Matrix Transforms of Summability Domains of Normal Series-to-Series Matrices

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Abstract: In the present paper matrix transforms of summability domains cs_A of normal series-to-series matrices A are investigated. Let M be a matrix and B a triangular series-to-series matrix. Necessary and sufficient conditions for M to be a transform from cs_A into cs_B are found. For application the special case, when A is a series-to-series Riesz matrix, are studied.

Keywords: Matrix transforms, normal series-to-series matrices, conservative and regular series-to-series matrices, Riesz matrix.

1. INTRODUCTION

In this paper matrix transforms of summability domains of normal series-to-series matrices are investigated. Let ω be the set of all sequences with real or complex entries, $c \subset \omega$ the set of all convergent sequences and $c_0 \subset c$ the set of all null sequences. For every $x = (x_k) \in \omega$ we denote

$$Sx = (X_n), X_n := \sum_{k=0}^{n} x_k, \lim S x := \lim_n X_n .$$

Throughout this paper, we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. Let

$$cs := \{x \in \omega | Sx \in c\}, cs_0 := \{x \in cs | Sx \in c_0\}.$$

Let $A = (a_{nk})$ be a matrix with real or complex entries. We say that a sequence x is A^{ser}-Asummable if the series

$$A_n x := \sum_k a_{nk} x_k$$

are convergent and $Ax := (A_n x) \in cs$. If the series $A_n x$ are convergent and $Ax \in c$, then we say that x is A^{seq}-summable. The sets of all A^{ser}- and A^{seq}-summable sequences we denote correspondingly by cs_A and c_A. A matrix $A = (a_{nk})$ is said to be normal if $A = (a_{nk})$ is lower triangular and $a_{nn} \neq 0$. A matrix A is called series-to-series conservative (shortly, Sr-Sr conservative) if $\mathrm{cs} \subset cs_A$, and series-to-series regular (shortly, Sr-Sr regular) if

$$\lim S(Ax) = \lim S x$$

for every $x \in cs$. Similarly, if for every $x \in cs$ (for every $x \in c$) we have $Ax \in c$, then A is called series-to-sequence conservative or Sr-Sq conservative (correspondingly sequence-to-sequence conservative or Sq-Sq conservative). If

$$\lim_n A_n x = \lim S x$$

for every $x \in cs$, then A is called series-to-sequence regular or Sr-Sq regular. If

$$\lim_n A_n x = \lim_n x_n$$

for every $x \in c$, then A is called sequence-to-sequence regular or Sq-Sq regular. Let $M = (m_{nk})$ be a matrix with real or complex entries and $B = (b_{nk})$ a triangular matrix with real or complex entries. We say that A and B are M^{ser}-consistent on cs_A if

$$\lim S[B(Mx)] = \lim S(Ax),$$

and M^{seq}-consistent on c_A if

$$\lim_n B_n(Mx) = \lim_n A_n x .$$

If $M = (\delta_{nk})$, where $\delta_{nk} = 1$ for $n = k$ and $\delta_{nk} = 0$ otherwise, M^{ser}-consistency and M^{seq}-consistency of A and B coincide with ordinary consistency of A and B.

The matrix transforms from c_A into c_B are studied in several works. First results for such transforms are obtained by Alpár (see [8], [9]), who found necessary and sufficient conditions for M to be transform from c_A into c_B if $A = C^{\alpha}$ and $B = C^{\beta}$ are series-to-sequence

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Cesáro matrices with orders $a>0$ and $\beta>0$. In 1986 Thorpe (see [13]) generalized the result of Alpár, taking instead of C^{β} arbitrary normal matrix B. Further generalization is presented in [7], where the author of the present paper considered this transform in the case where A is a reversible series-to-sequence matrix and B arbitrary triangular (series-to-sequence or sequence-to-sequence) matrix. In [6] this problem is studied for non-triangular B and in [6] also necessary and sufficient conditions for M^{seq}-consistency of A and B are found. Later in 1994 (see [5]) above-mentioned results are generalized for the case where A is a Sr-Sq regular or Sq-Sq regular perfect matrix and B is a triangular matrix. In 2009 (see [4]) the transform from c_A into cs_B are investigated in the case, where the elements of normal A, triangular B and arbitrary M are continuous linear operators from a Banach space X into a Banach space Y. In [1-3] some classes of matrices M, transforming c_A into cs_B, are characterized.

