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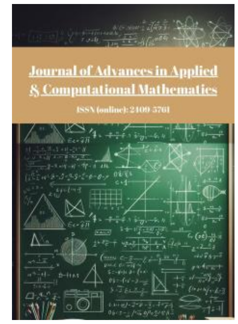

Hopf Bifurcation of a Delay Commensalism System with Density Dependent Birth Rates

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ABSTRACT

In ecology, commensalism and amensalism models are two kinds of important and interesting models. They have attracted much attention of ecologists and mathematicians in recent years. In this paper, we consider a two-species commensalism system with a discrete delay and density dependent birth rates.

First, we investigate the characteristic equation of the proposed system and study the distribution of its roots. We obtain that, when the delay τ is sufficiently small, the positive equilibrium is locally asymptotically stable, when τ increases to a critical value, the positive equilibrium loses its stability and a Hopf bifurcation occurs, as τ continues to increase, a family of periodic solutions bifurcate from the positive equilibrium. Then, by using the normal form theory and the center manifold theorem, we derive the precise formulae to determine the Hopf bifurcation direction and the stability of the bifurcating periodic solutions.

Numerical simulation results are included to support our theoretical analysis. We plot the trajectory graphs on $t-x$, $t-y$ plane respectively. We also plot phase graphs to illustrate the change of stability of the positive equilibrium and arise of periodic solution. In order to fully validate the occurrence of Hopf bifurcation, we use the numerical continuation package DDE-Biftool to generate bifurcation diagrams and accurately track stability changes of the positive equilibrium and periodic solution with respect to the delay parameter τ.

The commensalism model we propose considers both density dependent birth rate and time delay, it is of great practical and theoretical significance. The theoretical and numerical analysis that we do on the proposed system can make a supplement to the literature on the dynamics of delay commensalism systems.

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1. Introduction

In ecology, commensalism and amensalism are both biological interactions between two different species. For convenience and without loss of generality, let us denote one species by x and the other species by y. In a commensalism system, the density of y has a positive influence on the growth of density of x, while x has no influence on y. In a amensalism system, the density of y has a negative influence on the growth of density of x, while x has no influence on y [1-3]. In 2003, Sun and Wei [2, 3] first proposed a mathematical model to describe a commensalism or amensalism system

$$\begin{cases} \dfrac{dx}{dt} = r_1 x\left(1 - \dfrac{x}{k_1} + \dfrac{Ty}{k_1}\right), \\ \dfrac{dy}{dt} = r_2 y\left(1 - \dfrac{y}{k_2}\right), \end{cases}$$

where x and y refer to the populations of two species at time t, respectively; r_1, r_2, k_1, k_2 are positive constants. The model is said to be commensalism if T is positive and is said to be amensalism if T is negative. In [2, 3], the existence and stability (or instability) of all possible equilibria were studied.

In recent years, many topics on commensalism and amensalism systems have been studied, such as existence of equilibria, local and global stabilities of equilibria [4-9], global dynamics and phase diagrams [10-12], existence of positive periodic solutions [13, 14], bifurcation analysis of commensalism and amensalism systems with delays [15-19], dynamics of discrete amensalism and commensalism systems [20-22], dynamics of amensalism systems with fear effect [21-24], and with Allee effect [25-27], commensalism and amensalism systems with harvesting [28-32], seasonal commensalism system considering climate change [33], population density control for a commensal symbiosis model [34], commensalism system with distributed lags on time scales [35],

In 2018, Chen *et al*. [5] proposed the following two-species commensal symbiosis model with density dependent birth rate

$$\begin{cases} \frac{dx}{dt} = x\left(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}y\right), \\ \frac{dy}{dt} = y\left(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\right), \end{cases} \tag{1.1}$$

where x, y are the densities of the two species, respectively; all the parameters are positive. By constructing some suitable Lyapunov functions, they showed that under some suitable assumptions, all of the four equilibria may be globally asymptotically stable.

In fact, all of the above systems can be considered as extensions of the well-known logistic model, which was proposed in 1838 by Verhulst [36]:

$$\frac{1}{N(t)}\frac{dN(t)}{dt} = r\left(1 - \frac{N(t)}{K}\right),$$

where r is the intrinsic growth rate (birth rate minus death rate), K is the carrying capacity. The term $\frac{1}{N(t)}\frac{dN(t)}{dt}$ on the left side of the equation is per capita growth rate.

The fundamental idea of the logistic model is that the density of species has negative influence on the per capita growth rate. However, in reality, such an influence usually does not take place instantly, but with a time lag. In order to reflect the existence of time lag, delay logistic model for ecological systems was proposed in the 1940s and studied from then on [37, 38]:

$$\frac{dN(t)}{N(t)dt} = r\left(1 - \frac{N(t-\tau)}{K}\right). \tag{1.2}$$

Equation (1.2) can be rewritten as:

$$\frac{dN(t)}{dt} = N(t)(a - bN(t-\tau)), \tag{1.3}$$

where $a = r$, $b = r/K$, a is the intrinsic growth rate (birth rate minus death rate), b is called the density dependent coefficient.

In our recent work [15] in 2021, we studied the existence and properties of Hopf bifurcations for the commensalism system with a discrete delay and Beddington-DeAngelis functional response

$$\begin{cases} \dfrac{dx}{dt} = x(a_1 - b_1 x(t-\tau)), \\ \dfrac{dy}{dt} = y(a_2 - b_2 y + \dfrac{cx}{mx+ny+1}). \end{cases}$$

In our recent work [16] in 2022, we proposed and studied a two-species commensalism(amensalism) systems with distributed delays

$$\begin{cases} \dfrac{dx}{dt} = x\left(a_1 - b_1 \int_{-\infty}^{t} G_1(t-s)x(s)ds\right), \\ \dfrac{dy}{dt} = y\left(a_2 - b_2 y + \dfrac{c\left(\int_{-\infty}^{t} G_2(t-s)x(s)ds\right)^p}{1+\left(\int_{-\infty}^{t} G_2(t-s)x(s)ds\right)^p}\right), \end{cases} \tag{1.4}$$

where $t \geq 0$, $p \geq 1$, $c \in R$; $G_i(s): [0,\infty) \to [0,\infty)$ $(i = 1,2)$ are the delay kernels of a distributed delay and satisfy

$$G_i(s) \geq 0,\ \forall\ s \geq 0 \quad and \quad \int_0^{\infty} G_i(s)ds = 1, \qquad for\ \ i = 1,2.$$

If $c > 0$, then the first species has positive effect on the second species, system (1.4) is a commensalism system. If $c < 0$, the first species has negative effect on the second species, system (1.4) is an amensalism system. The effect of the first species on the second one is described by the Holling type functional response.

