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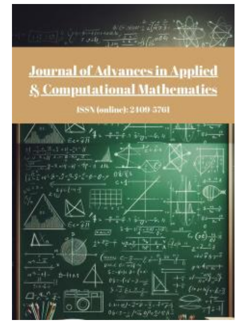

Vector Lattice Data Analysis: Fitting and Model Uncertainty

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ABSTRACT

Functional data analysis (FDA) is a popular research area of data analysis that is well-suited for modeling complex data structures such as time series data and images. In Linear Regression models, the random variables are often described using a finite--dimensional vector space, under the assumption that the random variables are represented by a finite set of parameters. FDA allows us to model random variables as functions. This can lead to a more flexible and expressive approach to the statistical model. Within FDA, the specific paper investigates the potential of vector lattices to enhance model flexibility and address model uncertainty. The limitations of finite-dimensional vector spaces in capturing the complexities of real-world random variables are discussed. An investigation is conducted into the concept of Vector Lattice Linear Regression Models (VLLM), highlighting their ability to effectively handle model uncertainty.

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1. Introduction

Functional data analysis (FDA) is the study of data whose ideal units of observation are functions given as several continuous domains, and the observed data are a sample of functions drawn from some population and sampled on a discrete grid, [1]. They presented a comprehensive overview of FDA, including key techniques such as mean and covariance functions, functional principal component analysis for dimension reduction and imputation, functional linear regression, and even nonlinear methods such as time warping and differential equations. Ramsay and Dalzell [2] defined the FDA as a four-step process: selecting the function space, specifying the analysis in functional terms, mapping observations to the function space, and interpreting the results back in the context of the original observations. This process makes use of powerful functional analysis techniques to analyse complicated, functional data.

The functional data are, by their very nature, infinitely dimensional which poses theoretical and computational difficulties that vary depending on how the functional data were gathered. This complexity causes difficulties not seen in traditional analysis, making it more complicated to represent observations and estimate parameters, [3]. Despite these difficulties, the infinite dimensionality of the data is a rich source of knowledge that offers numerous opportunities for research and data analysis, [1]. Although functional data has been extensively considered, Levitin *et al*. [8] pointed out that the increasing number of complex data gathering techniques in behaviour research demands an adjustment of our understanding and analysis of it. The FDA has shifted data analysis by providing an insight into complicated, function-based data such as complex curves, detailed images, and dynamic time series. For further information regarding FDA and its recent developments, see [4-7].

The FDA frequently allows researchers to ask questions that would be either mathematically impossible or computationally inefficient to ask using traditional statistical methods; as such, the FDA represents both uniqueness in analysis and simplicity in execution, [8]. Ullah and Finch [9] conducted a comprehensive search of 11 electronic databases between January 1995 and December 2010 in order to identify peer-reviewed FDA application studies. The main framework of FDA till now is the frame of Hilbert Spaces. These FDA models cannot explain the use of fitting arising by assuming the constant variable as an explanatory variable. Horvath and Kokoszka [5] addressed several fundamental concepts from operator theory and concentrated on the properties of random samples in a Hilbert space. In this paper, we do explain that spaces with order units are not appropriate for the FDA. The space of continuous functions is such a case, since the constant function is an order unit under the pointwise partial ordering. The pointwise partial ordering on a set of functions f and g is defined as $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in the domain of f and g. FDA modeling excludes continuous functions as a possible vector lattice for its models. On the other hand, the fact that L^p when $1 \leq p < \infty$ may be used in the FDA is interpreted since these spaces do not have order units with respect to the pointwise partial ordering.

The theory of ordered vector spaces, including vector lattices (Riesz spaces), started around 1935 from three distinct works by Riesz in Hungary, Kantorovitch in the Soviet Union, and Freudenthal in the Netherlands. For a summary on this progress, the reader may see Van der Walt [10]. Their contributions established the foundation for this important area of mathematics, which has applications in various fields like functional analysis, economics, and optimization. The functional analysis framework of vector lattices can be used to define the uncertainty in FDA models. For instance, the set of all potential functions that could fit a specific data set could be represented by a subspace of a vector lattice. Vector lattices in this paper are actually subspaces of function spaces and their lattice ordering is actually the pointwise partial ordering. Subspaces as ideals and bands are particularly important in the analysis of partially ordered vector spaces, according to Kalauch and Malinowski [11]. In this paper, we propose the decomposition of a vector lattice into ideals for the determination of level of uncertainty in regression models. Readers are encouraged to read the references [10-15], for more information on ordered linear spaces, vector-lattices, lattice- subspaces, positive projections, and Riesz estimators.

