wedge)\\ \mathfrak{L}(B\varsigma, T\varsigma, \wedge), \mathfrak{L}(A\zeta, T\varsigma, \wedge), \mathfrak{L}(S\zeta, B\varsigma, \wedge)\end{pmatrix} \leq 0,$$

where $F, G, H: R_+^6 \to R$ satisfies $(40), (43)$, and $(46)$ respectively. $A, B, S$ and $T$ have a unique common FP in case of $(A, S)$ and $(B, T)$ are occasionally weakly compatible.

## 6. Conclusion

In this manuscript, the authors introduced the concept of NSMSs and proved several common fixed point results for four mappings by utilizing the locally integrable function and occasionally weakly compatible mappings. These results generalized several fixed point results presented in [4, 7, 12, 14, 17, 24-27, 29-31, 33]. To show the validity of these results, the authors provided some non-trivial examples. This work can be extended in the framework of more generalized spaces and by increasing the number of mappings.

## Conflicts of Interest

The authors declare no conflict of interest.

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## Availability of Data and Materials

Data will be available on demand from corresponding author.

## Authors' Contributions

All authors contributed equally in this manuscript.

## References

[1] Zade LA. Fuzzy sets. Inform Control. 1965; 8(3): 338-53. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Kramosil I, Michálek J. Fuzzy metrics and statistical metric spaces. Kybernetika. 1975; 11(5): 336-44.

[3] George A, Veeramani P. On some result in fuzzy metric space. Fuzzy Sets Syst. 1994; 64: 395-9. https://doi.org/10.1016/0165-0114(94)90162-7

[4] Grabiec M. Fixed points in fuzzy metric spaces. Fuzzy Sets Syst. 1988; 27(3): 385-9. https://doi.org/10.1016/0165-0114(88)90064-4

[5] Hu XQ. Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces. Fixed Point Theory Appl. 2011; 2011: 1-14. https://doi.org/10.1155/2011/363716

[6] Deng Z. Fuzzy pseudo-metric spaces. J Math Analy Appl. 1982; 86(1): 74-95. https://doi.org/10.1016/0022-247X(82)90255-4

[7] Cho YJ, Sedghi S, Shobe N. Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces. Chaos Solitons Fractals. 2009; 39(5): 2233-44. https://doi.org/10.1016/j.chaos.2007.06.108

[8] Javed K, Uddin F, Ishtiaq U, Park C, Arshad M. On ordered theoretic controlled fuzzy metric spaces. Int J Nonlinear Analy Appl. 2023; 14(4): 1-14. https://doi.org/0.22075/IJNAA.2023.28472.3902

[9] Atanassov KT. Intuitionistic Fuzzy Sets. In: Atanassov KT, Ed., Intuitionistic Fuzzy Sets. Studies in Fuzziness and Soft Computing, vol 35. Heidelberg: Physica; 1999. https://doi.org/10.1007/978-3-7908-1870-3_1

[10] Park JH. Intuitionistic fuzzy metric spaces. Chaos Solitons Fractals. 2004; 22(5): 1039-46. https://doi.org/10.1016/j.chaos.2004.02.051

[11] Alaca C, Turkoglu D, Yildiz C. Fixed points in intuitionistic fuzzy metric spaces. Chaos Solitons Fractals. 2006; 29(5): 1073-8. https://doi.org/10.1016/j.chaos.2005.08.066

[12] Sharma V, Joshi MC, Kumar S. Fixed point theorems for contractive and weakly compatible mapping in complete intuitionistic fuzzy metric space. J Anal. 2021; 29: 1375-90. https://doi.org/10.1007/s41478-021-00317-6

[13] Davvaz B, Jan N, Mahmood T, Ullah K. Intuitionistic fuzzy graphs of $n^{th}$ type with applications. J Intell Fuzzy Syst. 2019; 36(4): 3923-32. https://doi.org/10.3233/JIFS-181123

[14] Kumar S, Vats RK, Singh V, Garg SK. some common fixed point theorems in intuitionistic fuzzy metric spaces. Int J Math Anal. 2010; 4(26): 1255-70.