Motivated by the above works, we let the intrinsic growth rate be a function of population density N in the delay logistic model (1.3) and divide it into two parts: $\frac{r}{a+N} - d$, where $\frac{r}{a+N}$ is the birth rate and $d > 0$ is the death rate. We also consider a discrete time delay in order to reflect the fact that the dynamics at present may depend on the history of the system and propose the following model

$$\begin{cases} \dfrac{dx}{dt} = x\left(\dfrac{r_1}{a_1+x} - d_1 - b_1 x(t-\tau)\right), \\ \dfrac{dy}{dt} = y\left(\dfrac{r_2}{a_2+y} - d_2 - b_2 y + cx\right), \end{cases} \tag{1.5}$$

where, (i) $t \geq 0$, $x(t)$ and $y(t)$ are the densities of the two species at time t, respectively; (ii) parameters r_i, a_i, b_i, d_i $(i = 1,2)$ and c are all positive; (iii) $\frac{r_1}{a_1+x}$ and $\frac{r_2}{a_2+y}$ are birth rates of the first and the second species, respectively, the birth rates of the two species decrease as the densities of the two species increase; (iv) d_1, d_2 are the death rates of the first and the second species, respectively; (v) b_1, b_2 are the density dependent coefficients of the first and the second species, respectively. Parameter $c > 0$ means that the first species has positive effect on the second species, then system (1.5) is a commensalism system. The commensalism model we propose considers both density dependent birth rate and time delay, it has great novelty and is of great practical and theoretical significance.

The remainder of this paper is organized as follows. In Section 2, we study the distribution of roots of the characteristic equation, and study the existence of local Hopf bifurcation of system (1.5) based on the existence and stability results of equilibria obtained in [5]. In Section 3, we study the properties of Hopf bifurcation and derive the formulae determining the direction of Hopf bifurcation and stability of the bifurcating periodic solution. In Section 4, numerical simulations are given to verify the theoretic analysis. The conclusion is made in Section 5.

2. Existence of Hopf Bifurcation

In this section, we study the existence of local Hopf bifurcation of system (1.5) based on the existence and stability results of equilibria obtained in [5].

When $\tau = 0$, delay system (1.5) reduces to the following ordinary differential system of the form (1.1):

$$\begin{cases} \frac{dx}{dt} = x\left(\frac{r_1}{a_1+x} - d_1 - b_1 x\right), \\ \frac{dy}{dt} = y\left(\frac{r_2}{a_2+y} - d_2 - b_2 y + cx\right). \end{cases} \tag{2.1}$$

The existence and stability of the equilibria of system (2.1) have been fully investigated in [5], we copy the results here (one can refer to Theorem 2.1 of [5]).

Theorem 2.1.

1. Assume that $r_1/a_1 > d_1$ and $r_2/a_2 > d_2 - cx^*$, then system (2.1) admits a unique positive equilibrium $E_4(x^*, y^*)$, which is globally asymptotically stable, where

$$\begin{aligned} x^* &:= \frac{\sqrt{(d_1+a_1b_1)^2 - 4b_1(a_1d_1 - r_1)} - (d_1 + a_1 b_1)}{2b_1}, \\ y^* &:= \frac{\sqrt{(d_2+a_2b_2-cx^*)^2 - 4b_2(a_2d_2 - r_2 - ca_2x^*)} - (d_2 + a_2 b_2 - cx^*)}{2b_2}; \end{aligned} \tag{2.2}$$

2. Assume that $r_1/a_1 > d_1$ and $r_2/a_2 < d_2 - cx^*$, then system (2.1) admits a nonnegative boundary equilibrium $E_3(x^*, 0)$, which is globally asymptotically stable;

3. Assume that $r_2/a_2 > d_2$ and $r_1/a_1 < d_1$, then system (2.1) admits a nonnegative boundary equilibrium $E_2(0, \hat{y})$, which is globally asymptotically stable, where

$$\hat{y} := \frac{\sqrt{(d_2 + a_2b_2)^2 - 4b_2(a_2d_2 - r_2)} - (d_2 + a_2b_2)}{2b_2};$$

4. System (2.1) always admits a boundary equilibrium $E_1(0,0)$. Assume that $r_2/a_2 < d_2$ and $r_1/a_1 < d_1$, then $E_1(0,0)$ is globally asymptotically stable.

In the rest of this paper, we always make the following assumption:

(H1) $r_1/a_1 > d_1$ and $r_2/a_2 > d_2 - cx^*$, where x^* is given in (2.2).

We now study the dynamics around $E_4(x^*, y^*)$ under the case $\tau > 0$. We move $E_4(x^*, y^*)$ of system (1.5) to the origin by letting $\bar{x} = x - x^*$, $\bar{y} = y - y^*$, and denoting $\bar{x}$ and $\bar{y}$ still by x and y, respectively. Then system (1.5) is transformed into

$$\begin{cases} \frac{dx}{dt} = \left(\frac{r_1 a_1}{(a_1+x^*)^2} - d_1 - b_1 x^*\right)x - b_1 x^* x(t-\tau) + f_1(x(t), x(t-\tau)), \\ \frac{dy}{dt} = cy^* x + \left(\frac{r_2 a_2}{(a_2+y^*)^2} - d_2 - 2b_2 y^* + cx^*\right)y + f_2(x(t), y(t)), \end{cases} \tag{2.3}$$

where f_1 and f_2 are high order terms,

$$\begin{aligned} f_1 &= \left(\frac{a_1 + x^*}{a_1 + x^* + x} - 1\right)\frac{r_1 x}{a_1 + x^*} + \left(\frac{a_1 + x^*}{a_1 + x^* + x} - 1 + \frac{x}{a_1 + x^*}\right)\frac{r_1 x^*}{a_1 + x^*} - b_1 x x(t-\tau), \\ f_2 &= \left(\frac{a_2 + y^*}{a_2 + y^* + y} - 1\right)\frac{r_2 y}{a_2 + y^*} + \left(\frac{a_2 + y^*}{a_2 + y^* + y} - 1 + \frac{y}{a_2 + y^*}\right)\frac{r_2 y^*}{a_2 + y^*} - b_2 y^2 + cxy. \end{aligned}$$

