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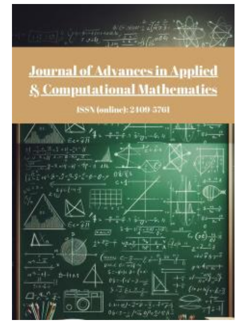

On Continuities via P-Statistical Convergence

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ABSTRACT

This study investigates the concepts of P-statistical continuity (statistical continuity with respect to power series method) and P-statistical ward continuity (statistical ward continuity with respect to power series method) within the framework of power series methods, which extend the scope of statistical convergence beyond classical matrix methods. In the background, the limitations of traditional methods in capturing generalized continuity behaviors are explored and the use of power series as a versatile tool is motivated. Connections between these specialized forms of continuity and standard continuity are established, providing proofs and detailed properties. The results include several foundational theorems characterizing P-statistical continuity and ward continuity under various settings. These findings contribute to a more profound comprehension of continuity concepts within the context of regular summability methods.

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1. Introduction

Continuity is a fundamental concept extending across mathematical analysis, real analysis, functional analysis, and topology. At its core, continuity for a function $f: R \to R$ at a point $x_0^{'} \in R$ is classically equivalent to sequential continuity at that point. The study of functional continuity has a rich history, beginning with a pivotal problem posed by Robbins in 1946 [1]:

"A function $f: R \to R$ which satisfies the property has to be linear".

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f(x_k) = f(x_0') \text{ whenever } \lim_n \frac{1}{n}\sum_{k=1}^{n} x_k = x_0', x_0' \in \mathrm{R},$$

In 1948, Buck published a solution to this problem [2], followed by solutions from five other researchers. This line of inquiry led to similar explorations involving various types of continuity concepts by several authors [3-10]. These studies often concluded that such continuous functions exhibit either linearity or continuity in the conventional sense.

Furthermore, continuity can be characterized by the criterion that the sequence $\{f(x_n)\}$ is a statistical Cauchy sequence whenever $\{x_n\}$ is statistically Cauchy. This perspective has motivated the study of continuity through sequences, where a function's continuity corresponds to its preservation of certain sequence types. Among the significant contributions in this area, Connor and Grosse-Erdmann [11] demonstrated that specific conditions imposed on sequence convergence can reinforce continuity. However, they also identified cases where this dichotomy does not necessarily apply [12-16].

The investigation of various types of continuities has constituted a central focus in mathematical analysis, resulting in a multitude of generalisations and extensions of the classical continuity concept. Among these generalisations, statistical convergence has emerged as a particularly efficacious tool for studying different modes of continuity. This concept, which extends the notion of ordinary convergence, provides a more refined methodology for analysing the behaviour of sequences and functions [17-22].

Although statistical convergence has been extensively studied, its application through power series methods remains underexplored. This paper addresses this gap by introducing and analyzing the concepts of P-statistical continuity (statistical continuity with respect to power series method) and P-statistical ward continuity (statistical ward continuity with respect to power series method) within the framework of power series methods, which extend the scope of statistical convergence beyond classical matrix methods [23-29]. In the background, the limitations of traditional methods in capturing generalized continuity behaviors are explored and the use of power series as a versatile tool is motivated. Connections between these specialized forms of continuity and standard continuity are established, providing proofs and detailed properties. The results include several foundational theorems characterizing P-statistical continuity and ward continuity under various settings. These findings contribute to a more profound comprehension of continuity concepts within the context of regular summability methods.

2. Preliminaries

Before presenting the main results, the foundational concepts and notation are introduced.

2.1. Statistical Convergence

Definition 2.1 [30] Let N be the set of natural numbers and $E \subseteq \mathrm{N}$, then the natural density of E, denoted by $\delta(E)$, is given by

$$\delta(E) := \lim_{k\to\infty} \frac{1}{k}\left|\{n \le k : n \in E\}\right|$$

whenever the limit exists, where $|.|$ denotes the cardinality of the set.

Definition 2.2 [31, 32] A sequence $x = \{x_n\}$ of numbers is statistically convergent to l provided that, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k}\left|\{n \le k : |x_n - l| \ge \varepsilon\}\right| = 0$$

that is,

$$E := E_k(\varepsilon) := \{n \le k : |x_n - l| \ge \varepsilon\}$$

has natural density zero. This is denoted by $st - \lim_n x_n = l$.

