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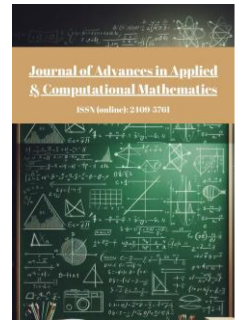

Fixed Point Theory in Graph-Structured Controlled Partial Metric Spaces with Applications to Integral Equations

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ABSTRACT

In this paper, we introduce and investigate a new class of mappings called generalised graph \phi-contractions within the setting of Generalised Hausdorff Controlled Partial Metric (GHCPM) spaces. This framework integrates the structure of a graph with a controlled partial metric, providing a natural generalization of classical fixed point theories. Our study extends previous results by incorporating mappings defined on collections of non-empty closed and bounded subsets of a GHCPM space, and introducing contractive conditions governed by an upper semi-continuous and non-monotonic function. By leveraging the graph structure on GHCPM, we define a generalised graph contraction as a mapping that respects the connectivity induced by the graph while satisfying a contractive inequality involving the Hausdorff controlled partial metric. We establish novel fixed point theorems for such contractions, which unify and extend several existing results in the literature. To illustrate the applicability and generality of our results, we demonstrate the existence of solutions for nonlinear integral equations of Fredholm type. Concrete examples demonstrating the existence of fixed points under the proposed framework are also provided. These results open new directions in the study of fixed point theory in generalized metric spaces with additional structure.

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1. Introduction

Fixed point theory is a fundamental area of mathematical analysis that studies points invariant under certain mappings or functions. Its significance lies in providing powerful tools for solving equations, modeling dynamic systems, and analyzing iterative processes across various disciplines, including economics, engineering, computer science, and physics [1, 2]. By guaranteeing the existence (and often uniqueness) of fixed points under specific conditions, fixed point theorems enable mathematicians and scientists to address complex problems such as differential and integral equations, optimization tasks, and stability analysis [3, 4]. Over time, the theory has evolved to encompass a broad spectrum of generalizations and applications, continually expanding its relevance and utility in both pure and applied mathematics [5]. See also [6-8] for more on the applications of fixed point theory.

While classical fixed point results are traditionally framed in standard metric spaces, recent research has extended these concepts to more general settings such as *controlled metric spaces*, *partial metric spaces*, and *graph-structured spaces*, to better model nonlinear and complex analytical systems. Each of these generalizations captures structural features that classical metrics cannot such as partial self-distance in partial metric spaces, variable scaling in controlled metrics, and relational dependencies in graph-based spaces.

Jachymski [9] introduced a novel graph-theoretic perspective on fixed point theory by replacing order relations with graph structures on metric spaces. This formulation provided a flexible framework for capturing relational structures beyond ordered or lattice frameworks. Building on this idea, Nazir *et al*. [10] investigated fixed points of set-valued contractions in partial metric spaces endowed with digraphs. Using the Pompeiu–Hausdorff partial metric, they established several fixed point results for multivalued contractions (including rational and generalized graph contractions). For further developments on graph-based fixed point theory, see [11-25].

In a parallel direction, Mlaiki *et al*. [26] introduced the concept of *controlled metric spaces*, which generalize b-metric spaces and have proven to be a powerful tool in studying the solvability of integral and differential equations. For more details on b-metric and controlled metric spaces, see [27-35]. Later, Alamgir *et al*. [36] constructed a Pompeiu–Hausdorff metric on collections of closed subsets of controlled metric spaces and established fixed point results under almost F-contractive conditions. Recently, Pamba and Tembo [37] proposed Suzuki-type contractions within the framework of controlled metric spaces, further broadening the applicability of fixed point principles. For additional recent advancements, see [9, 10, 38-46].

In applied mathematics, symbolic calculation methods, such as Lie symmetry analysis, He's semi-inverse variational method, and homotopy techniques, have traditionally been used to obtain exact solutions or conservation laws for nonlinear partial differential equations, including multidimensional systems like the Jimbo-Miwa equation and Burgers-type models [47-49]. While powerful, these symbolic methods often involve laborious and tedious calculations. Non-symbolic approaches based on fixed point theory, such as the framework developed in this paper, offer a simpler, more direct alternative. By focusing on existence and convergence properties rather than explicit formulas, these methods can circumvent complex algebraic manipulations and provide rigorous guarantees for broad classes of nonlinear integral and differential equations. This perspective motivates the use of generalized metric fixed point techniques as a practical and efficient tool for studying nonlinear problems.