Model structure and parameterization developments allow us to effectively reduce the model's overall uncertainty, leading to a more accurate and useful analysis, [16]. Model Uncertainty may be defined in a general way as follows: Model uncertainty arises when system gains or other parameters are not precisely known, or can vary over a given range. The range of a data set may be represented by finite or infinite unions or intersections of closed intervals $[a, b]$, where $a, b \in E$ and E is a vector lattice. This is exactly the approach of this paper, which is mainly devoted to regression models. However, this approach which is consistent to the partial ordering of E as a lattice, may be useful since it does not require topological arguments. Topological implications do arise on the other hand, relying on this approach. A non-trivial point of the paper is that vector lattices having order-units may not be a useful framework for linear regression models, since they do 'absorb' any other variable. This is actually a finding relying on the order-structure of such a class of vector lattices.

In this paper, we introduce the concept of Vector Lattice Linear Regression model (VLLM), which extends traditional linear regression to the context of vector lattices. The space of fitted values, error space, and VLLM subspace with regard to a particular subspace of explanatory variables are defined. This framework allows us to evaluate model uncertainty by examining how well the VLLM captures all variability in the data. Since ideals and subl-attices provide a more comprehensive understanding of model uncertainty, we investigate their relevance in VLLM in more detail. Riesz estimators are also introduced as a valuable tool for estimating model coefficients within vector lattices. We aim to provide insights into model uncertainty in FDA regression models by using these approaches. Additionally, the paper highlights the potential of Yudin bases, a special class of vector lattices, as a promising framework for further research in VLLMs.

2. Preliminaries

Let us recall some essential notions about ordered linear spaces: If E is a non-trivial vector space, then a *partial -ordering* is a binary relation $\geq$ between the elements of this vector space. The properties of such a binary relation are the following:

1. $x \geq x$, for any $x \in E$ (reflexive)
2. If $x \geq y$ and $y \geq x$ for some $x, y \in E$, then $x = y$ (antisymmetric)
3. If $x \geq y$ and $y \geq z$, then $x \geq z$ (transitive)
4. If $x \geq y$ and $\lambda \in R_+$, then $\lambda \cdot x \geq \lambda \cdot y$, where $\cdot$ denotes the scalar product, being defined on E and R_+ denotes the set of the real numbers.
5. If $x \geq y$ for some $x, y \in E$, then $x + z \geq y + z$, for any $z \in E$. Also, if $x \geq y$ for some $x, y \in E$, then $x - z \geq y - z$, for any $z \in E$.

The last two properties denote that $\geq$ is *compatible to the linear structure of* E.

Then the vector space, being endowed with a partial ordering relation $\geq$, which satisfies the above properties, is a *partially ordered vector space*. The set

$$E_+ = \{x \in E | x \geq 0\},$$

is called *positive cone* of this partially vector space. 0 denotes the zero element of the vector space E. Let us suppose that E is a partially ordered vector space, being endowed with the partial ordering $\geq$. Let us suppose some $x, y \in E$, where E is a partially ordered vector space, endowed by $\geq$. The *supremum* of $\{x, y\}$ where $x, y \in E$, with respect to $\geq$ is some $u \in E$, such that $u \geq x, u \geq y$, while for any other $r \in E$, such that $r \geq x, r \geq y$, then $r \geq u$. The *infimum* of $\{x, y\}$ where $x, y \in E$, with respect to $\geq$ is some $n \in E$, such that $x \geq n, y \geq n$, while for any other $m \in E$, such that $x \geq m, y \geq m$, then $n \geq m$. If the infimum and the supremum of $\{x, y\}$, where x, y belong in E, then E is called *vector -lattice*. In this case, the infimum of $\{x, y\}$ is denoted by $x \wedge y$ and the supremum of $\{x, y\}$ is denoted by $x \vee y$. If S is some non -trivial subspace of a vector -lattice, such that $x \vee y \in S$ and $x \wedge y \in S$ for any $x, y \in S$, then S is called *sub-*

lattice of E. A sub-lattice is actually a sub-space of a vector -lattice, which is a vector- lattice as well. Both the supremum and the infimum of any two elelemtns of S belongs in S. As it is well –known, |x|=sup{x,-x} and an *ideal* A is a subspace of a vector lattice, which has the following property : For any a in A, and any d in E, such that |d| is less or equal to a (with respect to the partial ordering that makes E a vector lattice), we obtain that d is an element of A.