The linearization of system (2.3) at $(0,0)$ is given by

$$\begin{cases} \frac{dx}{dt} = \left(\frac{r_1 a_1}{(a_1+x^*)^2} - d_1 - b_1 x^*\right) x - b_1 x^* x(t-\tau), \\ \frac{dy}{dt} = c y^* x + \left(\frac{r_2 a_2}{(a_2+y^*)^2} - d_2 - 2b_2 y^* + c x^*\right) y. \end{cases} \tag{2.4}$$

From [39, 40] we know that the characteristic equation of system (2.4) at $(0,0)$ is

$$\begin{vmatrix} \lambda - \frac{r_1 a_1}{(a_1+x^*)^2} + d_1 + b_1 x^* + b_1 x^* e^{-\lambda\tau} & 0 \\ -c y^* & \lambda - \frac{r_2 a_2}{(a_2+y^*)^2} + d_2 + 2b_2 y^* - c x^* \end{vmatrix} = 0. \tag{2.5}$$

Suppose that $\lambda = i\omega$ with $\omega > 0$ is a root of (2.5), substituting $\lambda = i\omega$ into (2.5), we can obtain

$$\left(i\omega - \frac{r_1 a_1}{(a_1 + x^*)^2} + d_1 + b_1 x^* + b_1 x^* e^{-i\omega\tau}\right)\left(i\omega - \frac{r_2 a_2}{(a_2 + y^*)^2} + d_2 + 2b_2 y^* - c x^*\right) = 0.$$

Since $\left(i\omega - \frac{r_2 a_2}{(a_2+y^*)^2} + d_2 + 2b_2 y^* - c x^*\right) \neq 0$, it can only be

$$i\omega - \frac{r_1 a_1}{(a_1+x^*)^2} + d_1 + b_1 x^* + b_1 x^* e^{-i\omega\tau} = 0. \tag{2.6}$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} -\frac{r_1 a_1}{(a_1+x^*)^2} + d_1 + b_1 x^* + b_1 x^* \cos\omega\tau = 0, \\ \omega - b_1 x^* \sin\omega\tau = 0. \end{cases} \tag{2.7}$$

From the first equation of (2.7) we have $\cos\omega\tau = -\frac{d_1+b_1x^*}{b_1(a_1+x^*)} < 0$. Then $\lambda = i\omega$ is a root of (2.5) if and only if $-\frac{d_1+b_1x^*}{b_1(a_1+x^*)} > -1$, which is equivalent to

(H2) $d_1 < a_1 b_1$.

If (H2) holds, the solution of (2.7) is given by

$$\omega_0 := \frac{x^*}{a_1+x^*}\sqrt{(a_1 b_1 + d_1 + 2b_1 x^*)(a_1 b_1 - d_1)}, \tag{2.8}$$

and the corresponding τ is given by

$$\tau = \tau_k := \frac{1}{\omega_0}\left[\arccos\left(-\frac{d_1+b_1x^*}{b_1(a_1+x^*)}\right) + 2k\pi\right],\ k = 0,1,2,\cdots. \tag{2.9}$$

If (H2) is violated, then (2.5) has no purely imaginary root, by the conclusions in Sections 2.1 and 2.2 of [36] (see also [41]), we know that all the roots of (2.5) have negative real parts.

Next, we suppose (H2) holds and verify the so-called transversality condition around E_4 at $\tau = \tau_k$, denote $J_1 = \frac{r_1 a_1}{(a_1+x^*)^2} - d_1 - b_1 x^*$. By differentiating two sides of (2.5) with respect to τ, we can obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1 - b_1 x^* \tau e^{-\lambda\tau}}{b_1 x^* \lambda e^{-\lambda\tau}} = \frac{1}{\lambda(J_1 - \lambda)} - \frac{\tau}{\lambda}.$$

When $\tau = \tau_k$ (i.e. $\lambda = \pm i\omega_0$), we have

$$Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\right) = Re\left(\frac{1}{\lambda(J_1-\lambda)}\right) = Re\left(\frac{J_1\lambda+\lambda^2}{\lambda^2(J_1^2-\lambda^2)}\right) = \frac{1}{J_1^2+\omega_0^2} > 0. \tag{2.10}$$

This indicates that the transversality condition holds.

Then, by the Hopf bifurcation theorem for delay differential equations (see for example, Theorem 3.1.1 and Corollary 3.1.1 of [36]), Lemma 2.2 of [42] and summarizing the aforementioned arguments, we have the following results on the stability of E_4 of system (1.5).

Theorem 2.2.

If (H2) is violated, then the positive equilibrium $E_4 = (x^*, y^*)$ of system (1.5) is asymptotically stable for all $\tau > 0$.

If (H2) holds, let τ_k be defined in (2.9), then the following conclusions hold true:

(i) The positive equilibrium $E_4 = (x^*, y^*)$ of system (1.5) is asymptotically stable if $\tau \in [0, \tau_0)$, while it is unstable if $\tau > \tau_0$;

(ii) System (1.5) undergoes a Hopf bifurcation at the positive equilibrium $E_4 = (x^*, y^*)$ when $\tau = \tau_k \ (k = 0,1,2,\cdots)$.

3. Direction of Hopf Bifurcation and Stability of Bifurcating Periodic Solutions

In this section, we shall study the direction of Hopf bifurcation and the stability of bifurcating periodic solutions by applying the normal form method and center manifold theorem for the general functional differential equations developed in [43]. We divide this section into 3 parts.