Building on this motivation, the present study aims to extend the analytical scope of fixed point techniques by incorporating richer structural settings. Despite numerous advances, the existing literature still treats *partial metric spaces*, *controlled metric spaces*, and *graph-based spaces* mostly in isolation. There remains a lack of a unified analytical framework that can jointly accommodate (i) the self-distance feature of partial metrics, (ii) the flexible scaling behavior of controlled metrics, and (iii) the relational dependencies encoded by graph structures. Moreover, most existing contraction conditions are governed by monotonic or continuous control functions, which can be overly restrictive for modeling systems exhibiting non-monotonic dynamics or irregular responses.

Motivated by these observations, this paper introduces a new hybrid structure, namely the *controlled partial metric space endowed with a graph*, and defines the corresponding *generalised Pompeiu–Hausdorff controlled partial metric (GHCPM)* on families of nonempty closed and bounded subsets. Within this setting, we establish a new class of set-valued contractions, termed *generalised graph ϕ-contractions*, governed by a control function ϕ that is *upper* *semicontinuous* and *non-monotonic*. This formulation extends and unifies several existing contraction concepts such as those of Nazir *et al*. [10] (graph-based partial metrics) and Alamgir *et al*. [36] (controlled metric contractions) into a broader and more flexible framework. The introduction of non-monotonic control functions in this hybrid setting constitutes a novel analytical direction not previously explored in the literature.

The principal contributions of this work are summarized as follows:

- We develop the foundational theory of GHCPM spaces by defining and analyzing the generalised Pompeiu–Hausdorff controlled partial metric.
- We establish new fixed point theorems for multivalued operators satisfying graph-based ϕ -contractive conditions in this generalized setting.
- We provide illustrative examples and an application to a class of nonlinear Fredholm-type integral equations, thereby demonstrating the practical relevance and adaptability of our results.

Section 2 presents the preliminary concepts related to controlled partial metrics, graph structures, and function spaces. In Section 3, we introduce the notion of generalised graph ϕ -contractions and derive our main fixed point results. Section 4 discusses applications to nonlinear integral equations and provides illustrative examples. Finally, Section 5 concludes the paper with a summary and future research directions.

2. Preliminaries

2.1. Partial Metric Space

Definition 1

Let X be a non-empty set. A mapping $p: X \times X \to [0, +\infty)$ is called a partial metric and (X, p) is called a partial metric space if the following are satisfied for all $x, y, z \in X$.

1. $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
2. $p(x,y) = p(y,x)$;
3. $p(x,x) \leq p(x,y)$;
4. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Example 2

A trivial example of a partial metric space is the pair (R^+, p), where $p: R^+ \times R^+ \to R^+$ is defined as $p(x,y) = max\{x,y\}$.

We follow Abbas and Nazir in [39] for more examples. See also [43, 44, 50-52] .

In this paper, let $CB^p(X)$ denote a collection of non-empty closed and bounded subsets, and for $A, B \in CB^p(X)$ and $x \in X$, define $\delta_p(x, A) = inf\{p(x,a): a \in A\}$, $\delta_p(A, B) = sup\{p(a, B): a \in A\}$ and $\delta_p(B, A) = sup\{p(b, A): b \in B\}$.

Proposition 3 [40]

Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:

1. $\delta_p(A, A) = sup\{p(a,a): a \in A\}$;
2. $\delta_p(A, A) \leq \delta_p(A, B)$;
3. $\delta_p(A, B) = 0$ implies that $A \subseteq B$;
4. $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - inf_{c \epsilon C}\, p(c, c)$.

Definition 4

Let (X, p) be a partial metric space and $\{\eta_n\}_{n\geq 1}$ be a sequence in X. Let τ_p be the topology that the partial metric generates.