3. Vector Lattice Linear Regression Model (VLLM)

Let us consider the following essential regression model:

$$y = a_0 h + a_1 x + e, \quad (1)$$

where $a_0, a_1 \in R$ and $y, h, x, e \in E$, where E is a function space. Model Uncertainty appears is the study of the case where the error term e is decomposed in the following way : $e = e_0 + e_1$. hence, lack-of-fitting is related both in two ways of explanation. The first is that however $y \neq a_0 e + a_1 x$ and the second is that even the way that e_0 is considered as the residual term, the lack- of- fitting is too great as well. A lot of lack-of-fitting measures are well-known. The interpretation of such a phenomenon is that the geometry of the function space implies the presence of another term e_1, such that $e = e_0 + e_1$. The term e_1 is not directly specified. In this way, we may say that Model Uncertainty appears in the regression model (1). In this case, we may wonder which variables we have to add in this regression model in order to increase goodness -of -fit. This is a rational question in any regression model. Another question is understanding the lack-of -fitting due to the geometry of the specific function space E. How less is the significance of the term e_1 ? Another question is whether the presence of the variables h and x reduces gooodness -of -fitting ? The same questions do arise in the case of including more variables in the regression equation.

Model Uncertainty is related to the essential questions in statistics and econometrics ”More or less variables ?” ”How many variables are necessary for explaining the behaviour of a data set ?” or ”Are these variables adequate for significant predictions ?” Even though we have to be careful not to overfit the model by adding too many additional variables, our model has to be able to account for any meaningful association discovered in the data. Model uncertainty can be used to compare the prediction accuracy of different models with different numbers of variables, as well as to identify variables that are important for explaining the data and variables that can be safely eliminated from the model.

A Vector Lattice Linear Regression Model (VLLM) is the following:

$$y = a_0 q + a_1 x_1 + \ldots. + a_m x_m + e, \quad (3.1)$$

where $y, q, x_1 \ldots, x_m, e \in E$ and $a_0, a_1, \ldots, a_m \in R$. The error term e is independent from $t, x_1 \ldots, x_m$ and q is usually the constant function whose any value is equal to 1. The structure of the span $X = [q, x_1, \ldots, x_m]$, which is a subspace of a vector lattice E, is important for the properties of the Linear Regression in Equation (3.1). The span X is supposed to be a subspace of S. Any VLLM expressed by the Equation 3.1 is denoted VLLM(S) for abbrevation. Let us suppose an infinite dimensional vector lattice E. Due to the applications, which are usually appear in economics, we suppose that this vector lattice is a subspace of $L^0(\Omega, F, P)$. The last vector space is the one of real–valued F -measurable functions, which are usually called random variables. Ω is a non–empty set, whose cardinality is equal to the one of real numbers, F is some σ -algebra, which contains subsets of F. These subsets are not only the empty set and the whole of Ω. Finally, P is an atom–less probability measure defined on F. Let us suppose that L is a subspace of the vector lattice E and $L^{\perp}$ the subspace of its disjoint elements, with respect to the lattice ordering. We remind that two elements x, y in a vector lattice E are disjoint if $|x| \wedge |y| = 0$, where 0 is the zero element of E.

4. Some Useful Lemmas

Lemma 4.1 $L^{\perp}$ is an ordered subspace as well, for any ordered subspace L of the vector lattice E.

Proof. If $x, y \in L^{\perp}$, then $|x| \wedge |z| = 0, |y| \wedge |z| = 0$ for any $z \in L$. Then $|x + y| \wedge |z| = |(x + y)| \wedge |z| \leq |x| \wedge |z| + |y| \wedge |z| = 0$, for any $z \in L$. Then, $x + y \in L$. We also suppose that $x \in L$ and $a \in R$. Then, $|a \cdot x| \wedge |z| = |a| \cdot (|x| \wedge |z|) = |a| \cdot 0 = 0$. $\cdot$ denotes the scalar product on the vector space E.

Lemma 4.2 $L + L^{\perp}$ is a direct sum for any ordered subspace L of E.