[15] Saadati R, Park JH. On the intuitionistic fuzzy topological spaces. Chaos Solitons Fractals. 2005: 27(2): 331-44. https://doi.org/10.1016/j.chaos.2005.03.019

[16] Wilson WA. On quasi-metric spaces. Am J Math. 1931; 53(3): 675-84. https://doi.org/10.2307/2371174

[17] Smarandache F. Neutrosophic sets, a generalization of the intuitionistic fuzzy sets. Int J Pure Appl Math. 2005; 24: 287-97.

[18] Kirişci M, Şimşek N. Neutrosophic metric spaces. Math Sci. 2020; 14: 241-8. https://doi.org/10.1007/s40096-020-00335-8

[19] Aanchal GV. Fixed Point Theorems in Neutrosophic Soft Metric Space. In: Sahni M, Merigó JM, Hussain W, León-Castro E, Verma RK, Sahni R. Eds. Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy. Advances in Intelligent Systems and Computing, vol. 1440. Singapore: Springer; 2023. https://doi.org/10.1007/978-981-19-9906-2_3

[20] Ishtiaq U, Javed K, Uddin F, Sen MDL, Ahmed K, Ali, MU. Fixed point results in orthogonal neutrosophic metric spaces. Complexity. 2021; 1-18. https://doi.org/10.1155/2021/2809657

[21] Rajan SS, Jeyaraman M, Smarandache F. Fixed point results for contraction theorems in neutrosophic metric spaces. Neutrosophic Sets Syst. 2020; 36: 309-18. https://doi.org/10.5281/zenodo.4065458

[22] Uddin F, Ishtiaq U, Saleem N, Ahmad K, Jarad F. Fixed point theorems for controlled neutrosophic metric-like spaces. AIMS Math. 2022; 7(12): 20711-39. https://doi.org/10.3934/math.20221135

[23] Ali U, Alyousef HA, Ishtiaq U, Ahmed K, Ali S. Solving nonlinear fractional differential equations for contractive and weakly compatible mappings in neutrosophic metric spaces. J Funct Spaces. 2023; Article ID 9868214. https://doi.org/10.1155/2023/9868214

[24] Aliouche A. A common fixed-point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type. J Math Analy Appl. 2006; 322(2): 796-802. https://doi.org/10.1016/j.jmaa.2005.09.068

[25] Merghadi F, Godet-Thobie C. Common fixed point theorems under contractive conditions of integral type in symmetric spaces. Demonstr Math. 2013; 46(4): 757-80. https://doi.org/10.1515/dema-2013-0489

[26] Sastry KPR, Murthy IK. Common fixed points of two partially commuting tangential selfmaps on a metric space. J Math Analy Appl. 2000; 250(2): 731-4. https://doi.org/10.1006/jmaa.2000.7082

[27] Aamri M, El Moutawakil D. Some new common fixed point theorems under strict contractive conditions. J Math Analy Appl 2002; 270(1): 181-8. https://doi.org/10.1016/S0022-247X(02)00059-8

[28] Al-Thagafi MA, Shahzad N. Generalized I-nonexpansive selfmaps and invariant approximations. Acta Math Sin. 2008; 24: 867-76. https://doi.org/10.1007/s10114-007-5598-x

[29] Pathak HK, Tiwari R, Khan MS. A common fixed-point theorem satisfying integral type implicit relations. Appl Math E-Notes. 2007; 7: 222-8.

[30] Djoudi A, Aliouche A. Common fixed-point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type. J Math Analy Appl. 2007; 329(1): 31-45. https://doi.org/10.1016/j.jmaa.2006.06.037

[31] Aliouche A, Popa V. Common fixed point theorems for occasionally weakly compatible mappings via implicit relations. Filomat. 2008; 22(2): 99-107.

