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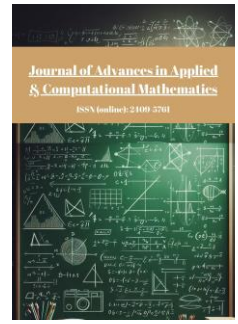

Convergence of θ-Milstein Method for Stochastic Differential Equations Driven by G-Brownian Motion

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ABSTRACT

Although numerical methods for classical stochastic differential equations (SDEs) driven by Brownian motion are well-established, research on numerical schemes for SDEs driven by G-Brownian motion (referred to as G-SDEs) remains limited. Most existing studies are confined to Euler-Maruyama-type methods, which achieve only a strong convergence order of one-half. To bridge this gap, this paper aims to develop higher-order numerical methods for G-SDEs. By combining the classical Milstein method with the G-Itô formula, we propose a novel θ-Milstein scheme for G-SDEs. Using tools from G-expectation theory and Taylor expansions, we prove that the proposed scheme achieves a strong convergence order of one under the L^r-norm, assuming Lipschitz conditions. Numerical experiments demonstrate that the θ-Milstein method yields smaller errors and attains a higher convergence order compared to the Euler-Maruyama method, confirming its effectiveness and potential for advancing numerical solutions of G-SDEs.

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1. Introduction

Classical stochastic differential equations (SDEs) driven by Brownian motion are widely used to model uncertain phenomena and play a crucial role in numerous scientific and industrial fields [1-3]. However, such models typically fail to account for ambiguous or uncertain probabilistic factors. In complex real-world environments, it is often difficult to construct ideal models where probability can be precisely determined. As a result, probability uncertainty has emerged as a significant and challenging area of research [4-6].

Motivated by financial challenges such as asset pricing, risk measurement, and decision-making under probability uncertainty, Peng [7, 8] established a time-consistent sublinear expectation framework and developed the theory of G-expectation. This theory provides the probabilistic foundation for defining G-Brownian motion, which in turn serves as the driving noise for G-SDEs. Since then, both the theoretical developments and practical applications of G-SDEs have garnered substantial research interest. Lin [9] explored G-SDEs with reflecting boundary conditions. Luo and Wang [10] proved that the integration of a G-SDE in R can be reduced to the integration of an ordinary differential equation parameterized by a variable in (Ω, F). Some properties such as pathwise properties, homeomorphic flows, the strong Markov property, and asymptotic estimates for G-SDEs have been investigated in [11-14].

Over the past decade, increasing attention has been devoted to stochastic stability and feedback control within the G-expectation framework (G-framework). A variety of stabilities, including moment stability, quasi-sure stability, and exponential synchronization, have been extensively studied. For instance, exponential stability for linear G-SDEs was analyzed in [15]; moment stability under Lyapunov-type conditions was investigated in [16]; asymptotic boundedness and exponential stability were established using the G-Lyapunov function method in [17]; asymptotic stability in distribution for highly nonlinear G-SDEs was developed in [18]. Additional results on stochastic stability and stochastic stabilization can be found in [19-23], among others. These contributions provide a foundation for advancing the applications of G-SDEs.

In practice, the successful application of G-SDEs depends critically on both qualitative and quantitative properties of their solutions. Although existing research has largely emphasized qualitative aspects, obtaining closed-form analytical solutions for these equations remains generally infeasible. Therefore, the analysis of numerical methods for G-SDEs is of significant practical importance. For classical SDEs driven by standard Brownian motion, there exists an extensive body of literature on numerical analysis [24-30]. In contrast, relatively few works have addressed the numerical analysis of G-SDEs, to the best of our knowledge. Yang and Li [31] introduced a θ-Euler-Maruyama (θ-EM) scheme for G-SDEs and discussed the p-th (for $p \in (0,1)$) moment exponential stability of the scheme under the global Lipschitz condition. They also derived convergence and stability results for the θ-EM scheme applied to neutral stochastic delay differential equations driven by G-Brownian motion in [32]. Under the local Lipschitz and linear growth conditions, Liu and Lu [33] examined the strong convergence of the EM scheme for G-SDEs. Deng *et al*. [34] established that the EM method is exponentially stable in mean-square if and only if the corresponding stochastic differential delay equations driven by G-Brownian motion (G-SDDEs) are exponentially stable in mean-square under the global Lipschitz condition. Moreover, Yuan and Zhu [35] further proved the practical mean-square exponential stability of the EM method for G-SDDEs under a condition that is less restrictive than the global Lipschitz requirement.

Based on the above discussion, research on numerical methods for G-SDEs has mainly focused on Euler-Maruyama-type schemes with order one-half. However, no work has been done on numerical methods for G-SDEs with higher convergence orders. The objective of this paper is to examine the strong convergence of higher-order schemes for G-SDEs. In the case of SDEs driven by standard Brownian motion, numerical schemes with first-order convergence such as Milstein-type methods have been extensively studied. Examples include the tamed Milstein [36], projected Milstein [37], symmetrized Milstein [38], implicit Milstein [39], truncated Milstein [40], positivity preserving Milstein [41] and semi-implicit projected Milstein [42]. By combining the classical Milstein method with G-expectation theory, we propose a θ-Milstein scheme for approximating the solution of G-SDEs. We establish the strong convergence order of this scheme in the L^r-norm for $r \geq 2$. Numerical experiments demonstrate that the proposed θ-Milstein method performs effectively in terms of both convergence and flexibility, yielding smaller errors than the classical Milstein method and achieving a higher convergence order compared to the EM method. Compared to previous studies on numerical methods for G-SDEs, the main contributions of this work are as follows:

• **Novel θ-Milstein Scheme**: We develop a new θ-Milstein scheme within the G-framework, which achieves a higher strong convergence order compared to conventional Euler-Maruyama-type methods.

• **First-Order Convergence**: For the first time, we establish the first-order strong convergence of the θ-Milstein method for G-SDEs in the L^r-norm setting.

The rest of this paper is organized as follows. In Section 2, we review some necessary mathematical preliminaries in the G-framework. Section 3 introduces the θ-Milstein scheme for G-SDEs. The convergence result of the scheme is established in Section 4. Numerical experiments are given in Section 5. Finally, Section 6 concludes this paper.

2. Preliminaries

Let R denote the one-dimensional Euclidean space. Denote the scalar product by $\langle\cdot,\cdot\rangle$ and the norm of x by $|x|$ for any $x \in R$. Denote by f' the first derivative of a function $f: R \to R$ and f'' the second derivative. Let $C^2(R; R_+)$ denote the space of continuous functionals $f: R \to R_+$ with continuous derivatives of orders up to 2. For two real numbers a and b, $a \vee b := max(a, b)$ and $a \wedge b := min(a, b)$. Let $(\Omega, H, \hat{\mathbb{E}})$ be a sublinear expectation space, where Ω is a given set, H is a linear space of real valued function defined on Ω. The space H can be considered as the space of random variables.

Definition 2.1 [8] A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}}: H \to R$ satisfying

1. Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ if $X \geq Y$;

2. Constant preserving: $\hat{\mathbb{E}}[c] = c$;

3. Sub-additivity: For any $X, Y \in H$, $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

4. Positivity homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

Definition 2.2 [8] A d-dimensional stochastic process $\{B(t)\}_{t\geq 0}$ on a sublinear expectation space $(\Omega, H, \hat{\mathbb{E}})$ is called a G-Brownian motion if the following properties are satisfied:

1. $B(0) = 0$;

2. for each $t, s \geq 0$, the increment $B(t+s) - B(t)$ and $B(s)$ are identically distributed and is independent from $(B(t_1), B(t_2), \cdots, B(t_n))$, for each $n \in N$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;

3. $\lim_{t\downarrow 0} \hat{\mathbb{E}}[\ |B(t)|^3] t^{-1} = 0$.

Let

$$G(a) := \frac{1}{2}\hat{\mathbb{E}}[aB(1)^2],\ \forall a \in R, \tag{1}$$

where $\overline{\sigma}^2 = \hat{E}[B(1)^2]$, $\underline{\sigma}^2 = -\hat{E}[-B(1)^2]$, $0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$. Denote by H_t the filtration generated by G-Brownian motion $\{B(t)\}_{t\geq 0}$. For some basic notations about G-Itô integral, one can refer the reference [8]. The following lemma and proposition are useful in our analysis.

Proposition 2.3 [8] Let $\xi_t \in M_G^2(0,T)$. Then, for any $t \in [0,T]$,

$$\hat{\mathbb{E}}\left[\int_0^T \xi_s dB(t)\right] = 0,$$

$$\hat{\mathbb{E}}\left[\int_0^T |\xi_s|^2 ds\right] \leq \int_0^T \hat{\mathbb{E}}|\xi_s|^2 ds,$$

$$\hat{\mathbb{E}}\left[\left|\int_0^T \xi_s dB(s)\right|^2\right] = \hat{\mathbb{E}}\left[\int_0^T |\xi_s|^2 d\langle B\rangle(s)\right] \le \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T |\xi_s|^2 ds\right].$$

Lemma 2.4 [11] Let $r \ge 2$ and $\xi_t \in M_G^r(0,T)$. Then, for any $t \in [0,T]$,

$$\hat{\mathbb{E}}\left|\sup_{s\le v\le t}\int_s^v \xi_u dB(u)\right|^r \le c_1(r,\overline{\sigma})\, |t-s|^{r/2-1}\, \hat{\mathbb{E}}\left|\int_s^t |\xi_u|^2 du\right|^{r/2},$$

$$\hat{\mathbb{E}}\left|\sup_{s\le v\le t}\int_s^v \xi_u d\langle B\rangle(u)\right|^r \le \overline{\sigma}^{2r}\, |t-s|^{r-1}\, \hat{\mathbb{E}}\int_s^t |\xi_u|^r du,$$

where $c_1(r,\overline{\sigma})$ is a constant dependent of r and $\overline{\sigma}$.

3. θ-Milstein Scheme for G-SDEs

For the sake of simplicity, we only discuss the case of scalar G-Brownian motion. In fact, our results can be generalized to the case of multi-dimensional G-Brownian motion. Consider the following one-dimensional G-SDE:

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dB(t) + h\big(X(t)\big)d\langle B\rangle(t),\ t > 0,$$

$$X(0) = X_0, \tag{2}$$

where $B(t)$ is a one-dimensional G-Brownian motion and $\langle B\rangle(t)$ is the quadratic variation process of the G-Brownian motion $B(t)$. Let $f, g, h\colon R \to R$ be Borel measurable functions. We impose the following hypotheses.