3.1. Transformation of the Original Equation (1.5)

Let $\tau = \tau_k + \mu$, where $\mu \in R$. Recall that system (2.3) can be transformed from system (1.5) by letting $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$ and dropping the bars. In system (2.3), let $u_1(t) = x(\tau t),\ u_2(t) = y(\tau t)$. Then, system (2.3) can be rewritten as a system of functional differential equations in $C([-1,0], R^2)$ of the form

$$\begin{cases} \frac{du_1(t)}{dt} = (\tau_k + \mu)\left[\left(\frac{r_1 a_1}{(a_1+x^*)^2} - d_1 - b_1 x^*\right) u_1(t) - b_1 x^* u_1(t-1) + f_1(u_1(t), u_1(t-1))\right], \\ \frac{du_2(t)}{dt} = (\tau_k + \mu)\left[c y^* u_1(t) + \left(\frac{r_2 a_2}{(a_2+y^*)^2} - d_2 - 2b_2 y^* + c x^*\right) u_2(t) + f_2(u_1(t), u_2(t))\right]. \end{cases} \tag{3.1}$$

For convenience, we denote

$$J_1 := \frac{r_1 a_1}{(a_1+x^*)^2} - d_1 - b_1 x^*, \quad J_2 := \frac{r_2 a_2}{(a_2+y^*)^2} - d_2 - 2b_2 y^* + c x^*.$$

For $\varphi := (\varphi_1, \varphi_2)^T \in C([-1,0], \mathrm{R}^2)$, define

$$\begin{aligned} F(\mu, \varphi) &:= (\tau_k + \mu)\begin{pmatrix} f_1(\varphi_1(0), \varphi_1(-1)) \\ f_2(\varphi_1(0), \varphi_2(0)) \end{pmatrix}, \\ L_\mu(\varphi) &:= (\tau_k + \mu)\begin{pmatrix} J_1 & 0 \\ c y^* & J_2 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} + (\tau_k + \mu)\begin{pmatrix} -b_1 x^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{pmatrix}, \end{aligned} \tag{3.2}$$

then, system (3.1) can be further rewritten as

$$\frac{du}{dt} = L_\mu(u_t) + F(\mu, u_t), \tag{3.3}$$

where, for $\theta \in [-1,0]$,

$$u(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad u_t(\theta) := u(t+\theta) = \begin{pmatrix} u_1(t+\theta) \\ u_2(t+\theta) \end{pmatrix}.$$

3.2. Center Manifold Equation and Its Poincaré Normal Form

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)\ (\theta \in [-1,0])$ whose elements are of bounded variation such that

$$L_\mu(\varphi) = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta),\ \varphi \in C([-1,0],\mathrm{R}^2). \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_k+\mu)\begin{pmatrix} J_1 & 0 \\ cy^* & J_2 \end{pmatrix}\delta(\theta) - (\tau_k+\mu)\begin{pmatrix} -b_1x^* & 0 \\ 0 & 0 \end{pmatrix}\delta(\theta+1),$$

where $\delta(\theta)$ is the Dirac function. Then (3.4) is satisfied.

If φ is any given function in $C([-1,0],R^2)$ and $u(\varphi)$ is the unique solution of the linearized equation $\frac{du}{dt} = L_\mu(u_t)$ of equation (3.3) with the initial function φ at zero, from Section 7.1 of [39] we know that the solution operator $T(t): C([-1,0],R^2) \to C([-1,0],R^2)$ with $t \geq 0$ is a strongly continuous semigroup of linear transformation on $[0,\infty)$ and the infinitesimal generator $A(\mu)$ is given by

$$A(\mu)\varphi(\theta) := \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\varphi(s), & \theta = 0. \end{cases} \tag{3.5}$$

For $\varphi \in C([-1,0],\mathrm{R}^2)$, define

$$R(\mu)\varphi(\theta) := \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu,\varphi), & \theta = 0. \end{cases}$$

Then, system (3.3) can be transformed into an operator differential equation of the form

$$\frac{du_t}{dt} = A(\mu)u_t + R(\mu)u_t. \tag{3.6}$$

Let R^{2*} be the 2-dimensional vector space of row vectors. For $\psi \in C([0,1],R^{2*})$, define

$$A^*\psi(s) := \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} \psi(-t)d\eta(t,0), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s),\varphi(\theta)\rangle := \overline{\psi(0)}\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi(\xi-\theta)}d\eta(\theta)\varphi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, then $A(0)$ and A^* are adjoint operators, $i\omega_0\tau_k$ are eigenvalues of $A(0)$, thus, they are also eigenvalues of A^*.

Suppose that q is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_k$, q^* is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_k$. By direct computing, we can choose

$$q(\theta) = (1,\alpha)^T e^{i\omega_0\tau_k\theta},\quad q^*(s) = D(1,\alpha^*)e^{i\omega_0\tau_k s} \tag{3.7}$$

such that $\langle q^*,q\rangle = 1,\ \langle q^*,\bar{q}\rangle = 0$, where

$$\alpha = \frac{cy^*}{i\omega_0 - J_2},\quad \alpha^* = -\frac{i\omega_0 + J_1 - b_1x^*e^{i\omega_0\tau_k}}{cy^*},\quad D = (1+\bar{\alpha}\alpha^* - b_1x^*\tau_k e^{i\omega_0\tau_k})^{-1}.$$

From (2.6) we know that $-i\omega_0 - J_1 + b_1x^*e^{i\omega_0\tau_k} = 0$, hence $\alpha^* = 0$, D can be simplified as $D = (1 - b_1x^*\tau_k e^{i\omega_0\tau_k})^{-1}$.

In what follows, we shall use the method in [43] and [44] to compute the coordinates describing the center manifold C_0 at $\mu = 0$. Denote the solution of equation (3.3) when $\mu = 0$ by u_t, $u_t(\theta) := u(t+\theta) = (u_1(t+\theta),\ u_2(t+\theta))^T$ and define

$$z(t) = \langle q^*,u_t\rangle,\quad W(t,\theta) = u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta) = u_t(\theta) - 2Re(z(t)q(\theta)). \tag{3.8}$$

On the center manifold C_0, we have $W(t,\theta) = W(z(t),\bar{z}(t),\theta)$, where

$$W(z(t),\bar{z}(t),\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots. \tag{3.9}$$

