Periodic Solutions for Damped Vibration Problems

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Abstract: In this paper we are concerned with the following damped vibration problem

$$\begin{cases} \ddot{u}(t)+g(t)\dot{u}(t)=\nabla V(t,u(t)),\ a.e.\ t\in[0,T] \\ u(0)=u(T), \quad \dot{u}(0)=\dot{u}(T), \end{cases}$$

where $T>0$, $g\in L^\infty(0,T;\mathbb{R})$ with $G(t)=\int_0^t g(s)ds$ and $G(T)=0$, $V(t,u)=\frac{1}{2}(L(t)u,u)-W(t,u)$ is T-periodic in t such that $L\in C(\mathbb{R},\mathbb{R}^{n^2})$ is a T-periodic, positive definite symmetric matrix and W satisfies the global Ambrosetti-Rabinowitz condition or is subquadratic at infinity. By use of the Mountain Pass Theorem or the genus properties in the critical theory, we establish some new criteria to guarantee the existence and multiplicity of periodic solutions. Recent results in the literature are generalized and significantly improved.

Keywords: Homoclinic solutions, critical point, variational methods, mountain pass theorem, genus.

1. INTRODUCTION

The purpose of this paper is to deal with the following damped vibration problem

$$\begin{cases} \ddot{u}(t)+g(t)\dot{u}(t)=\nabla V(t,u(t)),\ a.e.\ t\in[0,T] \\ u(0)=u(T), \quad \dot{u}(0)=\dot{u}(T), \end{cases} \tag{1.1}$$

where $T>0$, $g\in L^1(0,T;\mathbb{R})$ with $G(t)=\int_0^t g(s)ds$ and $G(T)=0$, $V(t,u)=\frac{1}{2}(L(t)u,u)-W(t,u)$ is T-periodic in t such that $L\in C(\mathbb{R},\mathbb{R}^{n^2})$ is a T-periodic, positive definite symmetric matrix and W satisfies the following assumption:

(*A*) $W(t,u)$ is measurable in t for every $u\in\mathbb{R}$ and continuously differentiable in u for a.e. $t\in[0,T]$ and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$ such that

$$|W(t,u)|\le a(|u|)b(t), \quad |W_u(t,u)|\le a(|u|)b(t)$$

for $u\in\mathbb{R}^n$ and a.e. $t\in[0,T]$.

When $g(t)\equiv 0$, (1.1) is the second order Hamiltonian systems. The existence of periodic solutions is one of the most important problems in the history of Hamiltonian systems. For the case that

$$V(t,u)=\frac{1}{2}(L(t)u,u)-W(t,u),$$

it has been intensively studied by many mathematicians *via* critical point theory, for example [1, 2, 6, 7, 14, 23, 24] and the references therein. For the case that $V(t,\cdot)$ is convex for a.e. $t\in[0,T]$ (not necessarily of the kind $V(t,u)=\frac{1}{2}(L(t)u,u)-W(t,u)$), Mawhin-Willem [9] studied the existence of solutions for the problem (1.1); for non-convex potential case, the existence and multiplicity of solutions has been also considered by many mathematicians, for instance, see [2, 15-19] and their references. Particularly, Antonacci [2] studied potential changing sign case, Tang-Wu [15-18] studied γ -quasisubadditive, subadditive, coercive potential cases and the cases that the nonlinearity grow sub linearly and subquadratically; in [19], the author studied the case that the nonlinearity grows linearly. Moreover, solutions for damped vibration problems with impulsive effects have also been extensively investigated by many authors, see for instance [4] and the references listed therein.

When $g(t)\not\equiv 0$, as far as the authors know, only the recent paper [20] dealt with the problem (1.1). In [20], for the first time the authors established the variational frame of (1.1) and then three existence theorems for periodic solutions are obtained. Motivated by [20], in the present paper, we consider the existence and multiplicity of solutions of (1.1) under two classes of assumptions on $V(t,u)$ which are not contained in [20] (see Remark 1 below).