Assumption 3.1 (Global condition) There exists a positive constant κ_1 such that for any $x_1, x_2 \in R$,

$$|f(x_1) - f(x_2)| \vee |g(x_1) - g(x_2)| \vee |h(x_1) - h(x_2)| \le \kappa_1 |x_1 - x_2|. \tag{3}$$

Assumption 3.2 There exists a positive constant κ_2 such that for any $x_1, x_2 \in R$,

$$|L^1 g(x_1) - L^1 g(x_2))| \le \kappa_2 |x_1 - x_2|, \tag{4}$$

where the operator L^1 is defined by $L^1 g(x) \coloneqq g(x)g'(x)$.

Assumption 3.3 Let f, g, h be two times continuously differentiable functions. There are positive constants K_2 such that for any $x \in R$,

$$|f'(x)| \vee |f''(x)| \vee |h(x)| \vee |h''(x)| \vee |g'(x)| \vee |g''(x)| \le K_2(1 + |x|). \tag{5}$$

Under Assumption 3.1, the G-SDE (1) has a unique continuous solution on $t > 0$, see [8]. We denote this true solution by $X(t)$. From (2) and (3), we have

$$|f(x)| \vee |g(x)| \vee |h(x)| \vee |L^1 g(x)| \le K_1(1 + |x|), \quad \forall x \in R, \tag{6}$$

where $K_1 \coloneqq (\kappa_1 \vee \kappa_2) + (|f(0)| \vee |g(0)| \vee |h(0)| \vee |L^1 g(0)|)$. For each $V \in C^2(R; R_+)$, we have the following G-Itô formula

$$dV\big(X(t)\big) = V'\big(X(t)\big)f\big(X(t)\big)dt + \left(\frac{1}{2}V''\big(X(t)\big)\big|g\big(X(t)\big)\big|^2 + V'\big(X(t)\big)h\big(X(t)\big)\right)d\langle B\rangle(t)$$

$$+V'\big(X(t)\big)g\big(X(t)\big)dB(t). \tag{7}$$

We now begin to introduce the idea of Milstein method for G-SDE (1). For a twice continuously differentiable function $g: R \to R$, the G-Itô formula (6) provides the representation

$$g\big(X(s)\big) = g(X_0) + \int_0^s g'\big(X(u)\big)f\big(X(u)\big)du + \int_0^s \left(\frac{1}{2}g''\big(X(u)\big)\big|g\big(X(u)\big)\big|^2 + g'\big(X(u)\big)h\big(X(u)\big)\right)d\langle B\rangle(u)$$

$$+ \int_0^s g'\big(X(u)\big)g\,\big(X(u)\big)dB(u), \quad 0 \le s \le t. \tag{8}$$

Inserting this into (1) gives that

$$X(t) = X_0 + \int_0^t f\,\big(X(s)\big)ds + \int_0^t h\,\big(X(s)\big)d\langle B\rangle(s) + g(X_0)B(t)$$

$$+ \int_0^t \int_0^s g'\,\big(X(u)\big)g\big(X(u)\big)dB(u)dB(s) + RR \tag{9}$$

with remainder term

$$RR = \int_0^t \int_0^s g'\,\big(X(u)\big)f\big(X(u)\big)du dB(s)$$

$$+ \int_0^t \int_0^s \left(\frac{1}{2}g''\big(X(u)\big)\big|g\big(X(u)\big)\big|^2 + g'\big(X(u)\big)h\big(X(u)\big)\right)d\langle B\rangle(u)dB(s).$$

For a sufficiently small $t > 0$, under some regularity of $g(X)$, by truncating the remainder term RR with order $3/2$, we have the following approximation

$$X(t) \approx X_0 + f(X_0)t + g(X_0)B(t) + h(X_0)d\langle B\rangle(t) + g'(X_0)g(X_0)\int_0^t \int_0^s dB(u)dB\,(s). \tag{10}$$

Moreover, we have the following multiple integrals of type

$$\int_0^t \int_0^s dB\,(u)dB(s) = \tfrac{1}{2}[|B(t)|^2 - \langle B\rangle(t)]. \tag{11}$$

Inserting this into (10) gives

$$X(t) \approx X_0 + f(X_0)t + g(X_0)B(t) + h(X_0)d\langle B\rangle(t) + \tfrac{1}{2}g'(X_0)g(X_0)[|B(t)|^2 - \langle B\rangle(t)]. \tag{12}$$

(12) provides an approximation expansion of order greater than $1/2$ for the G-Itô process $X(t)$ near X_0.

Now, we are ready to construct our numerical scheme for G-SDE (1). Fix any $T > 0$, let N be a positive integer and $\delta = T/N < 1$ a step size. Combining (12) with θ-method, we define the following θ-Milstein scheme

$$z_k = y_k + \theta f(z_k)\delta$$

$$y_{k+1} = y_k + f(z_k)\delta + g(z_k)\Delta B_k + h(z_k)\Delta\langle B\rangle_k + \tfrac{1}{2}L^1 g(z_k)(|\Delta B_k|^2 - \Delta\langle B\rangle_k),\ k \ge 0, \tag{13}$$

where $z_0 = X_0$, $y_0 = X_0 - \theta f(X_0)\delta$, $\theta \in [0,1]$, $t_k = k\delta$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, $\Delta\langle B\rangle_k = \langle B\rangle(t_{k+1}) - \langle B\rangle(t_k)$, $L^1 g(x) = g(x)g'(x)$. Inserting $y_k = z_k - \theta f(z_k)\delta$ into the second equation of (13), we get

$$z_{k+1} = z_k + \theta f(z_{k+1})\delta + (1-\theta)f(z_k)\delta + g(z_k)\Delta B_k + h(z_k)\Delta\langle B\rangle_k + \tfrac{1}{2}L^1 g(z_k)(|\Delta B_k|^2 - \Delta\langle B\rangle_k). \tag{14}$$

Define

$$\delta^* = \begin{cases} \dfrac{1}{8K_1\theta} \wedge 1, & \theta \in (0,1]; \\ 1, & \theta = 0. \end{cases}$$

In order to get a well-defined solution of (14), we assume that $\delta \le \delta^*$ which implies that the equation

$$z = y + \theta f(z)\delta$$

has a unique solution $z = F(y)$ for any $y \in R$ via the Banach fixed point theorem. For any $t \in [t_k, t_{k+1})$, define the continuous Milstein solution:

$$Y(t) = y_k + f(z_k)(t - t_k) + g(z_k)\big(B(t) - B(t_k)\big) + L^1 g(z_k) I_{t_k,t} + h(z_k)\big(\langle B\rangle(t) - \langle B\rangle(t_k)\big)$$
$$Z(t) = F\big(Y(t)\big) \tag{15}$$

with $Y(0) = X_0 - \theta\delta f(X_0)$, $I_{t_k,t} = \int_{t_k}^{t} \varDelta B(s) dB(s)$, where $\varDelta B(s)$ and $\underline{s}$ are defined by

$$\varDelta B(s) := B(s) - B(\underline{s}), \forall s \in [0,T],$$

$$\underline{s} := t_k, \ \ \forall s \in [t_k, t_{k+1}).$$

Moreover, (15) can be rewritten as the following equivalent form:

$$Y(t) = y_0 + \int_0^t f\left(Z(\underline{s})\right) ds + \int_0^t g\left(Z(\underline{s})\right) dB(s) + \int_0^t L^1 g\left(Z(\underline{s})\right) \varDelta B(s) dB(s) + \int_0^t h\left(Z(\underline{s})\right) d\langle B\rangle(s)$$
$$Z(t) = F\big(Y(t)\big). \tag{16}$$

Thus, we have $Y(t_k) = y_k$ and

$$Y(t) = Y(\underline{t}) + \int_{\underline{t}}^{t} f\left(Z(\underline{s})\right) ds + \int_{\underline{t}}^{t} g\left(Z(\underline{s})\right) dB(s) + \int_{\underline{t}}^{t} L^1 g\left(Z(\underline{s})\right) \varDelta B(s) dB(s) + \int_{\underline{t}}^{t} h\left(Z(\underline{s})\right) d\langle B\rangle(s), \tag{17}$$

which means $Y(t)$ and $Y(\underline{t})$ coincide with the discrete solution at the grid points. Let $\psi: R \to R$ be twice differentiable. Then the following Taylor formula

$$\psi(y) - \psi(y^*) = \psi'(y^*)(y - y^*) + \langle y - y^*, y - y^*\rangle \int_0^1 (1 - s)\, \psi''\big(y^* + s(y - y^*)\big) ds \tag{18}$$

holds. Replacing y and y^* in (18) by $Y(t)$ and $Y(\underline{t})$, respectively, we obtain from (17) that

$$\psi\big(Y(t)\big) - \psi\left(Y(\underline{t})\right) = \psi'\left(Y(\underline{t})\right)\left(\int_{\underline{t}}^{t} g\left(Z(\underline{s})\right) dB(s)\right) + \tilde{R}_1(\psi), \tag{19}$$

where

$$\tilde{R}_1(\psi) := \psi'\left(Y(\underline{t})\right)\left(\int_{\underline{t}}^{t} f\left(Z(\underline{s})\right) ds + \int_{\underline{t}}^{t} L^1 g\left(Z(\underline{s})\right) \varDelta B(s) dB(s) + \int_{\underline{t}}^{t} h\left(Z(\underline{s})\right) d\langle B\rangle(s)\right)$$

$$+|Y(s) - Y(\underline{s})|^2 \int_0^1 (1 - s)\, \psi''\left(Y(\underline{s}) + s\left(Y(s) - Y(\underline{s})\right)\right) ds.$$

Noting that $g'\left(Z(\underline{s})\right) g\left(Z(\underline{s})\right) = L^1 g\left(Z(\underline{s})\right)$, then we conclude from (19) that

$$g\big(Z(s)\big) - g\left(Z(\underline{s})\right) - L^1 g\left(Z(\underline{s})\right) \varDelta B(s) = \tilde{R}_1(g). \tag{20}$$

In what follows, C deontes a generic positive constant independent of step size δ, whose value may change from line to line.