In this paper in Section 2 necessary and sufficient conditions for M (with real or complex entries) to be transform from cs_A into cs_B for a normal series-to-series matrix A (with real or complex entries) and a triangular series-to-series matrix B (with real or complex entries) are established. Also in Section 2 the M^{ser} consistency of A and B on cs_A are investigated. In Section 3 for application the special case, when A is a series-to-series Riesz matrix, are studied.

2. MAIN RESULTS

Let throughout this Section $A=(a_{nk})$ be a normal series-to-series matrix with its inverse $A^{-1}=(\eta_{nk})$, $B=(b_{nk})$ a triangular series-to-series matrix and $M=(m_{nk})$ an arbitrary matrix. Throughout this paper, we use the following notations:

$$C_{sl}^n := \sum_{k=l}^{s} m_{nk}\eta_{kl}, \Delta_l C_{sl}^n := C_{sl}^n - C_{s,l+1}^n .$$

Theorem 2.1. *For all n the series $M_n x$ are convergent for every $x\in cs_A$ if and only if*

$$\text{there exist finite limits} \lim_s c_{sl}^n := c_{nl}, \tag{1}$$

$$\sum_l \left|\Delta_l c_{sl}^n\right| = O_n(1). \tag{2}$$

Moreover, for every $x\in cs_A$ hold the equalities

$$M_n x = \xi c_{n0} + \sum_l \Delta_l c_{nl}\left(Y_l - \xi\right) \tag{3}$$

with

$$Y_l := \sum_{k=0}^{l} y_k, \tag{4}$$

where $y_k := A_k x$ *and* $\xi := \lim_l Y_l$.

Proof. Necessity. Let all series $M_n x$ be convergent for every $x\in cs_A$. Then for every $x\in cs_A$ we have

$$k=\sum_{k=l}^{s} m_{nk}x_k = \sum_{l=o}^{s} c_{sl}^n y_l = \left(C^n\right)_s y,$$

where $y=(y_l)\in cs$ and $C^n := \left(c_{sl}^n\right)$. As by the normality of A for every $y\in cs$ there exists $x\in cs_A$ so that $Ax=y$, then the matrix C^n for every n transforms cs into c. In addition to it,

$$\lim_s \left(C^n\right)_s y = M_n x \tag{5}$$

for every $x\in cs_A$, where $y=Ax$. Consequently conditions (1) and (2) are fulfilled and equality (3) holds for each n by Theorem 1.3 of [10] (see also [11], p. 50).

Sufficiency. Let conditions (1) and (2) be fulfilled. Then by Theorem 1.3 of [10] the matrix C^n for every n transforms cs into c. As equalities (5) hold, then equalities (3) for every n also are satisfied by Theorem 1.3 of [10].

Now we prove the main result of this paper.

Theorem 2.2. *A matrix M transforms cs_A into cs_B if and only if conditions (1) and (2) are satisfied and*

$$\text{the series} \sum_t \gamma_{tl} \text{ are convergent for all } l, \tag{6}$$

$$\sum_l \left|\sum_{t=0}^{s} \Delta_l \gamma_{tl}\right| = 0(1), \tag{7}$$

where

$$\gamma_{tl} := \sum_{k=o}^{t} b_{tk} c_{kl}.$$

Proof. Necessity. Assume that M transforms cs_A into cs_B. Then all series $M_n x$ are convergent for every $x\in cs_A$. Hence conditions (1) and (2) are fulfilled and equalities (3) (where Y_l presented by (4)), hold for every $x\in cs_A$ by Theorem 2.1. It follows from equalities (3) that