For the statement of our first main result, $V(t,u)$ is supposed to satisfy the following conditions:

(H1) $V(t,u)=\frac{1}{2}(L(t)u,u)-W(t,u)$, where $L\in C(\mathbb{R},\mathbb{R}^{n^2})$ is a T-periodic, positive definite symmetric matrix;

(H2) there is a constant $\mu>2$ such that

$$0<\mu W(t,u)\le(u,W_u(t,u))$$

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for a.e. $t \in [0,T]$ and $u \in \mathbb{R}^n \setminus \{0\}$.

Theorem 1.1. If *(A)*, *(H1)* and *(H2)* are satisfied, then (1.1) possesses at least one nontrivial solution. Moreover, if we suppose that $W(t,u)$ is even in u, i.e.,

(H3) $W(t,-u) = W(t,u)$ for a.e $t \in [0,T]$ and $u \in \mathbb{R}^n$,

then (1.1) has infinitely many distinct solutions.

Remark 1. *(H2) is the so-called global Ambrosetti-Rabinowitz condition due to Ambrosetti and Rabinowitz, see [3].*

In [20], under some reasonable assumptions on $V(t,u)$, the authors obtained three existence theorems for solutions of (1.1). Explicitly, they assumed that

$$\lim_{|u|\to\infty} \inf \frac{V(t,u)}{|u|^2} > 0 \quad \text{uniformly for } a.e.\ t \in [0,T]$$

or

$$-\infty < \lim_{|u|\to\infty} \inf \frac{(V_u(t,u),u)}{|u|} \quad \text{uniformly for } a.e.\ t \in [0,T].$$

However, in view of (H2), we obtain that there exists a constant $\alpha_1 > 0$ such that (see Lemma 2.1 below)

$$W(t,u) \geq \alpha_1 |u|^\mu \quad a.e.\ t \in [0,T] \text{ and } |u| \geq 1,$$

which yields that

$$\frac{V(t,u)}{|u|^2} \to -\infty \quad \text{as} \quad |u| \to \infty$$

uniformly with respect to $a.e.\ t, since\, \mu > 2; and$

$$\frac{(V_u(t,u),u)}{|u|} \to -\infty \quad \text{as} \quad |u| \to \infty$$

uniformly with respect to $a.e.\ t$.

So Theorem 1.1 extends the conclusions in [20], in the sense that we deal with (1.1) under a class of assumptions on $V(t,u)$ where are not considered in [20].

In what follows, we consider the case that $W(t,u)$ is subquadratic at infinity. For the statement of our second result, $W(t,u)$ is supposed to satisfy the following conditions:

(H4) $W(t,0) = 0$ for $t \in [0,T]$, there exist $t_0 \in [0,T]$ and $\vartheta \in (1,2)$ such that

$$\lim_{(t,u)\to(t_0,0)} \frac{W(t,u)}{|u|^\vartheta} > 0;$$

(H5) $|\nabla W(t,u)| \leq b(t) |u|^{\vartheta-1}$ for all $t \in [0,T]$ and $u \in \mathbb{R}^n$, where $b : [0,T] \to \mathbb{R}^+$ is function such that $b \in L^\infty(0,T;\mathbb{R}^+)$.

Theorem 1.2. Under the assumptions of *(H4)* and *(H5)*, (1.1) has at least one nontrivial solution. Moreover, if W has an even symmetric in u, i.e., *(H3)*, then (1.1) has infinitely many nontrivial solutions.

This paper is organized as following. Section 2 is devoted to presenting some preliminary results. In section 3, we establish the proofs of Theorems 1.1 and 1.2.