It is noteworthy that every convergent sequence (in the classical sense) is also statistically convergent to the same limit. However, a statistically convergent sequence does not necessarily converge in the conventional sense.

2.2. Power Series Methods

Methods based on power series, such as the Abel and Borel techniques, offer more extensive convergence criteria than those associated with standard convergence. The main principles of these methods are as follows.

Definition 2.3 [33, 34] Let $\{p_n\}$ be a non-negative real sequence such that $p_0 > 0$ and such that the corresponding power series

$$p(u) := \sum_{n=0}^{\infty} p_n u^n$$

has radius of convergence R with $0 < R \le \infty$. If the limit

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n x_n = l$$

exists then it is said that $x = \{x_n\}$ is convergent in the sense of power series method to l.

Remark 2.4 The power series method is considered regular if and only if $\lim_{0<u\to R^-} \frac{p_n u^n}{p(u)} = 0$ holds for each n (for example Ref. [35]).

Remark 2.5 First, observe that when $R = 1$, it is straightforward to see that if $p_n = 1$ and $p_n = \frac{1}{n+1}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. Furthermore, when $R = \infty$ and $p_n = \frac{1}{n!}$, the power series method coincides with Borel summability method.

It is assumed throughout that the power series method is regular.

Definition 2.6 A sequence $\{x_n\}$ is quasi Cauchy with respect to power series methods if $\{\Delta x_n\}$ is convergent in the sense of power series method to 0, i.e.

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \Delta x_n = 0$$

where $\Delta x_n = x_{n+1} - x_n$ for every $n \in \mathrm{N}_0$.

2.3. P-Statistical Convergence

Recently, Ünver and Orhan [36] introduced the notion of P-statistical convergence (see also Ünver's earlier work for additional context [37]):

Definition 2.7 [36] Let $E \subset \mathrm{N}_0$. If the limit

$$\delta_P(E) := \lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n\in E} p_n u^n$$

exists, then $\delta_P(E)$ is said to be P-density of E.

It is worth noting that, from the definition of a power series method and P-density it is easy to verify that $0 \leq \delta_P(E) \leq 1$ whenever it exists.

Definition 2.8 [36] A sequence $x = \{x_n\}$ is statistically convergent with respect to power series method (P-statistically convergent) to l provided that, for any $\varepsilon > 0$

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n\in E_\varepsilon} p_n u^n = 0$$

where $E_\varepsilon = \{n \in \mathrm{N}_0 : |x_n - l| \geq \varepsilon\}$, that is $\delta_P(E_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case, it is denoted by $st_P - lim\, x_n = l$.

Example 2.9 Consider the power series method defined by

$$p_n = \begin{cases} 0, & n = 2k, \\ 1, & \text{otherwise,} \end{cases} k = 0,1,2,\ldots,$$

and let $x = \{x_n\}$ be given by

$$x_n = \begin{cases} n, & n = 2k, \\ 0, & \text{otherwise,} \end{cases} k = 0,1,2,\ldots.$$

It is straightforward to verify that x is P-statistically convergent to 0.

Definition 2.10 [25] A sequence $x = \{x_n\}$ is statistically Cauchy with respect to power series method (P-statistically Cauchy) provided that, for any $\varepsilon > 0$ there exists a number N such that

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |x_n - x_N| \geq \varepsilon\right\}\right) = 0.$$

3. Main Results

3.1. P-Statistical Continuity

In this section, a novel continuity concept based on P-statistical convergence is introduced. This concept establishes a continuity framework that utilises P-statistically convergent sequences. It is demonstrated that this approach offers an alternative perspective for analysing the continuity characteristics of functions.

Definition 3.1 Let $\{x_n\}$ be a sequence. The function $f: \mathrm{R} \to \mathrm{R}$ is P-statistical continuous at a point $x_0' \in \mathrm{R}$ if $st_P - lim\, f(x_n) = f(x_0')$ whenever $st_P - lim\, x_n = x_0'$.

It is important to note that this continuity definition cannot be derived from A-continuity for any regular summability matrix A as considered in [11].

Now, the examples are provided to show that the notion of P-statistical continuity is much weaker than classical continuity.