$$B_t(Mx) = \xi\gamma_{t0} + \sum_l \Delta_l \gamma_{tl}(Y_l - \xi) \quad (8)$$

for every $x \in cs_A$. By the normality of A for the sequence $e^0 := (1.0,0,...) \in cs$ there exists the sequence $\tilde{x} \in cs_A$ so that $A\tilde{x} = e^0$. This implies due to $\tilde{x} = \left(\left(A^{-1}\right)k^{e^0}\right)$ that

$$B_t(M\tilde{x}) = B_t\left[M\left(A^{-1}e^0\right)\right] = \gamma_{t0}.$$

Hence

$$\text{the series} \sum_t \gamma_{t0} \text{ is convergent}. \quad (9)$$

As every $Y = (Y_l) \in c$ can be presented in the form (4), where $y = (y_k) \in cs$ and by the normality of A for this y there exists $x \in cs_A$ so that $Ax = y$, then from (8) and (9) we get that the series

$$\sum_t \sum_l \Delta_l \gamma_{tl}(Y_l - \xi) \quad (10)$$

is convergent for every $Y = (Y_l) \in c$ with $\lim_l Y_l = \xi$. It's well-known (see, for example [11]) that every $Y = (Y_l) \in c$ with $\lim_l Y_l = \xi$ can be presented in the form

$$Y = Y^0 + \xi e; Y^0 = \left(Y_k^{\ 0}\right) \in c_0, e = (1,1,...).$$

Thus, the series (10) is convergent for each $Y^0 = (Y_k - \xi)$, i.e. the matrix $\Gamma := (\Delta_l \gamma_{ti})$ transforms c_0 into cs. Therefore condition (7) is satisfied and the series

$$\sum_l \Delta_l \gamma_{tl}$$

are convergent for all l by Proposition 43 of [12]. Consequently, condition (6) is fulfilled by (9).

Sufficiency. Let conditions (1), (2), (6) and (7) be fulfilled. Then all series $M_n x$ are convergent and equalities (3) are valid for every $x \in cs_A$ by Theorem 2.1. The validity of (3) implies also the validity of (8). It follows from conditions (6) and (7) that the matrix $\Gamma := (\Delta_l \gamma_{ti})$ transforms c_0 into cs. Therefore from (8) we get by condition (6) that M transforms cs_A into cs_B.

From Theorem 2.2 we get the following result.

Corollary 2.3. *Matrices A and B are M^{ser}-consistent if and only if conditions (1), (2) and (7) are satisfied and*

$$\sum_t \gamma_{tl} = 1 \ for\ all\ l. \quad (11)$$

Proof. Necessity. Let A and B are M^{ser}-consistent. Then conditions (1), (2) and (7) are fulfilled by Theorem 2.2 and equalities (8) are satisfied for every $x \in cs_A$, where

$$\lim S(Ax) = \xi. \quad (12)$$

Hence

$$\lim S\left[B(Mx)\right] = \xi \quad (13)$$

for every $x \in cs_A$. Let $\tilde{x} \in cs_A$ be a sequence, for which $A\tilde{x} = e^0$. As in this case $\lim S(A\tilde{x}) = 1$, then

$$\lim S\left[B(M\tilde{x})\right] = \sum_t \gamma_{t0} = 1. \quad (14)$$

Therefore, it follows from (8) and (13) that $\Gamma := (\Delta_l \gamma_{ti})$ transforms c_0 into cs_0. Hence

$$\sum_t \Delta_l \gamma_{tl} = 0$$

for all l by Proposition 54 of [12]. Consequently, with the help of (14) we have that condition (11) is satisfied.

Sufficiency. Let conditions (1), (2), (7) and (11) be satisfied. Them M transforms cs_A into cs_B by Theorem 2.2 and equalities (8) hold for every $x \in cs_A$. From conditions (7) and (11) it follows with the help of Proposition 54 of [12] that $\Gamma := (\Delta_l \gamma_{ti})$ transforms c_0 into cs_0. Consequently from (8) we get with the help of condition (7) that equality (13) holds for each $x \in cs_A$ satisfying equality (12), i.e. A and B are M^{ser}-consistent.

For a Sr-Sr-conservative matrix A we get the following necessary condition for M to be transform from cs_A to cs_B.