2. PRELIMINARY RESULTS

In order to establish the variational structure which enables us to reduce the solutions of (1.1) to find critical points of corresponding functional, we first introduce some notations to be used in the following. Let $(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R}^n$ denote the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is the induced norm. E denotes the Hilbert space of T-periodic functions on $[0,T]$ with values in $\mathbb{R}^n$ under the norm

$$\| u \|_E := \left(\int_0^T (|\dot{u}(t)|^2 + (L(t)u(t),u(t))dt \right)^{\frac{1}{2}}.$$

Let $L^p(0,T;\mathbb{R}^n)$ ($1 \leq p < +\infty$) denote the Banach spaces of the T-periodic functions on $[0,T]$ with values in $\mathbb{R}^n$ under the norm

$$\| u \|_p = \left(\int_0^T |u(t)|^p \ dt \right)^{\frac{1}{p}}.$$

In addition, $L^\infty(0,T;\mathbb{R}^n)$ denotes the space of T-periodic essentially bounded functions from $[0,T]$ to $\mathbb{R}^n$ equipped with the norm

$$\| u \|_\infty := \operatorname{ess\,sup}\{ |u(t)| : t \in [0,T] \}.$$

Due to the fact that $g \in L^\infty(0,T;\mathbb{R})$, then the norm of E is equivalent to the following norm defined by

$$\| u \| := \left(\int_0^T e^{G(t)} (|\dot{u}(t)|^2 + (L(t)u(t),u(t))dt \right)^{\frac{1}{2}}$$

Note that the embedding of E into $L^p(0,T;\mathbb{R}^n)$ is compact($1 \le p \le +\infty$). That is, for any $p \in [1,+\infty]$, there is a positive constant C_p such that

$$\| u \|_p \le C_p \| u \|, \quad \forall u \in E. \tag{2.1}$$

In order to give the proof of Theorem 1.1, we need the following preliminary results. We firstly recall some properties of the function $W(t,u)$ from [8].

Lemma 2.1. Assume that *(H2)* holds. Then, for a.e. $t \in [0,T]$, the following inequalities hold:

$$\begin{aligned} W(t,u) &\le W\left(t, \frac{u}{|u|}\right) |u|^\mu, \qquad if \quad 0 < |u| \le 1, \\ W(t,u) &\ge W\left(t, \frac{u}{|u|}\right) |u|^\mu, \qquad if \quad |u| \ge 1. \end{aligned} \tag{2.2}$$

Remark 2. *Form the first inequality of (2.2), it is obvious that*

$$|W(t,u)| = o(|u|^2) \quad \text{as} \ |u| \to 0$$

uniformly with respect to a.e. $t \in [0,T]$ *. That is, for any* $\epsilon > 0$*, there is* $\delta > 0$ *such that*

$$|W(t,u)| \le \epsilon |u|^2, \quad a.e.\ t \in [0,T] \ \text{and} \ |u| \le \delta. \tag{2.3}$$

Lemma 2.2. Set $m := \inf\left\{W(t,u) : t \in [0,T], |u| = 1\right\}$. Then, for every $\lambda \in \mathbb{R} \setminus \left\{0\right\}$ and $u \in E \setminus \left\{0\right\}$, we have

$$\int_0^T W(t, \lambda u(t)) dt \ge m |\lambda|^\mu \int_0^T |u(t)|^\mu \ dt - Tm. \tag{2.4}$$

Next we introduce more notations and some necessary definitions. Let $\mathcal{B}$ be a real Hilbert space, $I \in C^1(\mathcal{B},\mathbb{R})$ means that I is a continuously Frechet-differentiable functional defined on $\mathcal{B}$. I is said to satisfy the Palais-Smale condition (henceforth denoted by the (PS)-condition) if any sequence $\left\{u_j\right\} \subset \mathcal{B}$ for which $\left\{I(u_j)\right\}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$ possesses a convergent subsequence in $\mathcal{B}$. Let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denotes its boundary.

To obtain Theorem 1.1, we make use of the following well-known Mountain Pass Theorem and its $\mathbb{Z}_2$ version.