Example 3.2 Let $\{x_n\}$and $\{p_n\}$ be as in Example 2.9 and $f: [0,1] \to \mathrm{R}$ given with

$$f(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}.$$

Since $st_P-lim\,x_n=0$ and $st_P-lim\,f\,(x_n)=f(0)=0$, f is P-statistically continuous at a point 0. However f is not continuous at a point 0.

Example 3.3 Let $\{p_n\}$ be as in Example 2.9 and $f:[-2,0]\to \mathrm{R}$ given with $f(x)=[x]$ where $[x]$ denotes the greatest integer not exceeding x. It can be easily seen that f is not continuous at a point 0. Now, let $\{x_n\}$ given with

$$x_n=\begin{cases}-2+\frac{1}{n}, & n=2k,\\ 0, & \text{otherwise},\end{cases}.$$

Note that, $st_P-lim\,x_n=0$ and $\{x_n\}$ is neither statistically convergent nor convergent to 0. Also, observe that

$$f(x_n)=\begin{cases}-2, & n=2k,\\ 0, & \text{otherwise},\end{cases}$$

and $\{f(x_n)\}$ is P-statistically convergent to $f(0)=0$. Thus, f is P-statistically continuous at a point 0.

Lemma 3.4 If f and g are P-statistical continuous at a point $x_0'\in\mathrm{R}$, then $f+g$ is P-statistical continuous there.

Note that if f is P-statistical continuous, then cf is P-statistical continuous for any $c\in R$. Thus the set of P-statistical continuous functions forms a vector space.

Theorem 3.5 Every linear function $f:\mathrm{R}\to\mathrm{R}$ is P-statistical continuous.

Proof. If the function f is linear then, it has the form $f(x)=ax+b$ where a,b are constants. Let $\{x_n\}$ be a P-statistical convergent sequence at $x_0'\in R$. Thus, it is obtained that

$$st_P-lim\,f\,(x_n)=st_P-lim(ax_n+b)=ax_0'+b=f(x_0')$$

which was to be shown.

Nevertheless, it is possible to construct an example that demonstrates that the converse of the previous proposition does not universally hold true. The following example provides a clear illustration of this.

Example 3.6 Define the function $f:\mathrm{R}\to\mathrm{R}$ by $f(x)=x^2$. Since

$$st_P-lim\,x_n=x_0'\Rightarrow st_P-lim\,f\,(x_n)=st_P-lim\,x_n^2=(x_0')^2=f(x_0')$$

then, f is P-statistical continuous. However, f is not linear.

The following four types of continuity for functions $f:R\to R$ at a point $x_0'\in R$ are encountered in relation to P-statistically convergent sequences and convergent sequences:

1. $\lim_n x_n=x_0'\Rightarrow \lim_n f(x_n)=f(x_0')$ (the obvious continuity)

2. $st_P-\lim x_n=x_0'\Rightarrow st_P-\lim f\,(x_n)=f(x_0')$ (the P-statistical continuity)

3. $\lim_n x_n=x_0'\Rightarrow st_P-\lim\, f(x_n)=f(x_0')$

4. $st_P-\lim\, x_n=x_0'\Rightarrow \lim_n f(x_n)=f(x_0')$

It can be easily seen that (1) implies (3), (2) implies (3) and (4) implies (2). Now, the implication that (4) implies (1) is given.

Theorem 3.7 Let $\{x_n\}$ be a sequence and the function $f: \mathrm{R} \to \mathrm{R}$ has the following property: there exists such a point $x_0^{'} \in \mathrm{R}$ that

$$st_P - \lim x_n = x_0^{'} \Rightarrow \lim_n f(x_n) = f(x_0^{'}) \tag{3.1}$$

is valid. Then, f is a continuous function.

Proof. In light of the fact that the power series method is regular,

$$\lim_n x_n = x_0^{'} \Rightarrow st_P - \lim x_n = x_0^{'}.$$

Hence, it is obtained from (3.1) that

$$\lim_n x_n = x_0^{'} \Rightarrow \lim_n f(x_n) = f(x_0^{'}).$$

Theorem 3.8 If a function $f: \mathrm{R} \to \mathrm{R}$ is P-statistical continuous at a point $x_0^{'} \in \mathrm{R}$, then it is continuous there.