4. Main Results

In order to show the main results, we need some lemmas. We first state a known result by Yin *et al*. [20] as a lemma.

Lemma 4.1 Let Assumption 3.1 hold. Then for any $r \geq 2$, there exists a constant $C = C(r, T, \overline{\sigma})$ such that

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}|X(t)|^r\right] \leq C.$$

Now, we establish the boundedness of r-th moments of θ-Milstein solutions.

Lemma 4.2 Let Assumptions 3.1, 3.2 and 3.3 hold. Then for any $r \geq 2$ and $\delta \leq \delta^*$,

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}|Z(t)|^r\right] \vee \hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}|Y(t)|^r\right] \leq C \tag{21}$$

and

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}|Y(t)-Z(t)|^r\right] \leq C\delta^r, \tag{22}$$

where $C = C(\kappa_1, \kappa_2, r, T, \overline{\sigma}, \theta)$ is a positive constant independent of δ.

Proof: By the G-Itô formula and Assumption 3.1, we conclude from (15) that

$$\begin{aligned}|Y(t)|^2 &= |Y_0|^2 + \int_0^t 2\langle Y(s), f(Z(\underline{s}))\rangle ds + \int_0^t 2\langle Y(s), g(Z(\underline{s}))\rangle dB(s)\\ &\quad + \int_0^t \left(|g(Z(\underline{s})) + L^1 g(Z(\underline{s}))\Delta B(s)|^2 + 2\langle Y(s), h(Z(\underline{s}))\rangle\right) d\langle B\rangle(s)\\ &\leq |Y_0|^2 + \int_0^t \left(|Y(s)|^2 + K_1(1+|Z(\underline{s})|^2)\right) ds + \int_0^t 2\langle Y(s), g(Z(\underline{s}))\rangle dB(s)\\ &\quad + \int_0^t \left(3(1+|Z(\underline{s})|^2) + 2|L^1 g(Z(\underline{s}))\Delta B(s)|^2 + |Y(s)|^2\right) d\langle B\rangle(s).\end{aligned} \tag{23}$$

For any $r \geq 2$ and $t_1 \in [0, T]$, we have

$$\frac{1}{4^{r-1}}\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|Y(t)|^{2r}\right] \leq |Y_0|^r + \Pi_1(t) + \Pi_2(t) + \Pi_3(t). \tag{24}$$

where

$$\Pi_1(t) = \hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}\left|\int_0^t\left(2|Y(s)|^2 + (1+3\overline{\sigma}^2)K_1^2(1+|Z(\underline{s})|^2)\right)ds\right|^r\right] \tag{25}$$

$$\Pi_2(t) = 2^r\overline{\sigma}^r\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}\left|\int_0^t |L^1 g(Z(\underline{s}))\Delta B(s)|^2\, ds\right|^r\right] \tag{26}$$

$$\Pi_3(t) = 2^r \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} \left|\langle Y(s), g(Z(\underline{s}))\rangle dB(s)\right|^r\right]. \tag{27}$$

By the elementary inequalities, we have

$$\Pi_1(t) \le T^{r-1}\hat{\mathbb{E}}\int_0^{t_1}\left|\left(2|Y(s)|^2 + (1+3\overline{\sigma}^2)K_1^2(1+|Z(\underline{s})|^2)\right)\right|^r ds$$

$$\le C + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Y(u)|^{2r}\right]ds + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Z(u)|^{2r}\right]ds. \tag{28}$$

By Peng *et al*. [Ref. 8, Proposition 3.1.6, p.52], we have

$$\hat{\mathbb{E}}|\Delta B(s)|^{2r} = \hat{\mathbb{E}}|B(s) - B(\underline{s})|^{2r} \le \hat{\mathbb{E}}|B(\delta)|^{2r} \le \lfloor 2r\rfloor!!\,\overline{\sigma}^{2r}\delta^r.$$

According to the Hölder inequality, we have

$$\Pi_2(t) \le 2^r\overline{\sigma}^r T^{r-1}\hat{\mathbb{E}}\left[\int_0^{t_1}|L^1 g\left(Z(\underline{s})\right)\Delta B(s)|^{2r}ds\right]. \tag{29}$$

Since $\Delta B(s) = B(s) - B(\underline{s})$ is independent of $Z(\underline{s})$, we have

$$\hat{\mathbb{E}}\left[|L^1 g(Z(\underline{s}))\Delta B(s)|^{2r}\right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\left|L^1 g\left(Z(\underline{s})\right)\Delta B(s)\right|^{2r}\right]_{x=Z(\underline{s})}\right]$$

$$\le C\delta^r\hat{E}\left(1+\left|Z(\underline{s})\right|^{2r}\right). \tag{30}$$

Substituting this into (29) gives

$$\Pi_2(t) \le C\,\delta^r\int_0^{t_1}\hat{\mathbb{E}}\left(1+\left|Z(\underline{s})\right|^{2r}\right)ds \le C + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Z(u)|^{2r}\right]ds. \tag{31}$$

By Lemma 2.4 and (6), we have

$$\Pi_3(t) \le C\hat{\mathbb{E}}\left|\int_0^{t_1}(|Y(s)|^2 + |g(Z(\underline{s}))|^2)ds\right|^r \le C\hat{\mathbb{E}}\int_0^{t_1}(|Y(s)|^{2r} + |g(Z(\underline{s}))|^{2r})ds$$

$$\le C\hat{\mathbb{E}}\int_0^{t_1}(1 + |Y(s)|^{2r} + |Z(\underline{s})|^{2r})ds$$

$$\le C + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Y(u)|^{2r}\right]ds + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Z(u)|^{2r}\right]ds. \tag{32}$$

Substituting (8), (31) and (12) into (24), we get

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1}|Y(t)|^{2r}\right] \le C + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Y(u)|^{2r}\right]ds + C\int_0^{t_1}\hat{\mathbb{E}}\left[\sup_{0\le u\le s}|Z(u)|^{2r}\right]ds. \tag{33}$$

From (6), we have

$$\langle Z(t), f(Z(t))\rangle \le K_1|Z(t)|(1+|Z(t)|) \le K_1(1+|Z(t)|)^2 \le 2K_1(1+|Z(t)|^2). \tag{34}$$

From this and $Y(t) = Z(t) - \theta f\left(Z(t)\right)\delta$, we have

$$|Y(t)|^2 = |Z(t)|^2 - 2\theta\delta\langle Z(t), f(Z(t))\rangle + \theta^2|f(Z(t))|^2\delta^2$$

$$\geq |Z(t)|^2 - 4K_1\theta\delta(1+|Z(t)|^2) = (1-4K_1\theta\delta)|Z(t)|^2 - 4K_1\theta\delta.$$

Thus, we have

$$(1-4K_1\theta\delta)|Z(t)|^2 \leq |Y(t)|^2 + 4K_1\theta\delta. \tag{35}$$

For any $\delta \leq \delta^*$, we have

$$0 < \frac{1}{1-4K_1\theta\delta} \leq 2. \tag{36}$$

From this and (35), we have

$$|Z(t)|^2 \leq \frac{|Y(t)|^2+4K_1\theta\delta}{1-4K_1\theta\delta} \leq 2(|Y(t)|^2 + 4K_1\theta\delta) \leq (2+4K_1)(1+|Y(t)|^2). \tag{37}$$

By (33) and (37), we get

$$\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|Y(t)|^{2r}\right] \leq C + C\int_0^{t_1}\widehat{\mathbb{E}}\left[\sup_{0\leq u\leq s}|Y(u)|^{2r}\right]ds.$$

Using the Gronwall inequality, we get the desired assertion (21). Combining this with the fact that $Z(t)-Y(t) = \theta f\big(Z(t)\big)\delta$, which means

$$|Z(t)-Y(t)| \leq K_1(1+|Z(t)|)\delta, \tag{38}$$

we get the other assertion (22). Thus, the proof is complete.

Applying assumptions 3.1, 3.2, 3.3 and combining with lemmas 4.1 and 4.2, we have the following lemma.

Lemma 4.3 Let Assumptions 3.1, 3.2 and 3.3 hold. Then for any $r \geq 2$ and $\delta \leq \delta^*$,

$$\sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|f\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|g\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|h\big(X(t)\big)\right|^r \leq C$$

$$\sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|f'\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|g'\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|h'\big(X(t)\big)\right|^r \leq C$$

$$\sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|f''\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|g''\big(X(t)\big)\right|^r \vee \sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|h''\big(X(t)\big)\right|^r \leq C$$

$$\sup_{0\leq t\leq T}\widehat{\mathbb{E}}\left|L^1 g\big(X(t)\big)\right|^r \leq C \tag{39}$$

Moreover, replacing the true solution $X(t)$ by the Milstein solutions $Z(t)$ or $Y(t)$, the assertion (39) still holds.

Lemma 4.4 Let Assumptions 3.1, 3.2 and 3.3 hold. Then for any $r \geq 2$ and $\delta \leq \delta^*$,

$$\widehat{\mathbb{E}}|X(t) - X(\underline{t})|^r \leq C\delta^{r/2}, \quad \forall t \in [0,T], \tag{40}$$

and

$$\widehat{\mathbb{E}}|Y(t) - Y(\underline{t})|^r \leq C\delta^{r/2}, \quad \forall t \in [0,T], \tag{41}$$

where $C = C(\kappa_1, \kappa_2, r, T, \overline{\sigma}, \theta)$ is a positive constant independent of δ.