In fact, $z(t)$ and $\bar{z}(t)$ are local coordinates of center manifold C_0 in the direction of q^* and $\bar{q}^*$. Recall that $A(\mu)$ with $\mu = 0$ and A^* are adjoint operators, that is $\langle\psi, A(0)\varphi\rangle = \langle A^*\psi, \varphi\rangle$ for $(\psi,\varphi) \in D(A^*) \times D(A(0))$, hence we have

$$\begin{aligned}\frac{dz}{dt} &= \langle q^*, \frac{du}{dt}\rangle = \langle q^*, A(0)u_t + R(0)u_t\rangle = \langle A^*q^*, u_t\rangle + \langle q^*, R(0)u_t\rangle \\ &= \langle -i\omega_0\tau_k q^*, u_t\rangle + \bar{q}^*(0)F(0, W(z,\bar{z},0) + 2Re(zq(0))) \\ &= i\omega_0\tau_k z + \bar{q}^*(0)F(0, W(z,\bar{z},0) + 2Re(zq(0))) \\ &\overset{\text{def}}{=} i\omega_0\tau_k z + \bar{q}^*(0)F_0(z,\bar{z}),\end{aligned}$$

which can be rewritten in the following form:

$$\frac{dz}{dt} = i\omega_0\tau_k z(t) + g(z,\bar{z}), \tag{3.10}$$

with

$$g(z,\bar{z}) := \bar{q}^*(0)F_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.11}$$

By the results in Section 6.2 of [45] and letting

$$z = \xi + (\frac{g_{20}}{i\omega_0\tau_k})\frac{\xi^2}{2} - (\frac{g_{11}}{i\omega_0\tau_k})\xi\bar{\xi} - (\frac{g_{02}}{3i\omega_0\tau_k})\frac{\bar{\xi}^2}{2} + \cdots,$$

system (3.10) can be transformed into the Poincaré normal form

$$\frac{d\xi}{dt} = i\omega_0\tau_k\xi + c_1(0)\xi|\xi|^2 + O(|\xi|^5),$$

where

$$c_1(0) = \frac{i}{2\omega_0\tau_k}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}. \tag{3.12}$$

In the next subsection, we shall compute the coefficients g_{20}, g_{11}, g_{02}, and g_{21}.

3.3. Computation of the Coefficients g_{20}, g_{11}, g_{02}, g_{21} in the Poincaré Normal Form

By (3.8), we have

$$u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta))^T = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta),$$

this together with (3.7), (3.8) and (3.9) gives

$$\begin{aligned}u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\ u_{1t}(-1) &= e^{-i\omega_0\tau_k}z + e^{i\omega_0\tau_k}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3).\end{aligned}$$

By (3.2), we have

$$g(z,\bar{z}) = \bar{q}^*(0)F_0(z,\bar{z}) = \bar{D}(1,0)\tau_k\begin{pmatrix} f_1(u_{1t}(0), u_{1t}(-1)) \\ f_2(u_{1t}(0), u_{2t}(0))\end{pmatrix} = \bar{D}\tau_k f_1(u_{1t}(0), u_{1t}(-1)).$$

Writing the nonlinear terms f_1 and f_2 in terms of z and $\bar{z}$, we have

$$f_1(u_{1t}(0),u_{1t}(-1)) = M_{20}^{(1)}z^2 + M_{11}^{(1)}z\bar{z} + M_{02}^{(1)}\bar{z}^2 + M_{30}^{(1)}z^3 + M_{21}^{(1)}z^2\bar{z} + M_{12}^{(1)}z\bar{z}^2 + M_{03}^{(1)}\bar{z}^3 + O(|(z,\bar{z})|^4),$$
$$f_2(u_{1t}(0),u_{2t}(0)) = M_{20}^{(2)}z^2 + M_{11}^{(2)}z\bar{z} + M_{02}^{(2)}\bar{z}^2 + O(|(z,\bar{z})|^3).$$

Where

$$M_{20}^{(1)} := -\frac{a_1 r_1}{(a_1+x^*)^3} - b_1 e^{-i\omega_0\tau_k}$$

$$M_{11}^{(1)} := -\frac{2a_1 r_1}{(a_1+x^*)^3} - b_1(e^{-i\omega_0\tau_k} + e^{i\omega_0\tau_k})$$

$$M_{02}^{(1)} := -\frac{a_1 r_1}{(a_1+x^*)^3} - b_1 e^{i\omega_0\tau_k}$$

$$M_{21}^{(1)} := -\frac{a_1 r_1}{(a_1+x^*)^3}\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) + \frac{3a_1 r_1}{(a_1+x^*)^4}$$
$$-b_1\left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}(-1)}{2} + e^{i\omega_0\tau_k}\frac{W_{20}^{(1)}(0)}{2} + e^{-i\omega_0\tau_k}W_{11}^{(1)}(0)\right)$$

Comparing the coefficients with (3.11), we have

$$g_{20} = 2\bar{D}\tau_k M_{20}^{(1)}, \quad g_{11} = \bar{D}\tau_k M_{11}^{(1)}, \quad g_{02} = 2\bar{D}\tau_k M_{02}^{(1)}, \quad g_{21} = 2\bar{D}\tau_k M_{21}^{(1)}.$$

Thus, g_{20}, g_{11} and g_{02} in (3.12) are all expressed by the coefficients of system (1.5).

Next, we shall determine the expression of g_{21}, in order to do that, we need to compute $W_{20}^{(1)}(\theta)$ and $W_{11}^{(1)}(\theta)$. From (3.6), (3.8) and (3.10), we have

$$\frac{dW}{dt} = \begin{cases} A(0)W - 2Re[g(z,\bar{z})q(\theta)], & \theta \in [-1,0), \\ A(0)W - 2Re[g(z,\bar{z})q(0)] + F_0, & \theta = 0. \end{cases} \tag{3.13}$$

On the other hand, on the center manifold C_0 close enough to the origin, we have

$$\begin{aligned}\frac{dW}{dt} &= W_z\frac{dz}{dt} + W_{\bar{z}}\frac{d\bar{z}}{dt} = (W_{20}(\theta)z + W_{11}(\theta)\bar{z})\frac{dz}{dt} + (W_{11}(\theta)z + W_{02}(\theta)\bar{z})\frac{d\bar{z}}{dt} \\ &= (W_{20}(\theta)z + W_{11}(\theta)\bar{z})(i\omega_0\tau_k z + g(z,\bar{z})) + (W_{11}(\theta)z + W_{02}(\theta)\bar{z})(-i\omega_0\tau_k\bar{z} + \bar{g}(z,\bar{z})).\end{aligned}$$