Lemma 2.3. ([12, Theorem 2.2]) Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B},\mathbb{R})$ satisfying *(PS)*-condition. Suppose $I(0) = 0$ and

(A1) there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$, and

(A2) there is an $e \in \mathcal{B} \setminus B_\rho$, such that $I(e) \le 0$.

Then I possesses a critical value $c \ge \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \left\{g \in C([0,1],\mathcal{B}) : \ g(0) = 0, g(1) = e\right\}.$$

Lemma 2.4. ([12, Theorem 9.12]) Let $\mathcal{B}$ be an infinite dimensional real Banach space and $I \in C^1(\mathcal{B},\mathbb{R})$ be even, satisfy *(PS)*-condition and $I(0) = 0$. If $\mathcal{B} = Y \oplus X$, where Y is finite dimensional, and I satisfies

(A3) there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap X} \ge \alpha$ and

(A4) for each finite dimensional subspace $\tilde{\mathcal{B}} \subset \mathcal{B}$, there is an $R = R(\tilde{\mathcal{B}})$ such that $I \le 0$ on $\tilde{\mathcal{B}} \setminus B_{R(\tilde{\mathcal{B}})}$,

then I has an unbounded sequence of critical values.

To deal with the existence of solutions of (1.1) under the assumptions of Theorem 1.2, we appeal to the following well-known result, see for example [12].

Lemma 2.5. Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B},\mathbb{R})$ satisfying the *(PS)*-condition. If I is bounded from below, then $c = \inf_{\mathcal{B}} I(u)$ is a critical value of I.

To obtain the existence of infinitely many solutions of (1.1) under the assumptions of Theorem 1.2, we shall use the ``genus" properties. Therefore, we recall the following definitions and results, see [12, 13].

Let $\mathcal{B}$ be a Banach space, $I \in C^1(\mathcal{B},\mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset \mathcal{B} - \{0\} : A \, \text{is closed in} \ \mathcal{B} \, \text{and symmetric with respect to} \ 0\},$$
$$K_c = \{u \in \mathcal{B} \ : \ I(u) = c, I'(u) = 0\}, I^c = \{u \in \mathcal{B} \ : \ I(u) \le c\}.$$

Definition 2.6. For $A \in \Sigma$, we say genus of A is j (denoted by $\gamma(A) = j$) if there is an odd map $\psi \in C(A, \mathbb{R}^j \setminus \{0\})$ and j is the smallest integer with this property.

Lemma 2.7. Let I be an even C^1 functional on $\mathcal{B}$ and satisfy the *(PS)*-condition. For any $j \in \mathbb{N}$, set

$$\Sigma_j = \{A \in \Sigma \ : \ \gamma(A) \geq j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u).$$

I. If $\Sigma_j \neq \phi$ and $c_j \in \mathbb{R}$, then c_j is a critical value of I;

II. if there exists $r \in \mathbb{N}$ such that

$$c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R},$$

and $c \neq I(0)$, then $\gamma(K_c) \geq r + 1$.

Remark 3. *From Remark 7.3 in [12], we know that if* $K_c \subset \Sigma$ *and* $\gamma(K_c) > 1$*, then* K_c *contains infinitely many distinct points, i.e.,* I *has infinitely many distinct critical points in* $\mathcal{B}$.

3. PROOFS OF THE MAIN RESULTS

Firstly, we are going to establish the corresponding variational framework associated to the problem (1.1). To this end, define the functional $I : \mathcal{B} = E \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_0^T e^{Q(t)}[\frac{1}{2} \mid \dot{u}(t) \mid^2 + V(t, u(t))]dt \\ &= \frac{1}{2} \parallel u \parallel^2 - \int_0^T e^{Q(t)} W(t, u(t))dt. \end{aligned} \tag{3.1}$$

Under the assumptions of Theorems 1.1 and 1.2, from Theorems 2.1 and 2.2 in [20], we have the following lemma.