Proof: Using Lemmas 2.4 and 4.3, we get from (2) that

$$\hat{\mathbb{E}}|X(t)-X(\underline{t})|^r = C\hat{\mathbb{E}}\left|\int_{\underline{t}}^{t} f\left(X(s)\right)ds\right|^r + C\hat{\mathbb{E}}\left|\int_{\underline{t}}^{t} g\left(X(s)\right)dB(s)\right|^r + C\hat{\mathbb{E}}\left|\int_{\underline{t}}^{t} h\left(X(s)\right)d\langle B\rangle(s)\right|^r$$

$$\leq C\delta^{r-1}\int_{\underline{t}}^{t}\hat{\mathbb{E}}|f(X(s))|^r ds + C\delta^{r/2-1}\int_{\underline{t}}^{t}\hat{\mathbb{E}}|g(X(s))|^r ds + C\overline{\sigma}^{2r}\delta^{r-1}\int_{\underline{t}}^{t}\hat{\mathbb{E}}|h(X(s))|^r ds$$

$$\leq C\delta^{r/2}, \quad t\in[0,T]. \tag{42}$$

From (17), it follows that

$$Y(t)-Y(\underline{t}) = f\left(Z(\underline{t})\right)(t-\underline{t}) + g\left(Z(\underline{t})\left(B(t)-B(\underline{t})\right)\right) + L^1 g\left(Z(\underline{t})\right)I_{\underline{t},t} + h\left(Z(\underline{t})\right)\left(\langle B\rangle(t)-\langle B\rangle(\underline{t})\right) \tag{43}$$

where $I_{\underline{t},t} = \int_{\underline{t}}^{t}\int_{\underline{t}}^{s} dB\,(u)dB(s)$. Therefore,

$$\hat{\mathbb{E}}|Y(t)-Y(\underline{t})|^r \leq C\delta^r\hat{\mathbb{E}}\left[|f(Z(\underline{t}))|^r\right] + C\hat{\mathbb{E}}\left[|g(Z(\underline{t}))(B(t)-B(\underline{t}))|^r\right]$$

$$+C\hat{\mathbb{E}}\left[|L^1 g(Z(\underline{t}))I_{\underline{t},t}|^r\right] + C\hat{\mathbb{E}}\left[|h(Z(\underline{t}))(\langle B\rangle(t)-\langle B\rangle(\underline{t}))|^r\right]. \tag{44}$$

Note that $B(t)-B(\underline{t})$ is independent from $Z(\underline{t})$, hence

$$\hat{\mathbb{E}}\left[|g(Z(\underline{t}))[B(t)-B(\underline{t})]|^r\right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[g(x)|B(t)-B(\underline{t})|^r\right]_{x=Z(\underline{t})}\right] \tag{45}$$

$$\leq C\delta^{r/2}\hat{\mathbb{E}}\left[|g(Z(\underline{t}))|^r\right] \leq C\delta^{r/2}.$$

Moreover,

$$\hat{\mathbb{E}}|I_{\underline{t},t}|^r = \hat{\mathbb{E}}\left|\frac{|B(t)-B(\underline{t})|^2-(\langle B\rangle(t)-\langle B\rangle(\underline{t}))}{2}\right|^r$$

$$\leq \tfrac{1}{2}\left(\hat{\mathbb{E}}|B(\delta)|^{2r}+\hat{\mathbb{E}}|\langle B\rangle(\delta)|^{2r}\right) \leq \tfrac{1}{2}(\lfloor 2r\rfloor!!+1)\overline{\sigma}^{2r}\delta^{2r} =: c_2(r,\overline{\sigma})\delta^r. \tag{46}$$

Similarly, we have

$$\hat{\mathbb{E}}|L^1 g(Z(\underline{t})I_{\underline{t},t})|^r = \hat{\mathbb{E}}\left[\hat{E}\left[|L^1 g(x)|^r I_{\underline{t},t}\right]_{x=Z(\underline{t})}\right] \leq c_2(r,\overline{\sigma})\delta^r\hat{\mathbb{E}}\left[|L^1 g(Z(\underline{t}))|^r\right] \leq C\delta^r \tag{47}$$

and

$$\hat{\mathbb{E}}|h(Z(\underline{t})(\langle B\rangle(t)-\langle B\rangle(\underline{t}))|^r = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[|h(x)|^r(\langle B\rangle(t)-\langle B\rangle(\underline{t}))\right]_{x=Z(\underline{t})}\right]$$

$$\leq c_2(r,\overline{\sigma})\delta^r\hat{\mathbb{E}}|h(Z(\underline{t}))|^r \leq C\delta^r. \tag{48}$$

Plugging (45), (47) and (48) into (44) obtains (41). Thus, the proof is complete.

Lemma 4.5 Let Assumptions 3.1, 3.2 and 3.3 hold. Then for any $r\geq 2$ and $\delta\leq\delta^*$,

$$\hat{\mathbb{E}}|\tilde{R}_1(f)|^r \leq C\delta^r, \tag{49}$$

where $C = C(\kappa_1,\kappa_2,r,T,\overline{\sigma},\theta)$ is a positive constant independent of δ.

Proof: From (19), we obtain that

$$f(Y(t)) - f\left(Y(\underline{t})\right) = f'\left(Y(\underline{t})\right)\int_{\underline{t}}^{t} g\left(Z(\underline{s})\right)dB(s) + \tilde{R}_1(f), \tag{50}$$

where

$$\tilde{R}_1(f) = f'\left(Y(\underline{t})\right)\left(\int_{\underline{t}}^{t} f\left(Z(\underline{s})\right)ds + \int_{\underline{t}}^{t} L^1 g\left(Z(\underline{s})\right)\Delta B(s)dB(s) + \int_{\underline{t}}^{t} h\left(Z(\underline{s})\right)d\langle B\rangle(s)\right)$$

$$+ \underbrace{|Y(s) - Y(\underline{s})|^2 \int_0^t (1-s)\, f''\left(Y(\underline{s}) + s\left(Y(s) - Y(\underline{s})\right)\right)ds.}_{:=R_1(f)} \tag{51}$$

By the Hölder inequality and (41), we have

$$\hat{\mathbb{E}}|R_1(f)|^r \le \int_0^1 \hat{\mathbb{E}}\left[|Y(s) - Y(\underline{s})|^{2r}|f''\left(Y(\underline{s}) + s\left(Y(s) - Y(\underline{s})\right)\right)|^r\right]ds$$

$$\le \int_0^1 \left(\hat{\mathbb{E}}|Y(s) - Y(\underline{s})|^{4r}\right)^{1/2}\ \left(\hat{\mathbb{E}}|f''\left(Y(\underline{s}) + s\left(Y(s) - Y(\underline{s})\right)\right)|^{2r}\right)^{1/2} ds$$

$$\le C\left(1 + \hat{\mathbb{E}}|Y(t)|^{2r} + \hat{\mathbb{E}}|Y(\underline{s})|^{2r}\right)^{r/2}\delta^r \le C\delta^r. \tag{52}$$

From (19), we have

$$\hat{\mathbb{E}}|\tilde{R}_1(f)|^r \le C\delta^r\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))f(Z(\underline{s}))|^r\right] + C\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))L^1g(Z(\underline{s}))I_{\underline{t},t}|^r\right]$$

$$+C\hat{\mathbb{E}}[|R_1(f)|^r] + C\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))h(Z(\underline{s}))(\langle B\rangle(s) - \langle B\rangle(\underline{s}))|^r\right]. \tag{53}$$

By the Hölder inequality and Lemma 4.2, we have

$$\hat{\mathbb{E}}|f'(Y(\underline{s}))f(Z(\underline{s}))|^r \le \left(\hat{\mathbb{E}}\left|f'\left(Y(\underline{s})\right)|^{2r}\hat{\mathbb{E}}\right|f\left(Z(\underline{s})\right)|^{2r}\right)^{1/2} \le C. \tag{54}$$

Similarly, we have

$$\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))L^1g(Z(\underline{s}))|^r\right] \le C$$

and

$$\hat{\mathbb{E}}|I_{\underline{s},s}|^r \le C\delta^r.$$

Thus, we have

$$\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))L^1g(Z(\underline{s}))I_{\underline{s},s}|^r\right] \le C\delta^r, \tag{55}$$

and

$$\hat{\mathbb{E}}\left[|f'(Y(\underline{s}))h(Z(\underline{s}))(\langle B\rangle(s) - \langle B\rangle(\underline{s}))|^r\right]$$

$$= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[|f'(x_1)h(x_2)|^r|\langle B\rangle(s-\underline{s})|^r\right]_{x_1=Y(\underline{s}),x_2=Z(\underline{s})}\right]$$

$$\le \overline{\sigma}^{2r}\delta^r\hat{\mathbb{E}}|f'(Y(\underline{s}))h(Z(\underline{s}))|^r \le C\delta^r. \tag{56}$$

Substituting (52), (54), (55) and (56) into (53), we get the desired assertion (49).

Now, we establish the convergence result of the θ-Milstein method for G-SDEs as below.

Theorem 4.6 Let Assumptions 3.1, 3.2 and 3.3 hold, let $X(t)$ denote the solution of G-SDE (2), and let Y and Z represent the θ-Milstein approximations defined by (15). Then for any $r \geq 2$ and $\delta \leq \delta^*$,

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T} |X(t) - Y(t)|^r\right] \leq C\delta^r \tag{57}$$

and

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T} |X(t) - Z(t)|^r\right] \leq C\delta^r \tag{58}$$

where $C = C(\kappa_1, \kappa_2, r, T, \overline{\sigma}, \theta)$ is a positive constant independent of step size δ.