Proof. Suppose that f is discontinuous at $x_0' \in R$ then, there exists a sequence $\{x_n\}$ with $\lim_n x_n = x_0^{'}$ such that $\{f(x_n)\}$ is not convergent to $f(x_0')$. If $\{f(x_n)\}$ exists and $\lim_n f(x_n)$ is different from $f(x_0')$ then, it can be written that $\{f(x_n)\}$ has two subsequences such that

$$\lim_k f(x_{n_k}) = l_1, \lim_m f(x_{k_m}) = l_2 \left(\exists t, l_t \neq f(x_0^{'}), t = 1,2\right).$$

Since $\{x_{n_k}\}$ and $\{x_{k_m}\}$ are subsequences of $\{x_n\}$, the subsequences $\{x_{n_k}\}$ and $\{x_{k_m}\}$ are convergent and, they are P-statistical convergent to $x_0^{'} \in R$. So, it is obtained by hypothesis that

$$st_P - lim\, f\left(x_{n_k}\right) = st_P - lim\, f\left(x_{k_m}\right) = f(x_0').$$

This leads to a contradiction. If $\{f(x_n)\}$ is unbounded above, then it can be found that an n_1 such that $f(x_{n_1}) > f(x_0') + 2^1$. There exists a positive integer $n_2 > n_1$ such that $f(x_{n_2}) > f(x_0') + 2^2$. Then it can be chosen that $n_k > n_{k-1}$ such that $f(x_{n_k}) > f(x_0') + 2^k$. Inductively, a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ can be constructed such that $f(x_{n_k}) > f(x_0^{'}) + 2^k$. Hence,

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f(x_{n_k}) - f(x_0^{'})\right| \geq \frac{1}{2}\right\}\right) = 1 \neq 0.$$

On the other hand, since the sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, the subsequence $\{x_{n_k}\}$ is convergent so it is P-statistical convergent. Thus, by hypothesis, $st_P - lim\, f\left(x_{n_k}\right) = f(x_0^{'})$. This leads to a contradiction.

Theorem 3.9 A function $f: \mathrm{R} \to \mathrm{R}$ is P-statistical continuous at $x_0^{'} \in \mathrm{R}$ if and only if it is continuous there.

Proof. In view of the Theorem 3.8 it is enough to show that if a function $f: R \to R$ is continuous at $x_0' \in R$ then, it is P-statistical continuous at this point. Thus, let $st_P - lim\, x_n = x_0'$, and $\varepsilon > 0$. By continuity of f at x_0' there is a $\delta > 0$ such that $|x - x_0'| < \delta$ implies that $|f(x) - f(x_0')| < \varepsilon$. Hence, by the definition of P-statistical convergence, it is obtained that

$$\delta_P(\{n \in \mathrm{N}_0: |f(x_n) - f(x_0')| \geq \varepsilon\}) \leq \delta_P(\{n \in \mathrm{N}_0: |x_n - x_0'| \geq \delta\}) = 0,$$

so that $st_P - lim\, f\,(x_n) = f(x_0')$, and f is P-statistical continuous at x_0'.

It is a well-established result in the field of analysis that the uniform limit of a sequence of continuous functions remains continuous. This property similarly extends to P-statistical continuity.

Theorem 3.10 If $\{f_n\}$ is a sequence of P-statistical continuous functions defined on a subset S of R and $\{f_n\}$ is uniformly convergent to a function f, then f is P-statistical continuous on S.

Proof. Let $\{x_n\}$ be a P-statistical convergent sequence and $\varepsilon > 0$. Since $\{f_n\}$ is uniformly convergent, then there exists a positive integer N such that $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in S$, whenever $n \geq N$. As f_N is P-statistical continuous on S, it is given that

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f_N(x_n) - f_N(x_0')\right| \geq \frac{\varepsilon}{3}\right\}\right) = 0.$$