[32] Godet-Thobie C, Merghadi F. Common fixed point theorems under contractive condition of integral type in intuitionistic fuzzy semi-metric spaces. Demonstratio Mathematica. 2013; 46(4): 757-80. https://doi.org/10.1515/dema-2013-0489

[33] Saleem N, Ishtiaq U, Guran L, Bota MF. On graphical fuzzy metric spaces with application to fractional differential equations. Fractal Fract. 2022; 6(5): 238. https://doi.org/10.3390/fractalfract6050238

[34] Saleem N, Ahmad K, Ishtiaq U, De la Sen M. Multivalued neutrosophic fractals and Hutchinson-Barnsley operator in neutrosophic metric space. Chaos Solitons Fractals. 2023; 172: 113607. https://doi.org/10.1016/j.chaos.2023.113607

[35] Saeed M, Saeed MH, Shafaqat R, Sessa S, Ishtiaq U, Di Martino F. A theoretical development of cubic pythagorean fuzzy soft set with its application in multi-attribute decision making. *Symmetry.* 2022; *14*(12): 2639. https://doi.org/10.3390/sym14122639.

[36] Farhan M, Ishtiaq U, Saeed M, Hussain A, Al Sulami, H. Reich-type and (α, f)-contractions in partially ordered double-controlled metric-type spaces with applications to non-linear fractional differential equations and monotonic iterative method. *Axioms.* 2022; *11*(10): 573. https://doi.org/10.3390/axioms11100573

[37] Uddin F, Ishtiaq U, Javed K, Aiadi SS, Arshad M, Souayah N, Mlaiki N. A new extension to the intuitionistic fuzzy metric-like spaces. Symmetry. 2022; 14(7): 1400. https://doi.org/10.3390/sym14071400

[38] Bulut H, Khalid BJ. Optical soliton solutions of fokas-lenells equation via (m+ 1/g')-expansion method. J Adv Appl Comput Math. 2020; 7: 20-4. https://doi.org/10.15377/2409-5761.2020.07.3

[39] Li S, Yin H. A four step scheme approach to the forward-backward stochastic navier-stokes equations. J Adv Appl Comput Math. 2021; 8: 129-43. https://doi.org/10.15377/2409-5761.2021.08.10

[40] Uddin F, Din M, Ishtiaq U, Sessa S. Perov fixed-point results on F-contraction mappings equipped with binary relation. Mathematics. 2023; 11(1): 238. https://doi.org/10.3390/math11010238

[41] Ishtiaq U, Hussain A, Al Sulami H. Certain new aspects in fuzzy fixed point theory. AIMS Mathematics, 7(5), pp.8558-8573. 202; 7(5): 8558-73. https://doi.org/10.3934/math.2022477

[42] Ishtiaq U, Asif M, Hussain A, Ahmad K, Saleem I, Al Sulami H. Extension of a unique solution in generalized neutrosophic cone metric spaces. *Symmetry.* 2023; *15*(1): 94. https://doi.org/10.3390/sym15010094

[43] Uddin F, Ishtiaq U, Saleem N, Ahmad K, Jarad F. Fixed point theorems for controlled neutrosophic metric-like spaces. AIMS Math. 2022; 7: 20711-39. https://doi.org/10.3934/math.20221135

[44] Younis M, Singh D, Goyal A. A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable. J Fixed Point Theory Appl. 2019; 21: 1-17. https://doi.org/10.1007/s11784-019-0673-3

[45] Younis M, Singh D, Radenović S, Imdad M. Convergence theorems for generalized contractions and applications. Filomat. 2020; 34(3): 945-64. https://doi.org/10.2298/FIL2003945Y

[46] Younis M, Singh D, Asadi M, Joshi V. Results on contractions of Reich type in graphical b-metric spaces with applications. Filomat. 2019; 33(17): 5723-35. https://doi.org/10.2298/FIL1917723Y