Proof: (2) can be rewritten as:

$$X(t) = X_0 + \int_0^t f\left(X(s)\right)ds + \int_0^t g\left(X(s)\right)dB(s) + \int_0^t h\big(X(s)\big)d\langle B\rangle(s) \tag{59}$$

By (59) and (15), we have

$$e(t) = e(0) + \int_0^t (f(X(s)) - f(Z(\underline{s})))ds + \int_0^t \sigma(s)dB(s) + \int_0^t (h(X(s)) - h(Z(\underline{s})))d\langle B\rangle(s), \tag{60}$$

where $e(t) \coloneqq X(t) - Y(t)$ and $\sigma(s) \coloneqq g\big(X(s)\big) - g\left(Z(\underline{s})\right) - L^1 g\left(Z(\underline{s})\right)\Delta B(s)$. Applying G-Itô formula to (60), we have

$$\begin{aligned}
|e(t)|^2 &= |e(0)|^2 + 2\int_0^t \langle e(s), f(X(s) - f(Z(\underline{s}))\rangle\, ds + 2\int_0^t \langle e(s), \sigma(s)\rangle\, dB(s) \\
&\quad + \int_0^t (|\sigma(s)|^2 + 2\langle e(s), h(X(s)) - h(Z(\underline{s}))\rangle)d\langle B\rangle(s) \\
&= |e(0)|^2 + 2\int_0^t \langle e(s), f(X(s)) - f(Y(s))\rangle ds + 2\int_0^t \langle e(s), f(Y(s)) - f(Y(\underline{s}))\rangle ds \\
&\quad +2\int_0^t \left\langle e(s), f\left(Y(\underline{s})\right) - f\left(Z(\underline{s})\right)\right\rangle ds + \int_0^t |\sigma(s)|^2\, d\langle B\rangle(s) \\
&\quad +2\int_0^t \langle e(s), \sigma(s)\rangle\, dB(s) + 2\int_0^t \langle e(s), h(X(s)) - h(Y(s))\rangle\rangle d\langle B\rangle(s) \\
&\quad +2\int_0^t \langle e(s), h(Y(s)) - h(Y(\underline{s}))\rangle d\langle B\rangle(s) + 2\int_0^t \langle e(s), h(Y(\underline{s})) - h(Z(\underline{s}))\rangle\rangle d\langle B\rangle(s) \\
&\leq \theta^2|f(X_0)|^2\delta^2 + 2\kappa_1\int_0^t |e(s)|^2 ds + \kappa_1\int_0^t |e(s)|^2 d\,\langle B\rangle(s) \\
&\quad + \underbrace{2\int_0^t |\sigma(s)|^2\, d\langle B\rangle(s)}_{\coloneqq J_1(t)} + \underbrace{2\int_0^t \langle e(s), \sigma(s)\rangle dB(s)}_{\coloneqq J_2(t)} \\
&\quad + \underbrace{2\int_0^t \langle e(s), f(Y(\underline{s})) - f(Z(\underline{s}))\rangle ds}_{\coloneqq J_3(t)} + \underbrace{2\int_0^t \langle e(s), f(Y(s)) - f(Y(\underline{s}))\rangle ds}_{\coloneqq J_4(t)} \\
&\quad + \underbrace{2\int_0^t \langle e(s), h(Y(\underline{s})) - h(Z(\underline{s}))\rangle\rangle d\langle B\rangle(s)}_{\coloneqq J_5(t)} + \underbrace{2\int_0^t \langle e(s), h(Y(s)) - h(Y(\underline{s}))\rangle\rangle d\langle B\rangle(s)}_{\coloneqq J_6(t)}.
\end{aligned} \tag{61}$$

For any $t_1 \in [0,T]$ and $r \geq 2$, we have

$$\frac{1}{8^{\frac{r}{2}-1}}\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|e(t)|^r\right] \leq \theta^r|f(X_0)|^r\,\delta^r + \kappa_1^r\hat{\mathbb{E}}\left[\sup\left|\int_0^t 2\,|e(s)|^2\,ds + \int_0^t 2\,|e(s)|^2\,d\langle B\rangle(s)\right|^r\right]$$

$$+\sum_{i=1}^{6}\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|J_i(t)|^{r/2}\right] \tag{62}$$

Apply G-Itô formula to $g\big(X(s)\big)$, we obtain

$$g\big(X(s)\big) = g\left(X(\underline{s})\right) + \int_{\underline{s}}^{s} L^1\,g\big(X(u)\big)dB(u) + \int_{\underline{s}}^{s} g'\big(X(u)\big)\,f\big(X(u)\big)du$$

$$+\int_{\underline{s}}^{s}\left(\frac{1}{2}g''\left(X(u)\big|g\big(X(u)\big)\big|^2\right) + g'\big(X(u)\big)h\big(X(u)\big)\right)d\langle B\rangle(u) \tag{63}$$

Inserting this into $\sigma(s) = g\big(X(s)\big) - g\left(Z(\underline{s})\right) - L^1 g\left(Z(\underline{s})\right)\Delta B(s)$, we get

$$\sigma(s) = g(X(\underline{s})) - g(Z(\underline{s})) + \int_{\underline{s}}^{s}(L^1 g(X(u)) - L^1 g(Z(\underline{u})))dB(u)$$

$$+\int_{\underline{s}}^{s} g'\big(X(u)\big)f\big(X(u)\big)du + \int_{\underline{s}}^{s}\big[g'\big(X(u)\big)h\big(X(u)\big)\big]\,d\langle B\rangle(u)$$

$$+\int_{\underline{s}}^{s}\frac{1}{2}g''\left(X(u)\right)\big|g\big(X(u)\big)\big|^2 d\langle B\rangle(u)$$

$$= g(X(\underline{s})) - g(Z(\underline{s})) + \int_{\underline{s}}^{s}(L^1 g(X(u)) - L^1 g(X(\underline{u})))dB(u)$$

$$+\int_{\underline{s}}^{s}(L^1 g(X(\underline{u})) - L^1 g(Z(\underline{u})))dB(u) + \int_{\underline{s}}^{s}\big[g'\big(X(u)\big)h\big(X(u)\big)\big]\,d\langle B\rangle(u)$$

$$+\int_{\underline{s}}^{s}\frac{1}{2}g''\left(X(u)\right)\big|g\big(X(u)\big)\big|^2 d\langle B\rangle(u) \tag{64}$$

According to Assumptions 3.1 and 3.2, we have

$$\frac{\hat{\mathbb{E}}|\sigma(s)|^r}{6^{r-1}} \leq \hat{\mathbb{E}}|g(X(\underline{s}) - g(Z(\underline{s}))|^r + \hat{\mathbb{E}}\left|\int_{\underline{s}}^{s}\left(L^1 g\big(X(u)\big) - L^1 g\left(X(\underline{u})\right)\right)dB(u)\right|^r$$

$$+\hat{\mathbb{E}}\left|\int_{\underline{s}}^{s}(L^1 g(X(\underline{u})) - L^1 g(Z(\underline{u})))dB(u)\right|^r + \hat{\mathbb{E}}\left|\int_{\underline{s}}^{s}\left(g'\big(X(u)\big)h\big(X(u)\big)\right)d\langle B\rangle(u)\right|^r$$

$$+\hat{\mathbb{E}}\left|\int_{\underline{s}}^{s}\frac{1}{2}g''\big(X(u)\big)|g(X(u)|^2\,d\langle B\rangle(u)\right|^r + \hat{\mathbb{E}}\left|\int_{\underline{s}}^{s}\frac{1}{2}g'\left(X(u)\right)f(X(u)du\right|^r$$

$$\leq \kappa_1^r\hat{\mathbb{E}}|X(\underline{s}) - Z(\underline{s})|^r + \kappa_1^r\delta^{r/2-1}\int_{\underline{s}}^{s}\hat{\mathbb{E}}|X(u) - X(\underline{u})|^r ds$$

$$+\kappa_2^r\delta^{r/2-1}\int_{\underline{s}}^{s}\hat{\mathbb{E}}|X(\underline{u}) - Z\big(\underline{u}\big)|^r ds + \big(1+\overline{\sigma}^{2r}\big)\delta^{r-1}\int_{\underline{s}}^{s}\hat{\mathbb{E}}|g'(X(\underline{u}))\,f\big(X(u)\big)|^r ds$$

$$+\left(\frac{\overline{\sigma}^2}{2}\right)^r \delta^{r-1} \int_{\underline{s}}^{s} \widehat{\mathbb{E}} \left|g''\big(X(u)\big)\big|g\big(X(u)\big)\big|^2\right|^r du. \tag{65}$$

By the Lemma 4.3 and the Hölder inequality, we have

$$\widehat{\mathbb{E}}\big|g'\big(X(u)\big)f\big(X(u)\big)|^r \vee \widehat{\mathbb{E}}\big|g''\big(X(u)\big)|g(X(u))|^2|^r \leq C.$$

Therefore,

$$\frac{\widehat{\mathbb{E}}|\sigma(s)|^r}{6^{r-1}} \leq C\widehat{\mathbb{E}}\big|X\big(\underline{s}\big) - Y\big(\underline{s}\big)|^r + C\widehat{\mathbb{E}}\big|Y\big(\underline{s}\big) - Z\big(\underline{s}\big)|^r + \kappa_1^r\delta^r$$

$$+\kappa_2^r\delta^{r/2-1} \int_{\underline{s}}^{s} \widehat{\mathbb{E}}|X(\underline{u}) - Z\big(\underline{u}\big)|^r ds + C\delta^{r/2-1} \int_{\underline{s}}^{s} \widehat{\mathbb{E}}|Y(\underline{u}) - Z\big(\underline{u}\big)|^r ds + C\delta^r. \tag{66}$$

With the aid of Lemma 4.2, we obtain

$$\widehat{\mathbb{E}}\big|\sigma(s)|^r \leq C\widehat{\mathbb{E}}\big|e\big(\underline{s}\big)|^r + C\delta^r + \kappa_1^r\delta^r + C\delta^{r/2}\widehat{\mathbb{E}}\big|e\big(\underline{s}\big)|^r + C\delta^{3r/2} \leq C\widehat{\mathbb{E}}\big|e\big(\underline{s}\big)|^r + C\delta^r. \tag{67}$$

By Lemma 2.4 and (67), we have

$$\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} |J_1(t)|^{r/2}\right] = \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} \left|\int_0^t |\sigma(s)|^2 d\langle B\rangle(s)\right|^{r/2}\right]$$

$$\leq T^{r/2-1}\overline{\sigma}^r \int_0^{t_1} \widehat{\mathbb{E}}|\sigma(s)|^r ds \leq C \int_0^{t_1} \widehat{\mathbb{E}}|\sigma(s)|^r ds + C\delta^r. \tag{68}$$

By the elementary inequality

$$2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}, \quad \forall a, b, \varepsilon > 0,$$

and Lemma 2.4, we get

$$\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} |J_2(t)|^{r/2}\right] = 2^{r/2}\, \widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} \left|\int_0^t \langle e(s), \sigma(s)\rangle dB(s)\right|^{r/2}\right]$$

$$\leq 2^{r/2}c_1(r,\overline{\sigma})\widehat{\mathbb{E}}\left(\int_0^{t_1} |\langle e(s), \sigma(s)\rangle|^2\, ds\right)^{r/4}$$

$$\leq 2^{r/2}c_1(r,\overline{\sigma})\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} |e(s)|^{r/2} \left|\int_0^t |\sigma(s)|^2\, ds\right|^{r/4}\right]$$

$$\leq \beta_1\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} |e(s)|^r\right] + C\widehat{\mathbb{E}}\left(\int_0^{t_1} |\sigma(s)|^2\, ds\right)^{r/2}$$

$$\leq \beta_1\widehat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1} |e(s)|^r\right] + C\widehat{\mathbb{E}}\left[\int_0^{t_1} |\sigma(s)|^r\, ds\right] + C\delta^r \tag{69}$$

where β_1 is a constant to be determined and (67) has been used. Similarly, we have