Proof. Suppose that $x \in L \cap L^{\perp}$ and $x \neq 0$. Hence $|x| \wedge |x| = |x| = 0$, which implies that $x = 0$, a contradiction. Then, the sum $L + L^{\perp}$ is a direct sum.

Lemma 4.3 If some ordered subspace L of a vector lattice contains an order–unit of E, then $L^{\perp} = \{0\}$.

Proof. If $x \in L^{\perp}$, then $|x| \wedge |y| = 0$ for any $y \in L$. Since $y \in L$, we get that $|y| \leq t \cdot e$ from the definition of an order–unit. Hence, $|x| \wedge (t \cdot e) = 0$, for any $t \in R_{+}$. This is true if and only if $x = 0$.

5. Model Uncertainty

S is a non-trivial subspace of some vector lattice containing the explanatory variables. We may suppose that the errors' component is the **disjoint complement** of S, where S is the ideal generated by $\{z_1, \ldots, z_n\}$ in E. Namely, $S^{\perp} = \{z \in E \mid |z| \wedge |y| = 0, \text{for any } y \in S\}$.

In the way indicated above, we give the following useful definitions and relevant main points of interest:

• **Space of Fitted Values (S):** This subspace of the vector lattice contains the explanatory variables and reflects the component of the model that explains the observed data.

• **Error Space with respect to S ($S^{\perp}$):** This disjoint complement of S represents the portion of the data left unexplained by the model, including the errors and noise inherent in the observations.

• **Subspace of VLLM with respect to S (VLLM(S)):** This represents the total VLLM with respect to S as the direct sum of S and $S^{\perp}$. It is actually the direct sum $S \oplus S^{\perp}$.

• **Model Uncertainty:** This occurs when the VLLM(S) does not cover the full vector lattice. this is the case of $E \neq S \oplus S^{\perp}$. Or else, there exists a residual space beyond any

VLLM(S).

• **Dimension of Model Uncertainty:** This representing the dimension of the complementary subspace of $S \oplus S^{\perp}$ in E. It denotes the cardinality of the additional independent components required to adequately explain the data beyond the existing model.

• **Failed VLLM(S):** The VLLM(S) fails if there exists one variable $q \in S$, which appears in any VLLM(S). Failure is an attempt to catch the case of the so-called 'over-fitting'. Of course, more than one single variable may appear in any VLLM(S). This is the case of an ideal generated by one or more elements the vector lattice of E.

6. Ideals and Sub–lattices in Vector Lattice Linear Regression

Ideals are examples of vector lattice subspaces, where the notion or Model Uncertainty is more clear. If we assume that E is a vector lattice, while S is an ideal of it, then $S \oplus S^{\perp}$ is a subspace of E, where $S^{\perp}$ is the subspace of the elements of the vector lattice E, being disjoint on the elements of S. The above Proposition shows which is the relation between ideals and sub -lattices. The motivation for this result is that sub- lattices may be also used as a class of subspaces useful to Data Analysis. While every ideal is a vector sub–lattice, a vector sub–lattice is not necessarily an ideal. An example is the partially linear functions, which form a sub–lattice but not an ideal. Ideals simplify the estimation of VLLM parameters. Such a space is the one of partially linear functions defined on $[0, T]$, where $T > 0$. This subpace of the real -valued functions defined on $[0, T]$ is a sub- lattice, while it is not an ideal.

Proposition An ideal I generated by an arbitrary finite set of linearly independent elements lying in a vector lattice is a sub–lattice of E.

Proof: This can be proven by looking at cases based on the number of generating elements:

• **Singleton ideal:** If $I(z)$ is generated by a single element $z \in E_+$, then it is a sub- lattice of E since the lattice operations on its elements are bounded by multiples of z. This means that there exists some t_1, t_2 non–zero real numbers such that

$$|y| \leq t_1 \cdot z \quad and \quad |x| \leq t_2 \cdot z \tag{6.1}$$

hence

$$x \vee y \leq |x| \vee |y| \leq max\{ t_1, t_2\} z \tag{6.2}$$

and

$$x \wedge y \leq |x| \wedge |y| \leq min\{ t_1, t_2\} z \tag{6.3}$$

• **Two-element ideal:** If $I(z_1, z_2)$ is generated by two elements z_1 and $z_2 \in E_+$, then the Riesz decomposition property allows for separate analysis of positive and negative parts of elements within the ideal. By applying lattice operations and comparing dimensions, it can be shown that the ideal is closed under these operations, making it a sub–lattice. This implies that if $x, y \in I(z_1, z_2)$, there exist r_1, r_2, q_1 and q_2, such that