Substituting the above equation and equation (3.9) into equation (3.13), comparing the coefficients of z^2 and $z\bar{z}$, we get

$$A(0)W_{20}(\theta) = \begin{cases} 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), & \theta \in [-1,0), \\ 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) - 2\tau_k\left(M_{20}^{(1)}, M_{20}^{(2)}\right)^T, & \theta = 0, \end{cases} \tag{3.14}$$

and

$$A(0)W_{11}(\theta) = \begin{cases} g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), & \theta \in [-1,0), \\ g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta) - \tau_k\left(M_{11}^{(1)}, M_{11}^{(2)}\right)^T, & \theta = 0. \end{cases} \tag{3.15}$$

By (3.5), we know that when $\theta \in [-1,0)$, (3.14) and (3.15) are ordinary differential equations of $W_{20}(\theta)$ and $W_{11}(\theta)$, with $W_{20}(\theta) = \left(W_{20}^{(1)}(\theta), W_{20}^{(2)}(\theta)\right)^T$, $W_{11}(\theta) = \left(W_{11}^{(1)}(\theta), W_{11}^{(2)}(\theta)\right)^T$, whose solutions are given by

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}e^{i\omega_0\tau_k\theta}}{\omega_0\tau_k}q(0) + \frac{i\bar{g}_{02}e^{-i\omega_0\tau_k\theta}}{3\omega_0\tau_k}\bar{q}(0) + E_1 e^{2i\omega_0\tau_k\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}e^{i\omega_0\tau_k\theta}}{\omega_0\tau_k}q(0) + \frac{i\bar{g}_{11}e^{-i\omega_0\tau_k\theta}}{\omega_0\tau_k}\bar{q}(0) + E_2, \end{aligned} \tag{3.16}$$

where, $E_1 = \left(E_1^{(1)}, E_1^{(2)}\right)^T \in \mathrm{R}^2$, $E_2 = \left(E_2^{(1)}, E_2^{(2)}\right)^T \in \mathrm{R}^2$ are two constant vectors to be determined. In what follows, we shall compute E_1 and E_2.

Consider the case when $\theta = 0$. By (3.5), (3.14) and (3.15), we have

$$\begin{aligned} \int_{-1}^{0} d\eta(\theta)W_{20}(\theta) &= 2i\omega_0\tau_k W_{20}(0) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - 2\tau_k\left(M_{20}^{(1)}, M_{20}^{(2)}\right)^T, \\ \int_{-1}^{0} d\eta(\theta)W_{11}(\theta) &= g_{11}q(0) + \bar{g}_{11}\bar{q}(0) - \tau_k\left(M_{11}^{(1)}, M_{11}^{(2)}\right)^T. \end{aligned} \tag{3.17}$$

Substituting (3.16) into (3.17), and noticing that

$$\left(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta)\right) q(0) = 0, \quad \left(-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta)\right) \bar{q}(0) = 0,$$

we have

$$\left(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta} d\eta(\theta)\right) E_1 = 2\tau_k\left(M_{20}^{(1)}, M_{20}^{(2)}\right)^T, \quad and \quad \int_{-1}^{0} d\eta(\theta) E_2 = -\tau_k\left(M_{11}^{(1)}, M_{11}^{(2)}\right)^T,$$

which lead to

$$\begin{aligned} E_1 &= \frac{2}{|B|}\begin{pmatrix} 2i\omega_0 - J_2 & 0 \\ cy^* & 2i\omega_0 - J_1 + b_1x^*e^{-2i\omega_0\tau_k} \end{pmatrix}\begin{pmatrix} M_{20}^{(1)} \\ M_{20}^{(2)} \end{pmatrix}, \\ E_2 &= -\frac{1}{J_2(J_1 - b_1x^*)}\begin{pmatrix} J_2 & 0 \\ -cy^* & J_1 - b_1x^* \end{pmatrix}\begin{pmatrix} M_{11}^{(1)} \\ M_{11}^{(2)} \end{pmatrix}, \end{aligned} \tag{3.18}$$

where

$$|B| = (2i\omega_0 - J_2)(2i\omega_0 - J_1 + b_1x^*e^{-2i\omega_0\tau_k}).$$

Substituting (3.18) into (3.16), we have

$$W_{11}^{(1)}(-1) = -\frac{ig_{11}e^{-i\omega_0\tau_k}}{\omega_0\tau_k} + \frac{i\bar{g}_{11}e^{i\omega_0\tau_k}}{\omega_0\tau_k} - \frac{M_{11}^{(1)}}{J_1 - b_1x^*}, \quad W_{11}^{(1)}(0) = -\frac{ig_{11}}{\omega_0\tau_k} + \frac{i\bar{g}_{11}}{\omega_0\tau_k} - \frac{M_{11}^{(1)}}{J_1 - b_1x^*},$$

$$W_{20}^{(1)}(-1) = \frac{ig_{20}e^{-i\omega_0\tau_k}}{\omega_0\tau_k} + \frac{i\bar{g}_{02}e^{i\omega_0\tau_k}}{3\omega_0\tau_k} + \frac{2M_{20}^{(1)}e^{-2i\omega_0\tau_k}}{2i\omega_0 - J_1 + b_1x^*e^{-2i\omega_0\tau_k}},$$

$$W_{20}^{(1)}(0) = \frac{ig_{20}}{\omega_0\tau_k} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k} + \frac{2M_{20}^{(1)}}{2i\omega_0 - J_1 + b_1x^*e^{-2i\omega_0\tau_k}}.$$

Therefore, $c_1(0) = \frac{i}{2\omega_0\tau_k}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}$ can be expressed in terms of the parameters of system (1.5).

By Theorem 2.2 in Section 6.2 of [45] and (2.10), it follows that if $Re(c_1(0)) < 0$, the Hopf bifurcation of system (1.5) is forward, while if $Re(c_1(0)) > 0$, the Hopf bifurcation is backward. On the other hand, for $k \geq 1$, at $\tau = \tau_k$, all the Hopf bifurcating periodic solutions are unstable. However, at $\tau = \tau_0$, the bifurcating periodic solution is stable if $Re(c_1(0)) < 0$, while unstable if $Re(c_1(0)) > 0$.