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1}|J_3(t)|^{r/2}\right]\le C\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1}\left|\int_0^t\langle e(\underline{s}),f\left(Y(\underline{s})\right)-f\left(Z(\underline{s})\right)\rangle\,ds\right|^{r/2}\right]$$

$$\le C\hat{\mathbb{E}}\left[\int_0^{t_1}|\langle e(s),f(Y(\underline{s}))-f(Z(\underline{s}))\rangle|^{r/2}ds\right]$$

$$\le C\int_0^{t_1}\hat{\mathbb{E}}|e(s)|^r ds+C\int_0^{t_1}\hat{\mathbb{E}}|f(Y(\underline{s}))-f\left(Z(\underline{s})\right)|^r ds$$

$$\le C\int_0^{t_1}\hat{\mathbb{E}}|e(s)|^r ds+C\int_0^{t_1}\hat{\mathbb{E}}|Y(\underline{s})-Z(\underline{s})|^r ds$$

$$\le C\int_0^{t_1}\hat{\mathbb{E}}|e(s)|^r ds+C\delta^r. \tag{70}$$

Using (19) and definition of $J_4(t)$ gives that

$$J_4(t)=2\int_0^t\langle e(s),f(Y(s))-f(Y(\underline{s}))\rangle ds$$

$$=2\int_0^t\langle e(s),f'(Y(\underline{s}))\int_{\underline{s}}^s g\left(X(\underline{u})\right)dB(u)+\tilde{R}_1(f)\rangle ds$$

$$=2\int_0^t\langle e(s),f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\rangle ds+2\int_0^t\langle e(s),\tilde{R}_1(f)\rangle ds$$

$$\le\int_0^t|e(s)|^2ds+\int_0^t|\tilde{R}_1(f)|^2ds+\underbrace{2\int_0^t\langle X(s)-Y(s),f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\rangle ds}_{=:J_{41}(t)}. \tag{71}$$

Inserting

$$e(s)=e(\underline{s})+\int_{\underline{s}}^s(f(X(u))-f(Z(\underline{u})))du+\int_{\underline{s}}^s\sigma(u)dB(u)+\int_{\underline{s}}^s(h(X(u))-h(Z(\underline{u})))du \tag{72}$$

into $J_{41}(t)$, we have the following decomposition

$$J_{41}=\sum_{i=1}^4 J_{41i}(t),$$

where

$$J_{411}(t):=2\int_0^t\left\langle e(s),f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\Delta B(s)\right\rangle ds \tag{73}$$

$$J_{412}(t):=2\int_0^t\left\langle\int_{\underline{s}}^s\sigma(u)dB(u),f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\Delta B(s)\right\rangle ds \tag{74}$$

$$J_{413}(t):=2\int_0^t\left\langle\int_{\underline{s}}^s(f(X(u))-f(Z(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds \tag{75}$$

$$J_{414}(t):=2\int_0^t\left\langle\int_{\underline{s}}^s(h(X(u))-h(Z(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds. \tag{76}$$

Due to the fact that $\Delta B(s)=B(s)-B(\underline{s})$ is independent from $Y(\underline{s})$ and $Z(\underline{s})$, we get that

$$\hat{\mathbb{E}}\left[\left|f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\Delta B(s)\right|^r\right] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[|f'(x_1)g(x_2)|^r|B(s)-B(\underline{s})|^r]_{x_1=Y(\underline{s}),x_2=Z(\underline{s})}\right]$$

$$\leq C\delta^{r/2}\hat{\mathbb{E}}\left[\left|f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\right|^r\right] \leq \delta^{r/2}, \tag{77}$$

where Lemma 4.3 has been used. With the aid of the Hölder inequality and Lemma 2.4, we get that

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|J_{412}(t)|^{r/2}\right] \leq C\hat{\mathbb{E}}\left[\int_0^{t_1}\left|\langle\int_{\underline{s}}^s \sigma(u)\,dB(u), f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\Delta B(s)\rangle\right|^{r/2}\right]ds$$

$$\leq C\int_0^{t_1}\left(\hat{\mathbb{E}}\left|\int_{\underline{s}}^s \sigma(u)\,dB(u)\right|^r \hat{\mathbb{E}}\left|f'\left(Y(\underline{s})\right)g\left(Z(\underline{s})\right)\Delta B(s)\right|^r\right)^{1/2}ds$$

$$\leq C\int_0^{t_1}\left(\delta^{r-1}\hat{\mathbb{E}}\int_{\underline{s}}^s|\sigma(u)|^r\,du\right)^{1/2}ds. \tag{78}$$

Inserting (67) into (78) gives

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq t_1}|J_{412}(t)|^{r/2}\right] \leq C\int_0^{t_1}\left(\delta^{r-1}\left(\int_{\underline{s}}^s\hat{\mathbb{E}}|\sigma(u)|^r\,du + C\delta^r\right)\right)^{1/2}ds$$

$$\leq C\int_0^{t_1}\left(\delta^r\left(\hat{\mathbb{E}}|e(\underline{s})|^r + C\delta^{2r}\right)\right)^{1/2}ds$$

$$\leq C\int_0^{t_1}\left(\left(\delta^r\hat{\mathbb{E}}|e(\underline{s})|^r\right)^{1/2} + C\delta^r\right)ds \leq C\int_0^{t_1}\hat{\mathbb{E}}|e(\underline{s})|^r ds + C\delta^r. \tag{79}$$

For any $t\in[0,T]$, define $n(t) := max\{n: t_n < t\}$ and

$$\overline{s} := \left(\begin{array}{ccc} t_{k+1} & : & t_k < s \leq t_{k+1}, \\ t & : & t_{n(t)} < s \leq t. \end{array}\right.$$

According to integration by parts formula, we have

$$\int_{t_k}^{t_{k+1}}(B(u)-B(t_k))du = t_{k+1}[B(t_{k+1})-B(t_k)] - \int_{t_k}^{t_{k+1}}u\,dB(u) = \int_{t_k}^{t_{k+1}}(\overline{u}-u)\,dB(u). \tag{80}$$

Moreover, we have

$$\frac{J_{411}(t)}{2} = \int_0^t\langle e(\underline{s}), f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\rangle ds = \int_0^t\int_{\underline{s}}^s\langle e(\underline{s}), f'(Y(\underline{s}))g(Z(\underline{s}))\rangle dB(u)ds$$

$$= \sum_{k=0}^{n(t)-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^s\langle e(t_k), f'(y_k)g(z_k)\rangle dB(u)ds + \int_{t_{n(t)}}^t\int_{t_{n(t)}}^s\langle e(t_{n(t)}), f'(y_{n(t)})g(z_{n(t)})\rangle dB(u)ds. \tag{81}$$

Inserting (80) into (81), we have

$$\frac{J_{411}(t)}{2} = \sum_{k=0}^{n(t)-1}\int_{t_k}^{t_{k+1}}(t_{k+1}-u)\langle e(t_k), f'(y_k)g(z_k)\rangle dB(u) + \int_{t_{n(t)}}^t(t-u)\langle e(t_{n(t)}), f'(y_{n(t)})g(z_{n(t)})\rangle dB(u)$$

$$= \int_0^t(\overline{u}-u)\langle e(\underline{u}), f'(Y(\underline{u}))g(Z(\underline{u}))\rangle dB(u). \tag{82}$$

Applying the Hölder inequality and Lemma 2.4, we have

$$\widehat{\mathbb{E}}\left[\sup_{0\le t\le t_1}|J_{411}(t)|^{r/2}\right]\le C\widehat{\mathbb{E}}\int_0^{t_1}\left|(\bar{u}-u)\left\langle e(\underline{u}),f'\left(Y(\underline{u})\right)g\left(Z(\underline{u})\right)\right\rangle\right|^{r/2}du$$

$$\le C\delta^{r/2}\int_0^{t_1}\widehat{\mathbb{E}}\left|\left\langle e(\underline{u}),f'\left(Y(\underline{u})\right)g\left(Z(\underline{u})\right)\right\rangle\right|^{r/2}du$$

$$\le C\delta^{r/2}\int_0^{t_1}\left(\widehat{\mathbb{E}}|e(\underline{u})|^r\widehat{\mathbb{E}}|f'\left(Y(\underline{u})\right)g\left(Z(\underline{u})\right)|^r\right)^{1/2}du$$

$$\le C\int_0^{t_1}\widehat{\mathbb{E}}|e(\underline{s})|^r ds+C\delta^r. \tag{83}$$

Moreover, we have the following decomposition

$$J_{413}(t)/2=\int_0^t\left\langle\int_{\underline{s}}^s(f(Y(\underline{u}))-f(Z(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds$$

$$+\int_0^t\left\langle\int_{\underline{s}}^s(f(X(u))-f(X(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds$$

$$+\int_0^t\left\langle\int_{\underline{s}}^s(f(X(\underline{u})-f(Y(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds$$

$$=:\widetilde{\Pi}_1(t)+\widetilde{\Pi}_2(t)+\widetilde{\Pi}_3(t). \tag{84}$$

By Lemma 2.4, we have

$$\widehat{\mathbb{E}}\left[\sup_{0\le t\le t_1}\left|\widetilde{\Pi}_2(t)\right|^{r/2}\right]=\widehat{\mathbb{E}}\left[\sup_{0\le t\le t_1}\left|\int_0^t\left\langle\int_{\underline{s}}^s(f(Y(\underline{u}))-f(Z(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle ds\right|^{r/2}\right]$$

$$\le C\widehat{\mathbb{E}}\left[\int_0^{t_1}\left|\left\langle\int_{\underline{s}}^s(f(Y(\underline{u}))-f(Z(\underline{u})))du,f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)\right\rangle\right|^{r/2}ds\right]$$

$$\le C\int_0^{t_1}\left(\widehat{\mathbb{E}}\left|\int_{\underline{s}}^s(f(Y(\underline{u}))-f(Z(\underline{u})))du\right|^r\widehat{\mathbb{E}}|f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s)|^r\right)^{1/2}ds$$