Lemma 3.1. $I \in C^1(E, \mathbb{R})$ and the critical point of I in E is a solution of (1.1). Moreover, one has

$$I'(u)v = \int_0^T e^{Q(t)}[(\dot{u}(t), \dot{v}(t)) + (\nabla V(t, u(t)), v(t))]dt,$$

which yields that

$$I'(u)v = \left\| u \right\|^2 - \int_0^T e^{Q(t)}(\nabla W(t, u(t)), u(t))dt. \tag{3.2}$$

Lemma 3.2. Under the conditions of *(H1)* and *(H2)*, I satisfies the *(PS)*-condition.

Proof. Assume that $\{u_j\}_{j \in \mathbb{N}} \subset E$ is a sequence such that $\{I(u_j)\}_{j \in \mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C > 0$ such that

$$\mid I(u_j) \mid \leq C \quad \text{and} \quad \parallel I'(u_j) \parallel_{E^*} \leq C \tag{3.3}$$

for every $j \in \mathbb{N}$. We firstly prove that $\{u_j\}_{j \in \mathbb{N}}$ is bounded. By (3.1) and (H2), we have

$$\parallel u_j \parallel^2 \leq 2I(u_j) + \frac{2}{\mu} \int_0^T e^{Q(t)}(\nabla W(t, u_j(t)), u_j(t))dt. \tag{3.4}$$

Combining (3.4) with (3.2), we obtain

$$\left(1 - \frac{2}{\mu}\right) \parallel u_j \parallel^2 \leq 2I(u_j) - \frac{2}{\mu} I'(u_j)u_j. \tag{3.5}$$

From (3.5), it follows that

$$\left(1 - \frac{2}{\mu}\right) \parallel u_j \parallel^2 \leq 2I(u_j) + \frac{2}{\mu} \parallel I'(u_j) \parallel_{E^*} \parallel u_j \parallel. \tag{3.6}$$

Combining (3.6) with (3.3), we get

$$\left(1 - \frac{2}{\mu}\right) \parallel u_j \parallel^2 - \frac{2}{\mu} C \parallel u_j \parallel - 2C \leq 0. \tag{3.7}$$

Since $\mu > 2$, (3.7) shows that $\{u_j\}_{j \in \mathbb{N}}$ is bounded. By the compactness of the embedding $E \subset C([0, T], \mathbb{R}^n)$, the sequence $\{u_j\}_{j \in \mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j \in \mathbb{N}}$, and there exists $u \in E$ such that

$$\begin{aligned} &u_j \rightharpoonup u, \text{ weakly in } E, \\ &u_j \to u, \text{ strongly in } C([0, T], \mathbb{R}^n). \end{aligned} \tag{3.8}$$

Hence

$$(I'(u_j) - I'(u))(u_j - u) \to 0,$$

and

$$\int_0^T e^{G(t)}(\nabla W(t, u_j(t)) - \nabla W(t, u(t)), u_j(t) - u(t))dt \to 0$$

as $j \to +\infty$.

On the other hand, an easy computation shows that

$$\begin{aligned} &(I'(u_j) - I'(u), u_j - u) = \int_0^T e^{Q(t)}[\mid \dot{u}_j(t) - \dot{u}(t) \mid^2 \\ &+ (L(t)(u_j(t) - u(t)), u_j(t) - u(t)]dt + \int_0^T e^{Q(t)}(\nabla W(t, u_j(t)) \\ &- \nabla W(t, u(t)), u_j(t) - u(t))dt, \end{aligned}$$

and so one deduces that

$$\int_0^T e^{Q(t)}[| \dot{u}_j(t) - \dot{u}(t) |^2 + (L(t)(u_j(t) - u(t)), u_j(t) - u(t))]dt \to 0,$$

which yields that $\| u_j - u \| \to 0$ and the (PS)-condition holds.

Lemma 3.3. If *(H4)* and *(H5)* hold, then I satisfies the *(PS)*-condition.