$$|x| \leq r_1 \cdot z_1 + r_2 \cdot z_2 \tag{6.4}$$

and

$$|y| \leq q_1 \cdot z_1 + q_2 \cdot z_2 \tag{6.5}$$

From the Riesz Decomposition Property there exist x_1, x_2 such that $x = x_1 + x_2$, such that

$$|x_1| \leq r_1 \cdot z_1, \quad |x_2| \leq r_2 \cdot z_2 \tag{6.6}$$

and y_1, y_2 such that $y = y_1 + y_2$ such that

$$|y_1| \leq q_1 \cdot z_1, \quad |y_2| \leq q_2 \cdot z_2 \tag{6.7}$$

hence

$$x \vee y \leq |x| \vee |y| \leq (|x_1| + |x_2|) \vee (|y_1| + |y_2|) \leq max\{ r_1, q_1\} z_1 + max\{ r_2, q_2\} z_2 \tag{6.8}$$

and

$$x \wedge y \leq |x| \wedge |y| \leq (|x_1| + |x_2|) \wedge (|y_1| + |y_2|) \leq min\{ r_1, q_1\} z_1 + min\{ r_2, q_2\} z_2 \tag{6.9}$$

• **n-element ideal:** The general case for an n-element ideal follows through induction.

7. Riesz Estimators

Let us consider that I is an ideal generated by the linearly independent positive elements $z_1, \dots, z_n$ of E. The Riesz estimator of $x \in E$ with respect to the ideal generated by $|z_i|, i = 1, \dots, n$ is the following element of I

$$a(x) = \sum_{i=1}^{n} \quad a_i(x)|z_i|, \tag{7.1}$$

where

$$a_i(x) = inf\{ t_i \in \mathrm{R} | |x| \leq t_i \cdot |z_i| \}, i = 1, \ldots. n. \quad (7.2)$$

The real numbers a_i are well–defined, due to the Riesz Decomposition Property. The reader may see in Jameson [12] about the Riesz Decomposition Property in vector lattices.

It is obvious that this is an element of E_+, where E_+ is the positive cone of the vector lattice being used as a model for the associated functional model.

Example about Determination of Riesz Estimators

Example 7.1 Let us consider $E = L^2([0,1], B, \lambda)$, a vector lattice with pointwise partial ordering, where B is the σ -algebra of Borel Sets for the interval $[0,1]$ of the real numbers and λ is the Lebesgue measure defined on $[0,1]$. Consider S, the ideal generated by $z_1(t) = 1$ and $z_2(t) = t$, where $t \in [0,1]$ and $x(t) = t^2$, for any $t \in [0,1]$. According to the above definition, $t_1 = 1$ and $t_2 = 1$. Moreover, we obtain that $a_1(x) = 1, a_2(x) = 1$, hence the Riesz estimator of x is the function $a(x) = 1 + t, t \in [0,1]$.

Some vector lattices are not appropriate for fitting VLLM(S). Specifically, vector lattices having order–units are not appropriate for VLLM(S), since the ideal generated by an order-unit is the entire space. Since 1 is an order-unit of $L^\infty(\Omega, F, P)$ Using L^2 or in general some $L^p(\Omega, F, P)$, where $1 \leq p < \infty$ has a motivation, which is implied by the fact that 1 is not an order–unit in these spaces if they are not finite–dimensional. Hence, the constant random variable 1 is an element of some non–trivial VLLM(S) in these vector lattices. The reader may see the exact definition of an order-unit in vector lattices in Jameson [12].

Example 1 about order -units: Let $E = L^\infty(\Omega, F, P)$, where Ω is a compact topological space, F is the sigma-algebra of Borel sets on Ω, and P is a probability measure on F. Since $S = E$ if S is a vector lattice containing an order -unit, then in such a case $S \oplus S^\perp = E$ because $S^\perp = \{0\}$. In this case we obtain a failed VLLM(S), since the only explanatory variable is some order unit e lying in E. This is actually the case of $L^\infty(\Omega, F, P)$ or the case of $C(\Omega)$ if Ω is a compact topological space. Then the constant function 1 is an order unit in these spaces under the pointwise partial ordering.