Also, it is given that

$$\{n \in \mathrm{N}_0 : |f(x_n) - f(x_0')| \geq \varepsilon\} \subset \left\{n \in \mathrm{N}_0 : |f(x_n) - f_N(x_n)| \geq \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{n \in \mathrm{N}_0 : |f_N(x_n) - f_N(x_0')| \geq \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{n \in \mathrm{N}_0 : |f_N(x_0') - f(x_0')| \geq \frac{\varepsilon}{3}\right\},$$

now, from the above inclusion it follows that

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f(x_n) - f(x_0')\right| \geq \varepsilon\right\}\right)$$

$$\leq \lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f(x_n) - f_N(x_n)\right| \geq \frac{\varepsilon}{3}\right\}\right)$$

$$+ \lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f_N(x_n) - f_N(x_0')\right| \geq \frac{\varepsilon}{3}\right\}\right)$$

$$+ \lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f_N(x_0') - f(x_0')\right| \geq \frac{\varepsilon}{3}\right\}\right)$$

$$= 0 + 0 + 0 = 0.$$

Hence

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : \left|f(x_n) - f(x_0')\right| \geq \varepsilon\right\}\right) = 0.$$

This completes the proof.

3.2. P-Statistical Ward Continuity

Definition 3.11 A sequence $x = \{x_n\}$ is statistical quasi Cauchy with respect to power series method (P-statistical quasi Cauchy) if $st_P - lim\, \Delta\, x_n = 0$, i.e., for any $\varepsilon > 0$,

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |\Delta x_n| \geq \varepsilon\right\}\right) = 0$$

where $\Delta x_n = x_{n+1} - x_n$ for every $n \in \mathrm{N}_0$.

Let a be a fixed constant in R. If $\{x_n\}$ is P-statistical quasi Cauchy, then so is the sequence $\{ax_n\}$. Furthermore, if $\{x_n\}$ and $\{y_n\}$ are P-statistical quasi Cauchy sequences, then $\{x_n + y_n\}$ is also P-statistical quasi Cauchy. It can thus be concluded that the set of all P-statistical quasi-Cauchy sequences constitutes a vector space within the larger space of all sequences. Furthermore, any P-statistical convergent sequence is also P-statistical quasi-Cauchy.

Theorem 3.12 Any convergent sequence is P-statistical quasi Cauchy.

Proof. Let $x = \{x_n\}$ be a convergent sequence with limit l. Then, for every $\varepsilon > 0$ there exists an $n_0 \in N$ such that $|x_n - l| < \frac{\varepsilon}{2}$ for $n \geq n_0$. Thus, for every $\varepsilon > 0$,

$$\left\{n \in \mathrm{N}_0 : |x_n - l| \geq \frac{\varepsilon}{2}\right\} \subseteq \{0,1,2,\ldots,n_0\}.$$

Hence

$$\sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |x_n - l| \geq \frac{\varepsilon}{2}\right\}\right) \leq \sum_{n=0}^{n_0} p_n u^n.$$

Also,

$$\frac{1}{p(u)}\sum_{n=0}^{\infty} p_n u^n \chi\left(\{n \in \mathrm{N}_0 : |\Delta x_n| \geq \varepsilon\}\right) \leq \frac{1}{p(u)}\sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |x_{n+1} - l| \geq \frac{\varepsilon}{2}\right\}\right)$$

$$+\frac{1}{p(u)}\sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |l - x_n| \geq \frac{\varepsilon}{2}\right\}\right)$$

$$\leq 2\frac{1}{p(u)}\sum_{n=0}^{n_0} p_n u^n.$$

Then, thanks to regularity of power series method, it is obtained that

$$\lim_{0<u\to R^-} \frac{1}{p(u)}\sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |\Delta x_n| \geq \varepsilon\right\}\right) \leq 2 \lim_{0<u\to R^-} \frac{1}{p(u)}\sum_{n=0}^{n_0} p_n u^n = 0.$$

The proof is finished.

Definition 3.13 A subset S of R is called P-statistically ward compact if any sequence of points in S has an P-statistical quasi Cauchy subsequence, i.e. whenever $x = \{x_n\}$ is a sequence of points in S, there is an P-statistical quasi Cauchy subsequence $\alpha = \{\alpha_k\} = \{x_{n_k}\}$ of x.

Theorem 3.14 A subset of R is P-statistically ward compact iff it is bounded.