Proof. Assume that $\{u_j\}_{j\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C > 0$ such that

$$| I(u_j) | \le C, \tag{3.9}$$

for every $j \in \mathbb{N}$. We firstly prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. From (3.1), (H5) and (3.9), it is easy to deduce that

$$\begin{aligned} \| u_j \|^2 &= 2I(u_j) + 2\int_0^T W(t, u_j(t))dt \\ &\le 2C + \frac{2}{\vartheta} C_\vartheta^\vartheta \| b \|_\infty \| u \|^\vartheta . \end{aligned} \tag{3.10}$$

Since $1 < \vartheta < 2$, the inequality (3.10) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. Then the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u \in E$ such that

$$u_j \rightharpoonup u \text{ weakly in } E,$$

which yields that

$$(I'(u_j) - I'(u))(u_j - u) \to 0 \quad \text{as } j \to +\infty. \tag{3.11}$$

On account of the continuity of $\nabla W(t,u)$ and $u_j \to u$ in $L^\infty(0,T;\mathbb{R}^n)$, it follows that

$$\int_0^T e^{G(t)}(\nabla W(t, u_j(t)) - \nabla W(t, u(t)), u_j(t) - u(t))dt \to 0 \tag{3.12}$$

as $j \to +\infty$. Consequently, in view of (3.11), (3.12) and the following equality

$$(I'(u_j) - I'(u), u_j - u) = \left\| u_j - u \right\|^2 - \int_0^T e^{G(t)} (\nabla W(t, u_j(t)) - \nabla W(t, u(t)), u_j(t) - u(t))dt,$$

it concludes that $\| u_j - u \| \to 0$ as $j \to +\infty$.

Proof of Theorem 1.1 We divide the proof of Theorem 1.1 into two steps.

Step 1 We give the proof of the first part of Theorem 1.1. Firstly, we show that there exist constants $\rho > 0$ and $\alpha > 0$ such that (A1) holds in Lemma 2.3. By (2.3), for any $\epsilon > 0$ there exists $\delta > 0$ such that $| W(t,u) | \le \epsilon | u |^2$ for a.e. $t \in [0,T]$ and $| u | \le \delta$. Assume that $u \in E$ such that $0 < \| u \|_\infty \le \delta$, then we have

$$\int_0^T W(t, u(t))dt \le \epsilon \int_0^T | u(t) |^2 \, dt \le \epsilon C_2 \| u \|^2,$$

and in consequence, combining this with (3.1), we obtain

$$I(u) \ge \frac{1}{2} \| u \|^2 - \epsilon C_2 \| u \|^2 = \left(\frac{1}{2} - \epsilon C_2\right) - \| u \|^2 . \tag{3.13}$$

Choose $\epsilon = \frac{1}{4C_2}$ and $\| u \| = \rho = \frac{\delta}{C_\infty}$, then (3.13) gives that

$$I \,|_{\partial B_\rho} \ge \alpha = \frac{\delta^2}{4C_\infty^2} > 0.$$

It remains to prove that there exists $e \in E$ such that $\| e \| > \rho$ and $I(e) \le 0$. By (2.4) and (3.1), we have, for every $\lambda \in \mathbb{R} \setminus \{0\}$ and $u \in E \setminus \{0\}$, the following inequality holds:

$$I(\lambda u) \le \frac{\lambda^2}{2} \| u \|^2 - m | \lambda |^\mu \int_0^T | u(t) |^\mu \, dt + Tm. \tag{3.14}$$

Take $Q \in E$ such that $\| Q \| = 1$. Since $\mu > 2$ and $m > 0$, (3.14) implies that there exists $\xi = \xi_Q > 0$ such that $\| \xi Q \| = \xi > \rho$ and $I(\xi Q) < 0$. By Lemma 2.3, I possesses a critical value $c \ge \alpha > 0$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{ g \in C([0,1], E) : \; g(0) = 0, \, g(1) = e \}.$$

Hence, there is $u \in E$ such that

$$I(u) = c \quad \text{and} \quad I'(u) = 0.$$

Since $c > 0$, u is one nontrivial solution of (1.1).