$$\le C\int_0^{t_1}\left(\delta^{r-1}\int_{\underline{s}}^s\widehat{\mathbb{E}}\left[|f(Y(\underline{u}))-f(Z(\underline{u}))|^r du\right]\delta^{r/2}\right)^{1/2}ds$$

$$\le C\int_0^{t_1}\left(\delta^{3r/2}\widehat{\mathbb{E}}\left|f\left(Y(\underline{s})\right)-f\left(Z(\underline{s})\right)\right|^r\right)^{1/2}ds\le C\delta^r \tag{85}$$

and

$$\widetilde{\Pi}_3(t)=\int_0^t\int_{\underline{s}}^s\langle f(X(\underline{u}))-f(Y(\underline{u})),f'(Y(\underline{s}))g(Z(\underline{s}))\rangle dB(u)ds$$

$$=\sum_{k=0}^{n(t)-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^s\langle f(X(t_k))-f(y_k),f'(y_k)g(z_k)\rangle dB(u)ds$$

$$+\int_{t_{n(t)}}^t\int_{t_{n(t)}}^s\langle f(X(\underline{t}))-f(Y_{n(t)}),f'(y_{n(t)})g(z_{n(t)})\rangle dB(u)ds$$

$$= \sum_{k=0}^{n(t)-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - s) \langle f(X(t_k)) - f(Y_k), f'(y_k) g(z_k) \rangle dB(s)$$

$$+ \int_{t_{n(t)}}^{t} (t - s) \left\langle f\left(X(\underline{s})\right) - f\left(Y(\underline{s})\right), f'(y_{n(t)}) g(z_{n(t)}) \right\rangle dB(s)$$

$$= \int_0^t (\bar{s} - s) \left\langle f\left(X(\underline{s})\right) - f\left(Y(\underline{s})\right), f'\left(Y(\underline{s})\right) g\left(Z(\underline{s})\right) \right\rangle dB(s). \tag{86}$$

In the same fashion as (83) was obtained, we also have

$$\hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} \left|\tilde{\Pi}_3(t)\right|^{r/2}\right] \le C\hat{\mathbb{E}} \int_0^{t_1} \left|(\bar{s} - s) \langle f\left(X(\underline{s})\right) - f'\left(Y(\underline{s})\right) g\left(Z(\underline{s})\right) \rangle\right|^{r/2} ds$$

$$\le C\delta^{r/2} \int_0^{t_1} \hat{\mathbb{E}}\left[\left|f\left(X(\underline{s})\right) - f\left(Y(\underline{s})\right)\right|^{r/2} \left|f'\left(Y(\underline{s})\right) g\left(Z(\underline{s})\right)\right|^{r/2}\right] ds$$

$$\le C\delta^{r/2} \int_0^{t_1} \hat{\mathbb{E}}\, |e(s)|^{r/2}\, ds \le C \int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |e(s)|^r\right] ds + C\delta^r \tag{87}$$

and

$$\hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} \left|\tilde{\Pi}_1(t)\right|^{r/2}\right] = \hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} \left|\int_0^t \left\langle \int_{\underline{s}}^s (f(Y(\underline{u})) - f(Z(\underline{u})))du, f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s) \right\rangle ds\right|^{r/2}\right]$$

$$\le C\hat{\mathbb{E}}\left[\int_0^{t_1} \left|\left\langle \int_{\underline{s}}^s (f(Y(\underline{u})) - f(Z(\underline{u})))du, f'(Y(\underline{s}))g(Z(\underline{s}))\Delta B(s) \right\rangle\right|^{r/2} ds\right]$$

$$\le C \int_0^{t_1} \left(\hat{\mathbb{E}}\left|(s - \underline{s}) f\left(Y(\underline{s})\right) - f\left(Z(\underline{s})\right)\right|^p \hat{\mathbb{E}}\left|f'\left(Y(\underline{s})\right) g\left(Z(\underline{s})\right) \Delta B(s)\right|^{r/2}\right)^{1/2} ds$$

$$\le \int_0^{t_1} \left(\Delta^{3r/2} \hat{\mathbb{E}}\left|Y(\underline{s}) - Z(\underline{s})\right|^p\right)^{1/2} ds \le C\delta^{5r/4} \le C\delta^r.' \tag{88}$$

Inserting (88), (87) and (85) into (84) gives

$$\hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |J_{413}(t)|^{r/2}\right] \le C \int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |e(s)|^r\right] ds + C\delta^r. \tag{89}$$

Similarly, we have

$$\hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |J_{414}(t)|^{r/2}\right] \le C \int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |e(s)|^r\right] ds + C\delta^r. \tag{90}$$

By (83), (85), (89) and (90), we obtain

$$\hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |J_{14}(t)|^{r/2}\right] \le C \int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0 \le t \le t_1} |e(s)|^r\right] ds + C\delta^r. \tag{91}$$

With the help of Lemma 4.5, we get from (7) that

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |J_4(t)|^{r/2}\right] \le C\int_0^{t_1} \hat{\mathbb{E}}[|e(s)|^r]ds + C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(s)|^r\right]ds + C\delta^r. \tag{91}$$

$$\le C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(s)|^r\right]ds + C\delta^r \tag{92}$$

In a similar fashion as (92) is obtained, we also can show that

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |J_5(t)|^{r/2}\right] \le C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(s)|^r\right]ds + C\delta^r. \tag{93}$$

and

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |J_6(t)|^{r/2}\right] \le C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(s)|^r\right]ds + C\delta^r. \tag{94}$$

By (68)- (70), (92)-(94), we conclude from (61) that

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(t)|^r\right] \le 8^{r/2-1}\sum_{i=1}^{6} \hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |J_i(t)|^{r/2}\right] + \theta^r |f(X_0)|^r \Delta^r$$

$$+\left(2\kappa_1\left(1+\overline{\sigma}^2\right)\right)^r T^{r-1}\int_0^{t_1} \hat{\mathbb{E}}|e(s)|^r ds)$$

$$\le 8^{r/2-1}\beta_1 \hat{\mathbb{E}}\left[\sup_{0\le u\le s} \hat{\mathbb{E}}|e(u)|^r\right] + C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le u\le s} |e(u)|^r\right]ds + C\delta^r. \tag{95}$$

If we choose an appropriate constant β_1 such that $8^{r/2-1}\beta_1 < 1$, then we conclude from (95) that we have

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le t_1} |e(t)|^r\right] \le C\int_0^{t_1} \hat{\mathbb{E}}\left[\sup_{0\le u\le s} |e(u)|^r\right]ds + C\delta^r.$$

By the Gronwall inequality, we have

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T} |X(t) - Y(t)|^r\right] \le C\delta^r. \tag{96}$$

Combining this with (22) of Lemma 4.2 gives

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T} |X(t) - Z(t)|^r\right] \le C\delta^r. \tag{97}$$

Thus, we complete the proof.

Remark 4.7 Compared with the convergence order of one-half for G-SDEs established in [Deng *et al*. (2019)] and [Yang and Li(2019)], our numerical scheme achieves a higher convergence order of one. When the term $L^1 g$ is omitted, the θ-Milstein scheme reduces to the θ-EM scheme for G-SDEs. Furthermore, by setting $\theta = 0$, $\overline{\sigma} = 1$, and $\underline{\sigma} = 1$, the scheme simplifies to the classical Milstein scheme for standard SDEs. The proposed θ-Milstein scheme for G-SDEs thus offers considerable flexibility, particularly in contexts where SDEs are driven by Brownian motion with distribution uncertainty.

5. Numerical Experiments

In this section, we will test the following schemes: θ-Milstein scheme (G-TMIL) (14);

• Euler-Maruyama scheme (G-EM)

$$Y_{k+1} = Y_k + f(Y_k)\delta + g(Y_k)\Delta B_k + h(Y_k)\Delta\langle B\rangle_k, \quad Y_0 = X_0;$$

• Backward Euler-Maruyama scheme (G-BEM)

$$Y_{k+1} = Y_k + f(Y_{k+1})\Delta + g(Y_k)\Delta B_k + h(Y_k)\Delta\langle B\rangle_k, \quad Y_0 = X_0;$$

• Milstein scheme (G-MIL)

$$Y_{k+1} = Y_k + f(Y_k)\Delta + g(Y_k)\Delta B_k + \frac{1}{2}L^1 g(Y_k)(|\Delta B_k|^2 - \Delta\langle B\rangle_k), \quad Y_0 = X_0.$$

The aim of the tests is to compare the performance of the schemes: their convergence orders, quantitative errors and computational costs. The experiments were performed on a Windows desktop computer with an Intel Core CPU i5-9400.

We will apply the above methods to a population growth model in the G-framework, i.e., the following linear G-SDE

$$dX(t) = aX(t)dt + bX(t)dB(t) + cX(t)d\langle B\rangle(t), \quad t > 0,$$

$$X(0) = X_0, \tag{98}$$

where a, b, and c are real constants. The explicit solution to this G-SDE is

$$X(t) = X_0 e^{at + bB(t) + (c - 0.5b^2)\langle B\rangle(t)}. \tag{99}$$

Moreover, if $b^2 + 2c < 0$ and $a + G(b^2 + 2c) < 0$, then the trivial solution of G-SDE (98) is exponentially stable in mean-square, see [16, Example 5.4].

Moreover, we use the method from [34] to approximate G-expectation. Let $B(t){:}\, N(0, [\underline{\sigma}^2, \overline{\sigma}^2]t)$. Denote by M the number of random sample and J the number of partition. Consider an equidistant partition $\underline{\sigma} = \sigma_1 < \cdots \sigma_j < \cdots < \sigma_J = \overline{\sigma}$. For $i = 1,2,\cdots M$ and $j = 1,2,\cdots J$, define z_k^{ji} by

$$z_{k+1}^{ji} = z_k^{ji} + \theta f(z_{k+1}^{ji})\delta + (1-\theta)f(z_k^{ji})\delta + g(z_k^{ji})\xi^{ji}(k) + h(z_k^{ji})\sigma_j^2\delta$$

$$+\tfrac{1}{2}L^1(g(z_k^{ji}))((\xi^{ji}(k))^2 - \sigma_j^2\delta), \; k = 0,1,2,\cdots \tag{100}$$

with $z_0 = X_0$, where $\xi^{ji}(k){:}\, N(0, \sigma_j^2\delta)$. Then

$$\widehat{\mathbb{E}}\, z_k \approx \max_{1\le j\le J} \frac{1}{M}\sum_{i=1}^{M} z_k^{ji}, \; k = 0,1,2, \cdots \tag{101}$$

the right term of (101) is termed the maximal sample average of z_k.