[47] Younis M, Singh D, Altun I, Chauhan V. Graphical structure of extended b-metric spaces: an application to the transverse oscillations of a homogeneous bar. Int J Nonlinear Sci Numer Simul. 2020; 23(7-8): 1239-52. https://doi.org/10.1515/ijnsns-2020-0126

[48] Younis M, Singh D, Abdou A.A. A fixed point approach for tuning circuit problem in dislocated b-metric spaces. Math Methods Appl Sci. 2020; 45(4): 2234-53. https://doi.org/10.1002/mma.7922

[49] Younis M, Sing D. On the existence of the solution of Hammerstein integral equations and fractional differential equations. J Appl Math Comput. 2022; 68: 1087-1105. https://doi.org/10.1007/s12190-021-01558-1

[50] Younis M, Singh D, Petruşel A. Applications of graph kannan mappings to the damped spring-mass system and deformation of an elastic beam. Discrete Dyn Nature Soc. 2019; (1): 1315387. https://doi.org/10.1155/2019/1315387

[51] Younis M, Singh D, Gopal D, Goyal A, Rathore MS. On applications of generalized F-contraction to differential equations. Nonlinear Funct Analy Appl. 2019; 24(01): 155-74.

[52] Younis M, Ahmad H, Chen L, Han M. Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations. J Geom Phys 2023; 192: 104955. https://doi.org/10.1016/j.geomphys.2023.104955

[53] Younis M, Singh D, Chen L, Metwali M. A study on the solutions of notable engineering models. Math Model Analy. 2022; 27(3): 492-509. https://doi.org/10.3846/mma.2022.15276

[54] Younis M, Singh D, Shi L. Revisiting graphical rectangular b-metric spaces. Asian Eur J Math. 2022; 15(04): 2250072. https://doi.org/10.1142/S1793557122500723

[55] Karapınar E, Shatanawi W, Mustafa Z. Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces. J Appl Math. 2012; (1): 951912. https://doi.org/10.1155/2012/951912

[56] Abdeljawad T, Karapınar E, Taş K. Existence and uniqueness of a common fixed point on partial metric spaces. Appl Math Lett. 2011; 24(11): 1900-4. https://doi.org/10.1016/j.aml.2011.05.014

[57] Hussain N, Karapınar E, Salimi P, Vetro P. Fixed point results for G m-Meir-Keeler contractive and G-(α, ψ)-Meir-Keeler contractive mappings. Fixed Point Theory Appl. 2013; 34: 1-14. https://doi.org/10.1186/1687-1812-2013-34

[58] Karapınar E, Erhan IM. Fixed point theorems for operators on partial metric spaces. Appl Math Lett. 2011; 24(11): 1894-9. https://doi.org/10.1016/j.aml.2011.05.013

[59] Karapınar E, Cvetković M. An inevitable note on bipolar metric spaces. AIMS Math. 2024; 9(2): 3320-31. https://doi.org/10.3934/math.2024162

[60] Karapınar E, Fulga, A. A fixed point theorem for Proinov mappings with a contractive iterate. Appl Math J Chin Univ. 2023; 38: 403-12. https://doi.org/10.1007/s11766-023-4258-y

[61] Karapinar E, Cvetkovic M. Remarks On Some Generalizations of θ-Contraction. UPB Sci Bull Series A, 2023; 85(2): 31-42.