Step 2 Now we give the proof of the second part of Theorem 1.1. Firstly, (H3) implies that that I is even and by the above assumptions, we know that $I \in C^1(E,\mathbb{R})$, $I(0)=0$ and I satisfies the (PS)-condition.

To apply symmetric Mountain Pass Theorem (see Lemma 2.4), it suffices to prove that I satisfies (A3) and (A4). (A3) is identically the same as in Step 1, so it is already proved. Now we prove (A4). Let $\tilde{E} \subset E$ be a finite dimensional subspace. From Step 1, for any $Q \in \tilde{E}$ such that $\| Q \| = 1$, there exists $\xi_Q > 0$ such that

$$I(\lambda Q) < 0 \quad \text{for every } |\lambda| \geq \xi_Q > 0.$$

Since $\tilde{E}$ is a finite dimensional subspace, we can choose an $R = R(\tilde{E}) > 0$ such that

$$I(u) < 0, \quad \forall\, u \in \tilde{E} \setminus B_R.$$

So, according to Lemma 2.4, possesses infinitely many distinct critical points, i.e., (1.1) has infinitely many distinct solutions.

Now we are in the position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 It is clear that $I(0)=0$, and by Lemma 3.3 we have known that I is a C^1 functional on E satisfying the (PS)-condition. On the other hand, in view of (H5), (2.1) and (3.1), we obtain that

$$I(u) \geq \frac{1}{2}\| u \|^2 - \frac{C_\vartheta^\vartheta}{\vartheta}\| b \|_\infty \| u \|^\vartheta, \tag{3.15}$$

which implies that I is bounded below on E. Hence by Lemma 2.5, $c = \inf_E I(u)$ is a critical value of I, namely, there is a critical point $u^* \in E$ such that $I(u^*) = c$ and $I'(u^*) = 0$. Moreover, this critical value c is a negative real number as the following argument will show, and so u^* is a nontrivial solution of (1.1) by Lemma 2.5.

In what follows, we focus our attention to obtain the existence of infinitely many solutions of (1.1). Now, we additionally have from (H3) that I is even. In order to apply Lemma 2.7 lemma, we prove that

$$\text{for any } j \in \mathbb{N} \text{ there exists } \varepsilon > 0 \text{ such that } \gamma(I^{-\varepsilon}) \geq j. \tag{3.16}$$

By (H4), there exist an open set $D \subset [0,T]$ with $t_0 \in D$, $\sigma > 0$ and $\eta > 0$ such that

$$W(t,u) \geq \eta |u|^\vartheta, \quad \forall (t,u) \in D \times \mathbb{R}^n, |u| \leq \sigma. \tag{3.17}$$

For any $j \in \mathbb{N}$, we take j disjoint open sets D_i such that $\bigcup_{i=1}^j D_i \subset D$. For $i = 1,2,\ldots,j$, let $u_i \in (W_0^{1,2}(D_i) \cap E) \setminus \{0\}$ with $\| u_i \| = 1$, and

$$E_j = \operatorname{span}\{u_1, u_2, \ldots, u_j\}, \quad S_j = \{u \in E_j : \| u \| = 1\}.$$

Then, for any $u \in E_j$, there exist $\lambda_i \in \mathbb{R}$, $i = 1,2,\ldots,j$ such that

$$u(t) = \sum_{i=1}^j \lambda_i u_i(t) \quad \text{for } t \in [0,T]. \tag{3.18}$$