Corollary 2.4. *Let A be a Sr-Sr-conservative. If M transforms cs_A into cs_B, then*

$$\sum_t g_{tk} = g_k \left(g_k \ is\ a\ finite\ number\right), \quad (15)$$

where

$$g_{tk} := \sum_{n=0}^{t} b_{tn} m_{nk}.$$

Proof. Let $e^k = (0,...,0,1,0,...)$ with number 1 in k-th position. Taking $e^k \in cs_A$, we get

$$\lim S\left[B\left(Me^k\right)\right]=\sum_t g_{tk}.$$

This implies the validity of the assertion of Corollary 2.4.

For a Sr-Sr-regular matrix A we get the following necessary condition for M^{ser}-consistency of A and B.

Corollary 2.5. *Let A be a Sr-Sr-regular. If A and B are M^{ser} -consistent on cs_P, then condition (15) is fulfilled with $g_k=1$.*

Proof follows from the fact that $\lim S\left(Ae^k\right)=1$ for a Sr-Sr-regular matrix A.

3. MATRIX TRANSFORMS OF SUMMABILITY DOMAINS OF RIESZ MATRICES

In this section we consider the case when A is a Riesz matrix. Let (p_n) be a sequence of nonzero complex numbers, $P_n=p_0+...+p_n\neq 0$ and let $P=(R,p_n)=(a_{nk})$ be the series-to-series Riesz matrix generated by (p_n), i.e. P is the normal matrix with

$$a_{nk}=\frac{P_{k-1}p_n}{P_nP_{n-1}}$$

(see [10], p. 113). Throughout this section, we assume that terms with negative indices are equal 0. The matrix P has the inverse matrix $P^{-1}=(\eta_{nk})$, where (see [10], p. 116)

$$\eta_{nk}:=\begin{cases}\dfrac{P_n}{p_n} & (k=n),\\ \dfrac{P_{n-2}}{p_{n-1}} & (k=n-1),\\ 0 & (k<n-1\,or\,k>n).\end{cases}\tag{16}$$

Theorem 3.1. *Let P be a Sr-Sr-conservative matrix. Then M transforms cs_P into cs_B if and only if condition (15) is fulfilled and*

$$\frac{P_s}{p_s}m_{ns}=O_n(1),\tag{17}$$

$$\frac{P_{s-2}}{p_{s-1}}m_{ns}=O_n(1),\tag{18}$$

$$\sum_{l=0}^{s}\left|\Delta_l\left(\frac{P_l}{p_l}\Delta_l m_{nl}\right)+\Delta_l m_{n,l+1}\right|=O_n(1),\tag{19}$$

$$\sum_l\left|\Delta_l\left(\frac{Pl}{pl}\sum_{t=0}^{s}\Delta_l g_{tl}\right)+\sum_{t=0}^{s}\Delta_l g_{t,l+1}\right|=O(1).\tag{20}$$

Proof. Necessity. Assume that M transforms cs_P into cs_B. Then for $A=P$ conditions (1), (2), (6) and (7) are satisfied by Theorem 2.2 and condition (15) is fulfilled by Corollary 2.4. With the help of (16), we get that

$$c_{nl}=\frac{P_l}{p_l}m_{nl}-\frac{P_{l-1}}{p_l}m_{n,l+1},\tag{21}$$

$$c_{sl}^n=\begin{cases}c_{nl} & (l\leq s-1),\\ \dfrac{P_s}{p_s}m_{ns} & (l=s),\\ 0 & (l>s).\end{cases}$$

Hence

$$\Delta_l c_{sl}^n\Big|_{(l\leq s-2)}=c_{sl}^n-c_{s,l+1}^n=\frac{P_l}{p_l}m_{nl}-\frac{P_{l-1}}{p_l}m_{n,l+1}-\frac{P_{l+1}}{p_{l+1}}m_{n,l+1}+\frac{P_l}{p_{l+1}}m_{n,l+2}$$

$$=\frac{P_l}{p_l}m_{nl}-\frac{P_{l+1}}{p_{l+1}}m_{n,l+1}-\frac{P_l-p_l}{p_l}m_{n,l+1}+\frac{P_{l+1}-p_{l+1}}{p_{l+1}}m_{n,l+2}$$

$$=\frac{P_l}{p_l}\Delta_l m_{nl}-\frac{P_{l+1}}{p_{l+1}}\Delta_l m_{n,l+1}+\Delta_l m_{n,l+1}.$$