[62] Karapınar E, Fulga A. Discussions on proinov-C b-contraction mapping on b-metric space. J Funct Spaces. 2023; (1): 1411808. https://doi.org/10.1155/2023/1411808

[63] Cvetković M, Karapinar E, Rakočević V, Yeşilkaya SS. Perov-type results for multivalued mappings. In: Pardalos PM, Rassias TM, Eds., Analysis geometry, nonlinear optimization and applications, World Scientific; 2023, pp. 215-253. https://doi.org/10.1142/9789811261572_0008

[64] Arshad M, Fahimuddin, Shoaib A, Hussain A. Fixed point results for α-ψ-locally graphic contraction in dislocated qusai metric spaces. Math Sci. 2014; 8: 79-85. https://doi.org/10.1007/s40096-014-0132-7

[65] Ali A, Uddin F, Arshad M, Rashid M. Hybrid fixed point results via generalized dynamic process for F-HRS type contractions with application. Physica A: Stat Mech Appl. 2020; 538: 122669. https://doi.org/10.1016/j.physa.2019.122669

[66] Mehmood M, Aydi H, Ali MU, Fahimuddin, Shoaib A, De La Sen M. Solutions of integral equations via fixed-point results on orthogonal gauge structure. Math Probl Eng. 2021; (1): 8387262. https://doi.org/10.1155/2021/8387262

[67] Almalki Y, Din FU, Din M, Ali MU, Jan N. Perov-fixed point theorems on a metric space equipped with ordered theoretic relation. Aims Math. 2022; 11: 20199-212. https://doi.org/10.3934/math.20221105

[68] Ali MU, Din FU. Discussion on α-contractions and related fixed point theorems in Hausdorff b-Gauge Spaces. Jordan J Math Stat. 2017; 10(3): 247-63.

[69] Ali A, Alansari M, Uddin F, Arshad M, Asif A, Basendwah GA. Set-valued SU-type fixed point theorems via gauge function with applications. J Math. 2021; (1): 6612448. https://doi.org/10.1155/2021/6612448

[70] Rasham T, Kutbi MA, Hussain A, Chandok S. Fuzzy dominated nonlinear operators with applications. J Intell Fuzzy Syst. 2024; In press, 1-15. https://doi.org/10.3233/JIFS-238250

[71] Rasham T. Separate families of fuzzy dominated nonlinear operators with applications. J Appl Math Comput. 2024; 1-26. https://doi.org/10.1007/s12190-024-02133-0

[72] Rasham T, Qadir R, Hasan F, Agarwal RP, Shatanawi W. Novel results for separate families of fuzzy-dominated mappings satisfying advanced locally contractions in b-multiplicative metric spaces with applications. J Inequal Appl. 2024; 57: 1-19. https://doi.org/10.1186/s13660-024-03115-3

[73] Rasham T, Mustafa A, Mukheimer A, Nazam M, Shatanawi, W. Novel results for two families of multivalued dominated mappings satisfying generalized nonlinear contractive inequalities and applications. Demonstratio Math. 2024; 57(1): 20230161. https://doi.org/10.1515/dema-2023-0161

[74] Rasham T, Nazam M, Agarwal P, Hussain A, Al Sulmi HH. Existence results for the families of multi-mappings with applications to integral and functional equations. J Inequal Appl. 2023; 82: 1-15. https://doi.org/10.1186/s13660-023-02991-5

[75] Rasham T, Shabbir MS, Nazam M, Musatafa A, Park C. Orbital b-metric spaces and related fixed point results on advanced Nashine–Wardowski–Feng–Liu type contractions with applications. J Inequal Appl. 2023: 69: 1-16. https://doi.org/10.1186/s13660-023-02968-4

[76] Rasham T, Noor N, Safeer M, Agarwal RP, Aydi H, De La Sen M. On dominated multivalued operators involving nonlinear contractions and applications. AIMS Math. 2024; 9(1): 1-21. https://doi.org/10.3934/math.2024001

[77] Rasham T, Saeed F, Agarwal RP, Hussain A, Felhi A. Symmetrical hybrid coupled fuzzy fixed-point results on closed ball in fuzzy metric space with applications. Symmetry. 2023; 15(1): 30. https://doi.org/10.3390/sym15010030

[78] Shoaib A, Mir U. Interpolative multivalued α∗-dominated contractive functions in dislocated b-metric spaces and some fixed point results. Math Sci. 2024; 18: 9-16. https://doi.org/10.1007/s40096-022-00480-2