Example 2 about bands: When S is a band of E, namely an order closed ideal of E, then the Model Uncertainty is equal to zero.

Example 3 about L^2: In the vector lattice

$$L^2([0,1], \mathrm{B}, \lambda),$$

consider the following VLLM:

$$y = a_0(x)1 + a_1(x)z_1 + e,$$

where $y, 1, z_1 \in L^2([0,1], B, \lambda)$, and $a_0(x), a_1(x) \in R$. The error term e is independent from 1, x_1 and 1 is the constant function whose any value is equal to 1. Assume that x_1 is a positive function in $L^2([0,1], B, \lambda)$. Due to the fact that we can always select the coefficient $a_1(x)$ to be positive, this VLLM does not admit model uncertainty. This indicates that if $z_1 = 0$, the observed value of y will always be greater than or equal to the predicted value of y.

Example 4 about L^2: In the infinite-dimensional vector lattice

$$L^2([0,1], \mathrm{B}, \lambda),$$

and a corresponding ideal generated by n elements of it, where n > 2 . We consider an associated VLLM(S), where S is the corresponding sub -lattice. The Riesz Estimator of the coefficients $a_i(x)$, where $i = 0,1, \ldots, n$, are provided by 7.2. Once the coefficients are determined, we can use this VLLM to make predictions for new values of x.

8. Yudin Bases as a Further Research Framework of Vector Lattice Regression Models

A class of Partially Ordered Linear Spaces of particular interest are the Vector Lattices with a Yudin Bases. These Partially Ordered Linear spaces are of particular interest, since they may extend research in vector lattice regression models (VLLMs). Their own characteristics provide promising possibilities for exploring new modelling approaches and addressing complex regression analysis issues.

Partially ordered spaces having Yudin Bases do not have order units, as is shown in Werner *et al*. [17] and in Aliprantis and Tourky [18]. Hence any span of elements lying in a Yudin Space provides a VLLM. In Abramovich *et al*. [13], the relationships between lattice-subspaces and positive projections are thoroughly investigated. The question is to show some examples of Yudin Bases. If we consider $c_{00}(G)$ and G is a class of positive–valued functions defined of some Ω this is a Yudin vector lattice, where lattice operations are the ones of the real numbers applied on the elements of G. c_{00} means that only a finite number of functions among G have non–zero coefficients. Hence, any Hamel Basis of positive real functions defined on G ”produces” a Yudin Lattice.

9. Conclusions

In FDA, vector lattices provide a powerful framework for investigating model uncertainty. This approach provides a straightforward way to analyse the level of uncertainty in a given model as well as indicate potential areas for improvement. The structure of vector lattices allows us to divide the model space into subspaces that correspond to the residual uncertainty, error terms, and fitted values. It also enables us to measure how well the model represents the observed data and to discover potential missing components that could improve the model’s explanatory power.

Riesz estimators prove to be a useful tool for estimating elements in vector lattices. Understanding their advantages and calculating methods are important for their efficient use in a wide range of settings. These estimation tools help researchers to approximate elements in a vector lattice based on their bounds and relationships with other elements. One can make informed predictions and inferences inside the vector lattice framework by effectively employing Riesz estimators. Our understanding of data analysis and modelling within this complex mathematical environment could improve with further investigations of Riesz estimators and their properties.

The research highlighted the need of addressing the geometry of the function space in FDA. It suggested that vector lattices without order units may be more appropriate for FDA modeling. The concept of model uncertainty can be extended to a variety of statistical models. VLLM models can be used to describe complex interactions between functions while accounting for model uncertainty. These models relies heavily on ideals as they allow for efficient estimation and interpretation of model coefficients. Considering the lattice structure, Riesz estimators provide an appropriate approach for estimating VLLM coefficients.

Furthermore, Tobit models represent a specific class of VLLMs. Tobit models are linear regression models with positive explanatory variables and exhibit disjoint supports. A key characteristic of these variables is that their values are often zero, either above or below a certain threshold. This property corresponds to the disjoint support of positive basis vectors within the sublattice generated by a design matrix columns. Therefore, Tobit models can be effectively understood and analyzed within the VLLM framework, leveraging the properties of vector lattices and Riesz estimators for improved model understanding and coefficient estimation.

Conflict of Interest

The authors declare that there is no conflict of interst.

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