This implies

$$\Delta_l c_{sl}^n\Big|_{(l\leq S-2)}=\Delta_l\left(\frac{P_l}{p_l}\Delta_l m_{nl}\right)+\Delta_l m_{n,l+1}.\tag{22}$$

It is easy to see that

$$\Delta_l c_{sl}^n\Big|_{(l=S-1)}=\frac{P_{s-1}}{p_{s-1}}m_{n,s-1}-\frac{P_s}{p_s}m_{n,s}-\frac{P_{s-2}}{p_{s-1}}m_{n,s},\tag{23}$$

$$\Delta_l c_{sl}^n\Big|_{(l=S)}=\frac{P_s}{p_s}m_{n,s}\tag{24}$$

and

$$\Delta_l c_{sl}^n\Big|_{(l>s)}=0.\tag{25}$$

Therefore conditions (17) and (19) are fulfilled by (2) and

$$\left|\frac{P_{s-1}}{p_{s-1}}m_{n,s-1}-\frac{P_s}{p_s}m_{n,s}-\frac{P_{s-2}}{p_{s-1}}m_{n,s}\right|=O_n(1).$$

Consequently, condition (18) is satisfied by (17).

Using (21), we get

$$\gamma_{tl} = \frac{P_l}{p_l} g_{tl} - \frac{P_{l-1}}{p_l} g_{n,l+1}. \tag{26}$$

Therefore, similarly to relation (22) it is possible to show that

$$\Delta_l \gamma_{tl} = \Delta_l \left(\frac{P_l}{p_l} \Delta_l g_{tl} \right) + \Delta_l g_{t,l+1}. \tag{27}$$

Thus, condition (20) is fulfilled by condition (7).

Sufficiency. Assume that conditions (15) and (17) - (20) are fulfilled and show that M transforms cs_P into cs_B. For this purpose it is sufficient to show that all conditions of Theorem 2.2 are satisfied for $A = P$. First we see that conditions (1) and (6) are fulfilled correspondingly by (21) and (26). As relations (22) - (25) hold, then condition (2) is fulfilled by (17) - (19). From relation (27) we get by (20) that condition (7) is also satisfied. Thus M transforms cs_P into cs_B by Theorem 2.2.

From Theorem 3.1 we get the following corollary.

Corollary 3.2. *Let P be a Sr-Sr-regular matrix. Then P and B are M^{ser}-consistent on cs_P if and only if condition (15) with $g_k = 1$ and conditions (17) - (20) are fulfilled.*

Proof. Conditions (15) and (17) - (20) are necessary and sufficient for M to be transform from cs_P *into* cs_B. Therefore conditions (1), (2), (6) and (7) are satisfied by Theorem 2.2. By the Sr-Sr-regularity of P we have that the relation $g_k = 1$ is necessary for M -consistency of P and B on cs_P. This relation implies by (26) that condition (11) is fulfilled. Consequently by Corollary 2.3 P and B are M^{ser}-consistent on cs_P.

It is well-known (see [10], p. 114 or [11]) that the existence of $\lim_n P_n \neq 0$ is necessary for P to be Sr-Sr-conservative and $\lim_n |P_n|$ is necessary for P to be Sr-Sr-regular. Therefore, from Theorem 3.1 we immediately get the following results.

Corollary 3.3. *If M transforms cs_P* into *cs_B for a Sr-Sr-conservative Riesz matrix P, then*

$$m_{ns} = O_n(p_s) \, and \, m_{ns} = O_n(p_{s-1}).$$

Corollary 3.4. *If M transforms cs_P into cs_B for a Sr-Sr-regular Riesz matrix P, then*

$$m_{ns} = o_n(p_s) \text{ and } m_{ns} = o_n(p_{s-1}).$$

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