Proof. Since power series method is regular, it is clear that any bounded subset of R is P-statistically ward compact . Suppose now that S is unbounded. First pick an element x_0 of S so that $x_0 > 1$. Then choose an element x_1 of S so that $x_1 > x_0 + 1$. Similarly choose an element x_2 of S so that $x_2 > x_1 + 2^1$. Inductively, elements of S can be chosen such that $x_{k+1} > x_k + 2^k$ for each $k \in N_0$. Take any subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$. Thus

$$\frac{1}{p(u)}\sum_{n=0}^{\infty} p_n u^n \chi\left(\{n \in \mathrm{N}_0 : |\Delta x_n| \geq 1\}\right) = 1 \neq 0.$$

Thus the sequence $\{x_n\}$ has no P-statistical quasi Cauchy subsequence as well. If it is unbounded below, then similarly, a sequence of points in S can be constructed that has no P-statistical quasi Cauchy subsequence. This completes the proof of the theorem.

Now, a new type of continuity is introduced, defined through P-statistical quasi-Cauchy sequences. This form of continuity can be seen as an extension of the classical concept, enabling the examination of the behavior of sequences under specific statistical conditions within a broader framework.

Definition 3.15 A function $f: R \rightarrow R$ is called P-statistical ward continuous on a subset S of R if it preserves P-statistical quasi Cauchy sequences of points in S, i.e. $\{f(x_n)\}$ is P-statistical quasi Cauchy whenever $\{x_n\}$ is a P-statistical quasi Cauchy sequence of points in S.

We note that this definition of continuity cannot be obtained through A-continuity for any regular summability matrix A considered in [11]. Also note that, this continuity is not subsequential method.

Lemma 3.16 If f and g are P-statistical ward continuous on a subset S of R, then $f + g$ is P-statistical ward continuous there.

Proof. Let f and g be P-statistical ward continuous on a subset S of R. Suppose that $\{x_n\}$ is a P-statistical quasi Cauchy sequence of points in S and $\varepsilon > 0$. Since f and g are P-statistical ward continuous on S,

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |\Delta f(x_n)| \geq \frac{\varepsilon}{2}\right\}\right) = 0,$$

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |\Delta g(x_n)| \geq \frac{\varepsilon}{2}\right\}\right) = 0.$$

Also, since

$$|(f+g)(x_{n+1}) - (f+g)(x_n)| \leq |\Delta f(x_n)| + |\Delta g(x_n)|,$$

it can be easily obtained

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\left(\left\{n \in \mathrm{N}_0 : |(f+g)(x_{n+1}) - (f+g)(x_n)| \geq \varepsilon\right\}\right) = 0.$$

This completes the proof.

Let a be a fixed constant in R. If f is P-statistical ward continuous, then af is also P-statistical ward continuous. However, the product of two P-statistical ward continuous functions is not necessarily P-statistical ward continuous. This can be illustrated by considering the product of the P-statistical ward continuous function $f(y) = y$ with itself, together with a P-statistical quasi-Cauchy sequence $\{\sqrt{n}\}$.

The proofs of the following theorems are omitted as they are similar to the proofs of Theorem 3.8 and Theorem 3.10, respectively.

Theorem 3.17 If a function $f: R \rightarrow R$ is P-statistical ward continuous on a subset S of R, then it is continuous on S.

Theorem 3.18 If $\{f_n\}$ is a sequence of P-statistical ward continuous functions defined on a subset S of R and $\{f_n\}$ is uniformly convergent to a function f, then f is P-statistical ward continuous on S.

4. Conclusion

In this paper, the concepts of P-statistical continuity and P-statistical ward continuity were explored within the framework of power series methods, extending the classical notions of continuity and statistical convergence. By analyzing these specialized forms of continuity, their foundational properties were established and their relationships with traditional continuity concepts were examined.

The results demonstrated that P-statistical continuity and ward continuity provide a richer framework for understanding generalized forms of convergence, offering a natural extension to existing theories. Furthermore, the incorporation of power series methods highlighted the flexibility and depth of these approaches in surpassing the limitations of classical matrix-based methods.

It is suggested that future research directions may include the exploration of analogous continuity concepts in other summability methods, with the subsequent extension of these results to more complex spaces [38-41]. Furthermore, the present framework may be subjected to further investigation in the context of double sequences [42, 43], as well as extending it to the fuzzy case [44, 45], which promises to yield a wealth of applications and insights in related mathematical fields.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Authors' Contributions

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors read and approved the final manuscript.

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