From which it follows that

$$\| u \|_\vartheta = \left(\int_0^T |u(t)|^\vartheta\right)^{1/\vartheta} = \left(\sum_{i=1}^j |\lambda_i|^\vartheta \int_{D_i} |u_i(t)|^\vartheta\, dt\right)^{1/\vartheta} \tag{3.19}$$

and

$$\begin{aligned} \| u \|^2 &= \int_0^T [|\dot{u}(t)|^2 + (L(t)u(t), u(t))]dt \\ &= \sum_{i=1}^j \lambda_i^2 \int_{D_i} [|\dot{u}_i(t)|^2 + (L(t)u_i(t), u_i(t))]dt \\ &= \sum_{i=1}^j \lambda_i^2 \int_0^T [|\dot{u}_i(t)|^2 + (L(t)u_i(t), u_i(t))]dt \\ &= \sum_{i=1}^j \lambda_i^2 \| u_i \|^2 = \sum_{i=1}^j \lambda_i^2. \end{aligned} \tag{3.20}$$

Since all norms of a finite dimensional norm space are equivalent, there is a constant $d = d(j) > 0$ such that

$$d \| u \| \leq \| u \|_\vartheta, \quad \forall u \in E_j. \tag{3.21}$$

Note that $W(t,0) = 0$, and so according to (3.17)-(3.21), we have

$$\begin{aligned} I(su) &= \frac{s^2}{2}\| u \|^2 - \int_0^T W(t, su(t))dt \\ &= \frac{s^2}{2}\| u \|^2 - \sum_{i=1}^j \int_{D_i} W(t, s\lambda_i u_i(t))dt \\ &\leq \frac{s^2}{2}\| u \|^2 - \eta s^\vartheta \sum_{i=1}^j |\lambda_i|^\vartheta \int_{D_i} |u_i(t)|^\vartheta\, dt \\ &= \frac{s^2}{2}\| u \|^2 - \eta s^\vartheta \| u \|_\vartheta^\vartheta \\ &\leq \frac{s^2}{2}\| u \|^2 - \eta (ds)^\vartheta \| u \|^\vartheta \\ &= \frac{s^2}{2} - \eta (ds)^\vartheta \end{aligned} \tag{3.22}$$

for all $u \in S_j$ and sufficient small $s > 0$. In this case (3.17) is applicable, since u is continuous on $\overline{D}$ and so $|s\lambda_i u_i(t)| \le \sigma, \forall t \in D, i = 1,2,\cdots,j$ can be true for sufficiently small s. Therefore, it follows from (3.22) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$I(\delta u) < -\varepsilon \quad \text{for } u \in S_j. \tag{3.23}$$

Let

$$S_j^\delta = \{\delta u \, : \, u \in S_j\}, \Omega = \{(\lambda_1, \lambda_2, ..., \lambda_j) \in \mathbb{R}^j \, : \, \sum_{i=1}^{j} \lambda_i^2 < \delta^2\}.$$

Then it follows from (3.23) that

$$I(u) < -\varepsilon, \quad \forall u \in S_j^\delta,$$

which, together with the fact that I is an even C^1 functional on E, yields that

$$S_j^\delta \subset I^{-\varepsilon} \in \Sigma,$$

where $I^{-\varepsilon}$ and Σ have been previously introduced in Section 2. On the other hand, it follows from (3.18) and (3.20) that there exists an odd homeomorphism $\psi \in C(S_j^\delta, \partial\Omega)$. By some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [12]), we infer

$$\gamma(I^{-\varepsilon}) \ge \gamma(S_j^\delta) = j, \tag{3.24}$$

so (3.16) follows. Set

$$c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u),$$

where Σ_j is defined in Lemma 2.7 lemma. It follows from (3.24) and the fact that I is bounded from below on E (see (3.15)), we have $-\infty < c_j \le -\varepsilon < 0$, which implies that, for any $j \in \mathbb{N}$, c_j is a real negative number. By lemma 2.7 lemma and Remark 3, I has infinitely many nontrivial critical points, and consequently, (1.1) possesses infinitely many solutions.

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