We are now in the position to state the following results.

Theorem 3.1.

Let τ_k and $c_1(0)$ be defined in (2.9) and (3.12) respectively. Then, the following conclusions on the Hopf bifurcations of system (1.5) hold true:

(i) If $Re(c_1(0)) < 0$ $(resp., Re(c_1(0)) > 0)$ holds, the Hopf bifurcation is forward (resp., backward);

(ii) The Hopf bifurcations at $\tau = \tau_k$, with $k \geq 1$, are always subcritical;

(iii) If $Re(c_1(0)) < 0$ $(resp., Re(c_1(0)) > 0)$ holds, the Hopf bifurcation at $\tau = \tau_0$ is supercritical (resp., subcritical).

Remark 3.1. The Hopf bifurcation with respect to parameter τ at $\tau = \tau_k$ is said to be backward (resp., forward) if there is a small amplitude periodic orbit for each $\tau \in (\tau_k - \varepsilon, \tau_k)$ (resp., $\tau \in (\tau_k, \tau_k + \varepsilon)$) where $\varepsilon > 0$ is a small constant, and is said to be supercritical (resp., subcritical) if the bifurcating periodic solutions are orbitally asymptotically stable (resp., unstable) [46].

Remark 3.2. From subsection 3.3, we see that $c_1(0)$ in (3.12) can be expressed in terms of the parameters of system (1.5). However, the sign of $c_1(0)$ is unclear since $\omega_0\tau$ in (2.9) is not a special value.

4. Numerical Simulations

In this section, we consider one special case of system (1.5) and give some numerical results to support our analytical conclusions obtained in previous sections.

Consider system (1.5) with $r_1 = 6,\ a_1 = 10,\ b_1 = 0.1,\ d_1 = 0.2, r_2 = 2,\ a_2 = 3,\ b_2 = 0.1,\ d_2 = 0.5,\ c = 0.1$, that is,

$$\begin{cases} \frac{dx}{dt} = x\left(\frac{6}{10+x} - 0.2 - 0.1x(t-\tau)\right), \\ \frac{dy}{dt} = y\left(\frac{2}{3+y} - 0.5 - 0.1y + 0.1x\right). \end{cases} \tag{4.1}$$

The values of parameters are chosen hypothetically and have no particular biological background. System (4.1) has a positive equilibrium $E_4 \approx (2.718,1.845)$ by (2.2). By (2.8) and (2.9), we have $\omega_0 \approx 0.252$, $\tau_0 \approx 7.730$. According to Theorem 2.2, the positive equilibrium E_4 is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. We take the initial condition to be $(x(t), y(t)) = (6,4)$ for $t \in [-\tau, 0]$.

Based on some methods and algorithms of numerical bifurcation analysis [47], we plot the following Figs. **1** and **2** using the DDE-Biftool, which is a MATLAB package for numerical bifurcation analysis of delay differential equations [48-50].

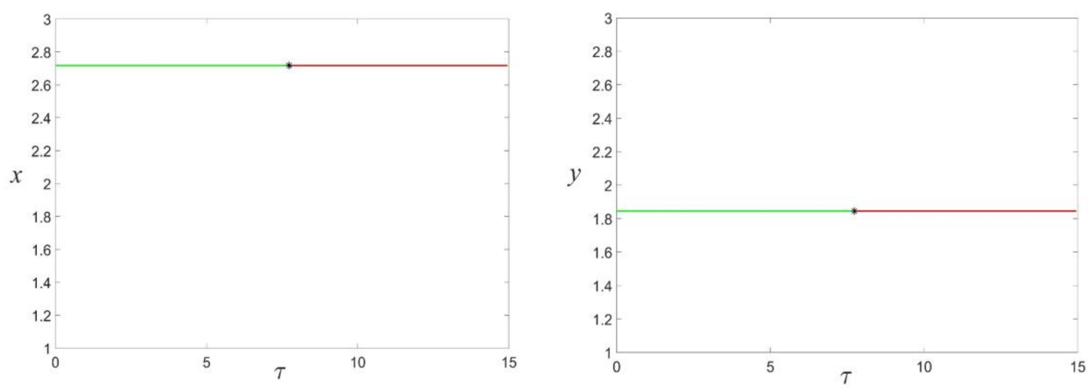

Figure 1: Stability change of the positive equilibrium.

Figure **1** contains two bifurcation diagrams, they show the change of stability of the positive equilibrium $E_4 \approx (2.718,1.845)$ at the critical value $\tau_0 \approx 7.730$ (which is marked as black asterisk in the diagrams). When $\tau < \tau_0$, E_4 is stable (corresponds to the green part of the branch), when $\tau > \tau_0$, E_4 is unstable (corresponds to the red part of the branch).

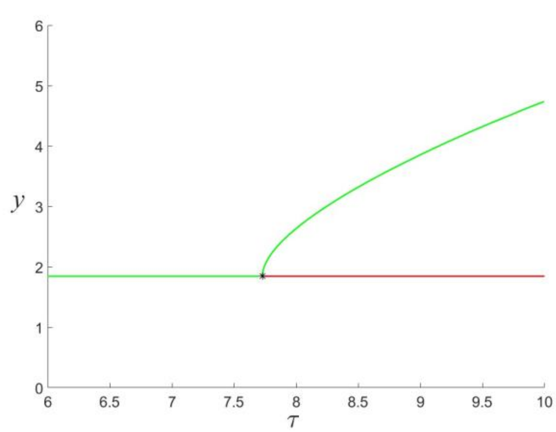

Figure 2: Bifurcating of periodic solution.

Figure **2** shows the bifurcating of a stable periodic solution (corresponds to the green curve over the straight line) when E_4 loses its stability. Here we only show the birfurcation diagram in the $\tau - y$ plane and omit the diagram in the $\tau - x$ plane.