1. We say that $\{\eta_n\}_{n\geq 1}$ converges to an element or a point $\eta \in X$ $w.r.t.$ the topology τ_p, if and only if $\lim_{n\to\infty} p(\eta, \eta_n) = p(\eta, \eta)$.
2. $\{\eta_n\} \subset X$ is a Cauchy sequence if $\lim_{n,m\to\infty} p(\eta_n, \eta_m)$ exists and if it is finite.
3. The set X is a complete set if every Cauchy sequence $\{\eta_n\} \subset X$ converges to a point $\eta \in X$ such that $\lim_{n,m\to\infty} p(\eta_n, \eta_m) = p(\eta, \eta)$. Note that this convergence is w.r.t. the topology τ_p.

Definition 5

Let (X, p) be a partial metric space. Then for $\Xi_1, \Xi_2 \in CB^p(X)$, we define the Hausdorff partial metric(H_p) induced by p as $H_p(\Xi_1, \Xi_2) = max\{\delta_p(\Xi_1, \Xi_2), \delta_p(\Xi_2, \Xi_1)\}$ where, $\delta_p(\Xi_1, \Xi_2) = sup\{p(\eta_1, \Xi_2): \eta_1 \in \Xi_1\}$ and $\delta_p(\eta_1, \Xi_2) = inf\{p(\eta_1, \eta_2): \eta_2 \in \Xi_2\}$.

Remark 6

Any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 2.6 in [40]).

Definition 7

A function $\phi: R^+ \to R^+$ is called an admissible ϕ -function if it satisfies the following conditions:

1. ϕ is upper semicontinuous;
2. ϕ is not necessarily monotonic;
3. $\phi(t) < t$ for all $t > 0$.

Definition 8

The set-valued mapping $T: CB^p(X) \to CB^p(X)$ is called a generalised graph ϕ -contraction whenever the following properties are satisfied:

1. e_T is an edge that links $T(\Xi_1)$ to $T(\Xi_2)$ whenever e is the preceding edge that links Ξ_1 and Ξ_2.
2. W_T is a path from $T(\Xi_1)$ to $T(\Xi_2)$ whenever W is a path from Ξ_1 to Ξ_2.
3. Take a function ϕ defined from R^+ into itself with the properties that: ϕ is upper semicontinuous, non - decreasing satisfying the inequality, $\phi(t) < t$ for every $t > 0$ such that if e is an edge from Ξ_1 to Ξ_2 we infer that

$H_p(T(\Xi_1), T(\Xi_2)) \leq \phi(M_p(\Xi_1, \Xi_2))$, where

$$M_p(\Xi_1, \Xi_2) = max\{H_p(\Xi_1, \Xi_2), H_p(\Xi_1, T(\Xi_1)), H_p(\Xi_2, T(\Xi_2)), \frac{H_p(\Xi_1, T(\Xi_2)) + H_p(\Xi_2, T(\Xi_1))}{2}\}$$

Remark 9

If for any converging sequence $\{x_n\}_{n\geq 1} \subset CB^p(X)$, that is $\lim_{n\to\infty} x_n = x$, we have an edge between the two consecutive terms x_n and x_{n+1}, then we can extract a subsequence $\{x_{n_k}\}_{k\in N}$ from $\{x_n\}$, from which one deduces that there also exists an edge that connects x_{n_k} and the limiting set X to each other. Here, we say that G is a graph having the property $(P^{\mathring{a}})$.

Theorem 10 below shows analogous fixed point results for set-valued self-maps on $CB^p(X)$ based on partial metric p with some conditions on graph contraction as obtained in [10]:

Theorem 10

Let (X, p) be a complete partial metric space equipped with a digraph G having both vertex and edge sets satisfying $V(G) = X$ and $\Delta \subseteq E(G)$, respectively. We assume that the map $T: CB^p(X) \to CB^p(X)$ is a generalised graph ϕ -contraction and also $F(T) \neq \emptyset$. Then, it holds that

1. the partial Hausdorff weight associated with $U, V \in F(T)$ vanishes whenever the non-empty set $F(T)$ is complete.
2. If $F(T) \neq \emptyset$, then $X_T \neq \emptyset$. Furthermore, for any $U \in F(T)$ implies that $H_p(U, U) = 0$.
3. Assume that $\tilde{G}$ has the property P^* and that $X_T \neq \emptyset$. Then the map T has a fixed point.
4. $F(T)$, the set of fixed points is a complete set if and only if $F(T)$ is reduced to a unit set(singleton).