5.1. Errors, Convergence Orders and Computational Costs

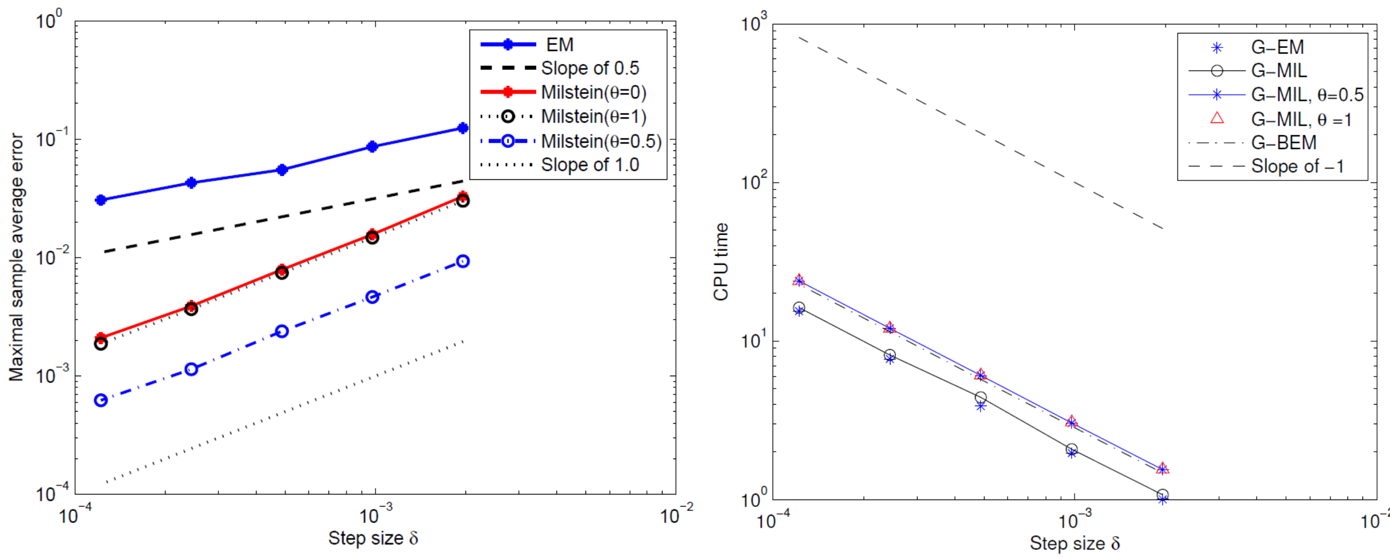
This subsection compares the strong convergence orders, maximum sample average of absolute errors, and computational costs among the methods described above.

Table 1: Maximal sample average errors e_δ^* and convergence orders of approximations for Example 5.1.

δ	2^{-9}	2^{-10}	2^{-11}	2^{-12}	2^{-13}	Order
G-EM	1.2447e-01	8.6450e-02	5.5201e-02	4.2845e-02	3.0626e-02	0.5064
G-BEM	1.1940e-01	8.5334e-02	5.4426e-02	4.2317e-02	3.0639e-02	0.4937
G-MIL	3.2937e-02	1.5678e-02	7.9239e-03	3.9469e-03	2.0871e-03	0.9948
G-MIL ($\theta = 1$)	3.0128e-02	1.4746e-02	7.4187e-03	3.6681e-03	1.8725e-03	1.0023
G-MIL ($\theta = 0.5$)	9.3195e-03	4.6409e-03	2.3877e-03	1.1399e-03	6.2310e-04	0.9831

Table 2: CPU times for the selected schemes for Example 5.1.

δ	2^{-9}	2^{-10}	2^{-11}	2^{-12}	2^{-13}	α	γ
G-EM	1.006s	1.968s	3.886s	7.726s	15.431s	0.0020	-0.9937
G-BEM	1.471s	2.905s	5.731s	11.458s	22.787s	0.0030	-0.9926
G-MIL	1.073s	2.079s	4.144s	8.159s	16.260s	0.0026	-0.9713
G-MIL ($\theta = 1$)	1.555s	3.077s	6.083s	12.018s	23.950s	0.0032	-0.9898
G-MIL ($\theta = 0.5$)	1.547s	3.041s	6.070s	12.019s	23.942s	0.0032	-0.9915



(**a**) Errors vs. δ (**b**) CPU times vs. δ

Figure 1: Simulations of Example 5.1.

Example 5.1 Consider scalar G-SDE (98) with $a = 2$, $b = 1$, $c = 0$, $X_0 = 1$, $\underline{\sigma} = 0.6$ and $\overline{\sigma} = 0.8$. In our numerical tests, we will focus on the error at the endpoint T, so we let

$$e_\delta^* := \hat{\mathbb{E}}|X(T) - Y(T)|,$$

where $\hat{\mathbb{E}}$ is approximated by the maximal sample average, and X and Y represent the true and numerical solutions, respectively. We set $T = 1$ and $M = 1000$. The true solution X is computed by (99).

Error Analysis: Table **1** and Fig. (**1**) present the maximum sample average errors and the experimentally observed convergence orders for the corresponding methods. The observed convergence orders for the G-EM and

G-BEM schemes are close to the theoretical value of 0.5, while those for the G-MIL and G-TMIL schemes are close to 1.0. For a fixed step size δ, the most accurate scheme is G-MIL, and the less accurate is G-EM.

Computational Cost Analysis: The computational costs, measured in CPU seconds, are presented in Table **2** for the selected schemes. We also provide a diagram of computing time versus step size in Fig. (**1**). We observe that the G-EM scheme is the fastest, while G-MIL with $\theta = 1$ is the slowest. Compared to the EM-type scheme, the Milstein-type scheme incorporates an additional term: $\frac{1}{2}L^1 g(Y_k)(|\Delta B_k|^2 - \Delta\langle B\rangle_k)$, which increases the computational cost but improves the convergence order. Assuming that the CPU runtime obeys a power law relation

$$y = \alpha\delta^{\gamma}, \quad \forall\delta \in (0,1],$$

the corresponding nonlinear fitting results for α and γ for each scheme are presented in the last two columns of Table **2**. We observe that the values of γ for all schemes are close to -1, indicating that the computational time of these schemes is approximately inversely proportional to the step size δ.

5.2. Stability

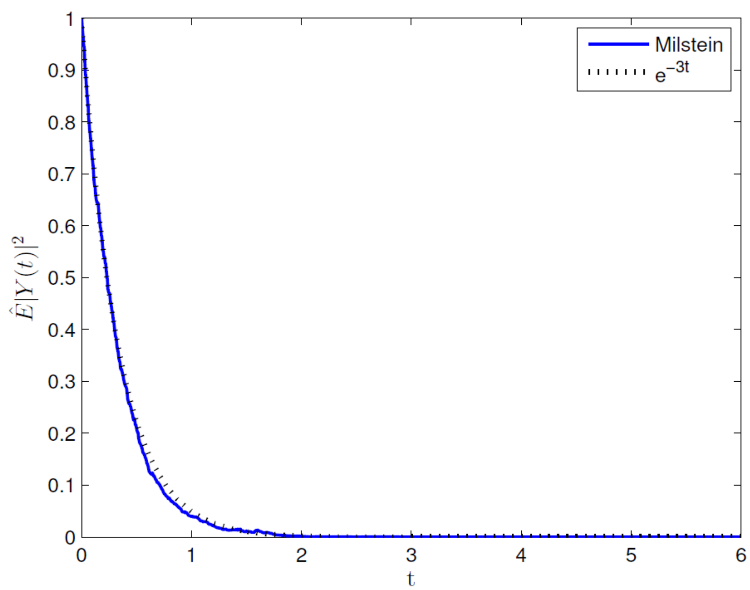


Figure 2: Simulation of $\hat{\mathbb{E}}|Y(t)|^2$ by Milstein scheme for Example 5.2.

This subsection tests the numerical stability of the θ-Milstein scheme of G-SDE (98).

Example 5.2 Consider G-SDE (98) with the following parameters

$$a = 2, \quad b = 1, \quad c = -5, \quad X_0 = 1, \quad \underline{\sigma}^2 = 0.8, \quad \overline{\sigma}^2 = 1.0. \tag{102}$$

Stability Analysis: According to [Li *et al.* (2016), Theorem 5.5], the solution of (102) is exponentially stable in mean-square with the Lyapunov exponent equal to -3, i.e.,

$$\hat{\mathbb{E}}|X(t)|^2 \leq |X_0|^2 e^{-3t}, \ \forall t \geq 0,$$

since

$$2a + 2G(b^2 + 2c) \leq 2a + b^2\overline{\sigma}^2 + G(4c) = 2 \times 2 + 1^2 + 2 \times 0.8 \times (-5) = -3 < 0.$$

Note that the corresponding unperturbed system

$$dX(t) = 2X(t)dt, \quad t > 0,$$

$$X(0) = 1, \tag{103}$$

is unstable. However, introducing the stochastic perturbation $X(t)dB(t) - 5X(t)d\langle B\rangle(t)$ to (103) results in system (102), which is exponentially stable in mean-square. We examine the stability of the Milstein scheme using a step size $\delta = 0.005$. With $\theta = 0.5$, the mean-square stability of the θ-Milstein method is illustrated in Fig. (**2**), demonstrating that the true solution is exponentially stable in mean-square.

6. Conclusion

This paper mainly investigates the convergence of the θ-Milstein scheme for G-SDEs. We first construct a Milstein-type scheme for G-SDEs according to the G-Itô formula and then establish the moment bound of the θ-Milstein solutions. Moreover, we prove the scheme converges strongly to the true solution with order one in the $L^r(\Omega; R)$ sense by the theory of G-expectation. Numerical experiments, including simulation of G-expectation, confirm the effectiveness of our theoretical results.

In future work, we will investigate numerical methods for G-SDEs with non-globally Lipschitz continuous coefficients.

Conflict of Interest

The authors declare no conflict of interest.

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