[79] Ahmad J, Shoaib A, Ayoob I, Mlaiki N. Common fixed points for (κGm)-contractions with applications. AIMS Math. 2024; 9(6): 15949-65. https://doi.org/10.3934/math.2024772

[80] Mehmood M, Shoaib A, Mlaiki N. Fixed point results on triple controlled quasi rectangular metric like spaces. AIMS Math. 2023; 8(5): 10049-66. https://doi.org/10.3934/math.2023509

[81] Shoaib A, Khaliq K. Fixed-point results for generalized contraction in K-sequentially complete ordered dislocated fuzzy quasimetric spaces. Fixed Point Theory Algorithms Sci Eng. 2022; 27: 1-22. https://doi.org/10.1186/s13663-022-00737-4

[82] Shahzad A, Shoaib A, Mlaiki N, Subhi Aiadi S. Results for fuzzy mappings and stability of fuzzy sets with applications. Fractal Fract. 2022; 6(10): 556. https://doi.org/10.3390/fractalfract6100556

[83] Shoaib A, Hassan Z. Results for multivalued mappings for Kannan type contractions in ordered pg-metric spaces. Appl Sci. 2022; 24: 245-60.

[84] Shoaib A, Kumam P, Sitthithakerngkiet K. Interpolative Hardy Roger's type contraction on a closed ball in ordered dislocated metric spaces and some results. AIMS Math. 2022; 7(8): 13821-31. https://doi.org/10.3934/math.2022762

[85] Achtoun Y, Radenović S, Tahiri I, Sefian ML. Exploring multivalued probabilistic ψ-contractions with orbits in b-Menger spaces. Vojnotehnički glasnik/Military Technical Courier. 2024; 72(2): 563-82.

[86] Zoto K, Šešum-Čavić V, Pantović M, Todorčević V, Zoto M, Radenović S. A Unified Approach and Related Fixed-Point Theorems for Suzuki Contractions. Symmetry. 2024; 16(6): 739. https://doi.org/10.3390/sym16060739

[87] Achtoun Y, Radenović S, Tahiri I, Sefian ML. The nonlinear contraction in probabilistic cone b-metric spaces with application to integral equation. Nonlinear Analy: Model Control. 2024; 1-12. https://doi.org/10.15388/namc.2024.29.35180

[88] Achtoun Y, Gardasević-Filipović M, Mitrović S, Radenović S. On Prešić-Type Mappings: Survey. Symmetry, 16(4), p.415. Symmetry. 2024; 16(4): 415. https://doi.org/10.3390/sym16040415

[89] Ozturk V, Radenovic S. Hemi metric spaces and Banach fixed point theorems. Appl Gen Topol. 2024; 25(1): 175-82. https://doi.org/10.4995/agt.2024.19780

[90] Anjum R, Abbas M, Safdar H, Din M, Zhou M, Radenović S. Application to activation functions through fixed-circle problems with symmetric contractions. Symmetry. 2024; 16(1): 69. https://doi.org/10.3390/sym16010069

[91] Moussaoui A, Melliani S, Radenovic S. A nonlinear fuzzy contraction principle via control functions. Filomat. 2024; 6: 1963-72. https://doi.org/10.2298/FIL2406963M

[92] Kadelburg Z, Radenović S. Some new observations on w-distance and F-contractions. Matematički vesnik. 2024; 76(1): 43-55. https://doi.org/10.57016/MV-UxZA2735

[93] Gopal D, Sintunavarat W, Ranadive AS, Shukla S. The investigation of k-fuzzy metric spaces with the first contraction principle in such spaces. Soft Comput. 2023; 27(16): 11081-9. https://doi.org/10.1007/s00500-023-07946-y