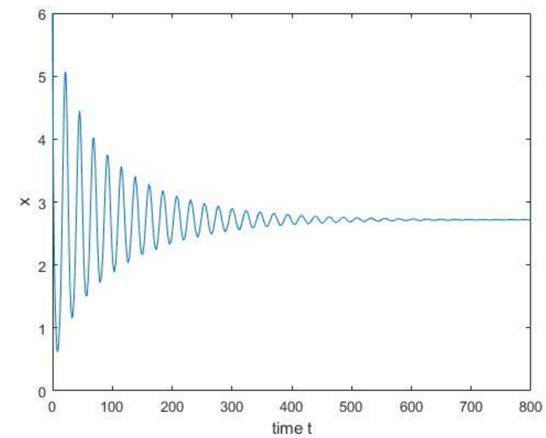

Figure 3: The trajectory graph on $t - x$ plane of system (4.1) with $\tau = 7$.

The numerical simulations of system (4.1) for $\tau = 7$ and $\tau = 8$ are shown in Figs. **3**–**8**, respectively.

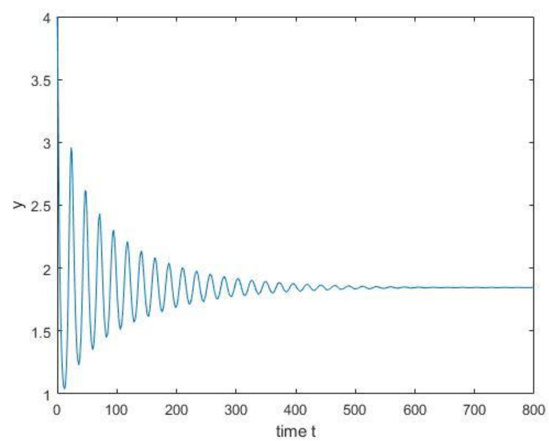

Figure 4: The trajectory graph on $t - y$ plane of system (4.1) with $\tau = 7$.

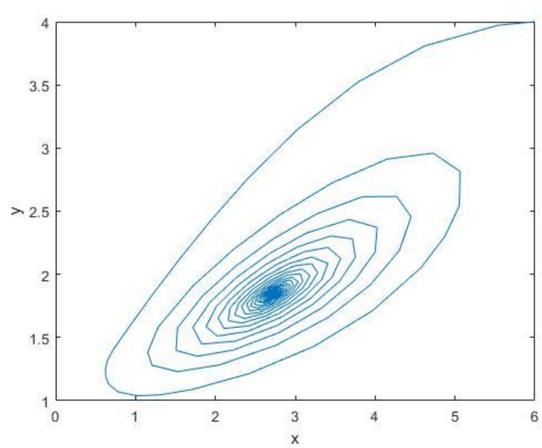

Figure 5: The phase graph of system (4.1) with $\tau = 7$.

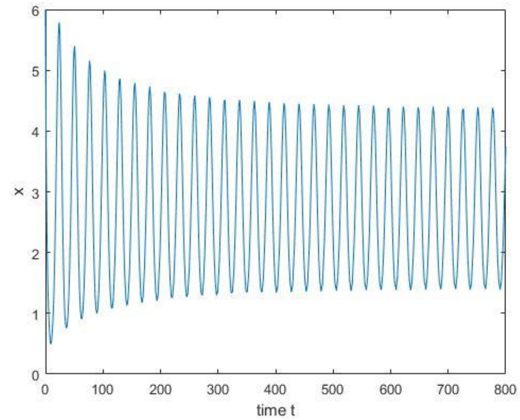

Figure 6: The trajectory graph on $t - x$ plane of system (4.1) with $\tau = 8$.

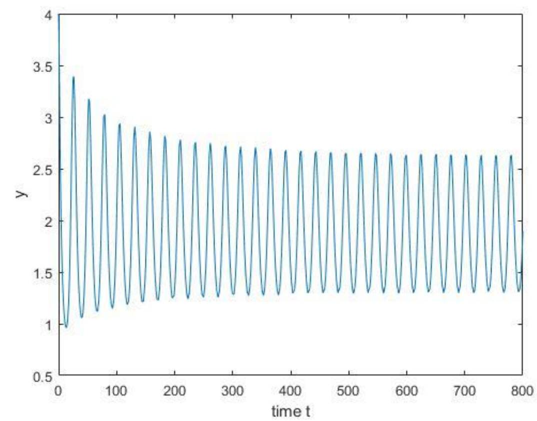

Figure 7: The trajectory graph on $t - y$ plane of system (4.1) with $\tau = 8$.

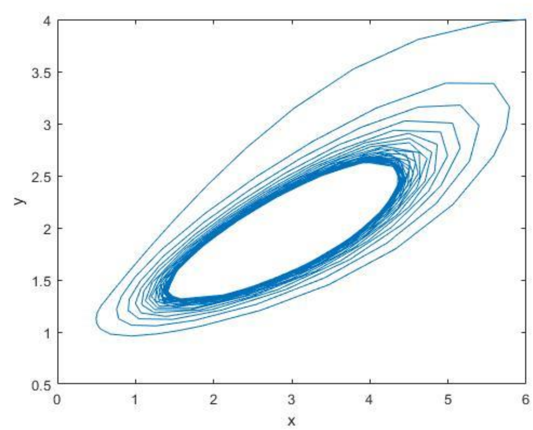

Figure 8: The phase graph of system (4.1) with $\tau = 8$.

It is clear that when $\tau \in [0, \tau_0)$, there is no periodic solution and E_4 is asymptotically stable; when τ increases to the critical value τ_0, E_4 loses its stability and Hopf bifurcation occurs. The Hopf bifurcation of system (4.1) is forward and supercritical.

5. Conclusions

In this paper, we have studied a 2-species commensalism system with a discrete delay and density dependent birth rates. Based on the results of existence and stability of equilibria obtained in [5], we have investigated the distribution of roots of the characteristic equation and obtained that, when the delay τ is sufficiently small, the positive equilibrium is locally asymptotically stable, and when τ increases to a critical value, the positive equilibrium loses its stability and a Hopf bifurcation occurs, as τ continues to increase, a family of periodic solutions bifurcate from the positive equilibrium. Then, by using the normal form theory and the center manifold reduction introduced by Hassard *et al*. [43], we have derived the formulae $c_1(0)$ in (3.12) determining the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions at critical values of τ. Finally, numerical simulations have been given to illustrate the results of theoretical analysis.

Conflict of Interest

The authors declare no potential conflict of interests.

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Authors' Contributions

Tianyang Li: Writing-original draft. Qiru Wang: Writing-review and editing.

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