[94] Saleh HN, Imdad M, Sintunavarat W. Fixed points which belong to the set of unit values of a suitable function on fuzzy metric spaces. Appl Gen Topol. 2023; 24(1): 9-24. https://doi.org/https://doi.org/10.4995/agt.2023.16924

[95] Turab A, Sintunavarat W. On the solution of the generalized functional equation arising in mathematical psychology and theory of learning approached by the Banach fixed point theorem. Carpathian J Math. 2023; 39(2): 541-51. https://doi.org/10.37193/CJM.2023.02.14

[96] Nithiarayaphaks W, Sintunavarat W. On approximating fixed points of weak enriched contraction mappings via Kirk's iterative algorithm in Banach spaces. Carpathian J Math. 2023; 39(2): 423-32. https://doi.org/10.37193/CJM.2023.02.07

[97] Rouzkard F, Imdad M. Common fixed points for hybrid pair of generalized non-expensive mappings by a three-step iterative scheme. Int J Nonlinear Analy Appl. 2024; 15(3): 91-102. https://doi.org/10.22075/IJNAA.2022.21245.3444

[98] Arif M, Imdad M. Coincidence point results on a metric space endowed with a locally T-transitive binary relation employing comparison functions. Miskolc Math Notes. 2024; 25(1): 63-78. https://doi.org/10.18514/MMN.2024.4114

[99] Arab R, Hazarika B, Imdad M, Das A. Common fixed point theorem of family of contraction maps and its applications in integral equations. Thai J Math. 2023; 21(2): 253-63.

[100] Asim M, Kumar S, Imdad M, George R. C*-algebra valued quasi metric spaces and fixed point results with an application. Appl Gen Topol. 2022; 23(2): 287-301. https://doi.org/10.4995/agt.2022.16783

[101] Maheshwaran K, Hussain RJ, Khan MS, Sessa S. Common Fixed Point Theorems for Mappings Satisfying (EA)-Property on Cone Normed B-Metric Spaces. Int J Analy Appl. 2024; 22: 1-16. https://doi.org/10.28924/2291-8639-22-2024-56

[102] Ali MU, Sessa S, Almalki Y, Alansari M. Fundamental characteristics of the product-operated metric spaces. Axioms. 2024; 13(2): 103. https://doi.org/10.3390/axioms13020103

[103] Asem V, Singh YM, Khan MS, Sessa S. On (α, p)-cyclic contractions and related fixed point theorems. Symmetry. 2023; 15(10): 1826. https://doi.org/10.3390/sym15101826

[104] Furqan S, Saleem N, Sessa S. Fuzzy n− controlled metric space. Int J Analy Appl. 2023; 21: 1-20. https://doi.org/10.28924/2291-8639-21-2023-101

[105] Narzary S, Das D. Singh YM, Khan MS, Sessa S. C*-algebra-valued partial modular metric spaces and some fixed point results. Symmetry. 2023; 15(6): 1135. https://doi.org/10.3390/sym15061135

[106] Zhou M, Secelean NA, Saleem N, Abbas M. Best proximity points for alternative p-contractions. J Inequal Appl. 2024; 4: 1-17. https://doi.org/10.1186/s13660-024-03078-5

[107] Rashid M, Saleem N, Bibi R, George R. Some multidimensional fixed point theorems for nonlinear contractions in C-distance spaces with applications. J Inequal Appl. 2024; 13: 1-17. https://doi.org/10.1186/s13660-024-03079-4

[108] Chand D, Rohen Y, Saleem N, Aphane M, Razzaque A. S-Pata-type contraction: a new approach to fixed-point theory with an application. J Inequal Appl. 2024; 59: 1-16. https://doi.org/10.1186/s13660-024-03136-y

[109] Alam KH, Rohen Y, Saleem N. Aphane M, Rzzaque A. Convergence of Fibonacci–Ishikawa iteration procedure for monotone asymptotically nonexpansive mappings. J Inequal Appl. 2024; 81: (2024). https://doi.org/10.1186/s13660-024-03156-8