We follow [10] for the proof and for some examples.

2.2. Controlled Metric Spaces

Definition 11 (Controlled metric)[26]

Let X be a non-empty set and $\alpha: X \times X \to [0, \infty)$. Then a mapping $d_\alpha: X \times X \to [0, \infty)$ is called a controlled metric, if for all $x, y, z \in X$, it satisfies the following axioms;

1. $d_\alpha(x, y) = 0$ if and only if $x = y$;
2. $d_\alpha(x, y) = d_\alpha(y, x)$
3. $d_\alpha(x, z) \leq \alpha(x, y) d_\alpha(x, y) + \alpha(y, z) d_\alpha(y, z)$.

The pair (X, d_α) is called a controlled metric space.

We define convergence, Cauchy and completeness in controlled metric spaces as follows:

Definition 12 [45]

Let (X, d_α) be a controlled metric space and $\{x_n\}_{n \leq 0}$ be a sequence in X. Then

1. the sequence $\{x_n\}$ converges to some x in X if for every $\varepsilon > 0$, there exists $N \in N$ such that $d_\alpha(x_n, x) < \varepsilon$ for all $n \geq N$. In this case we write $\lim_{n \to \infty} x_n = x$,
2. the sequence x_n is Cauchy if for every $\varepsilon > 0$, there exists $N \in N$ such that $d_\alpha(x_m, x_n) < \varepsilon$ for all $m, n \geq N$,
3. the controlled metric space (X, d_α) is called complete if every Cauchy sequence is convergent.

Definition 13 [45](Controlled partial metric type space)

Let X be a nonempty set and $\alpha: X \times X \to [1, \infty)$ be a mapping. The function $\zeta: X \times X \to [1, \infty)$ is called a controlled partial metric type if

1. $x_1 = x_2$ if and only if $\zeta(x_1, x_2) = \zeta(x_1, x_1) = \zeta(x_2, x_2)$,
2. $\zeta(x_1, x_2) = \zeta(x_2, x_1)$,
3. $\zeta(x_1, x_1) \leq \zeta(x_2, x_1)$,
4. $\zeta(x_1, x_2) \leq \alpha(x_1, x_3) \zeta(x_1, x_3) + \alpha(x_3, x_2) \zeta(x_3, x_2)$, for all $x_1, x_2, x_3 \in X$. The pair (X, ζ) is called a controlled partial metric type space.

Remark 14

A controlled partial metric space is more extensive than the controlled metric space and one can follow [45] for a counter example.

Throughout this text, we shall denote the class of closed bounded subsets of X in the setting of a controlled partial metric, ζ, by $CB^{\zeta}(X)$.

3. Main Results

3.1. Hausdorff Controlled Partial Metric Space

In this section, we present the main contributions of the paper within the framework of Generalised Hausdorff Controlled Partial Metric (GHCPM) spaces. We begin by introducing the Hausdorff controlled partial metric on families of nonempty closed and bounded subsets.

For a controlled partial metric space (X,ζ), let $x \in X$ and $A \subset X$. We define $\alpha(x,A) := \inf_{a\in A} \alpha(x,a)$, where $\alpha(x,a)$ is the control function defined in Definition 13. This extends the control function from points to subsets of X.

Lemma 15

Let (X,ζ) be a controlled partial metric space. Then $\zeta(x_1,A) \leq \alpha(x_1,x_3)\zeta(x_1,x_3) + \alpha(x_3,A)\zeta(x_3,A)$, for all $x_1,x_2,x_3 \in X$ and $a \in A \subset X$, where $\alpha(x_2,A) = \inf_{a\in A} \alpha(x_2,a)$.

Proof.

By axiom 4 in Definition 13 we have that

$$\zeta(x_1,a) \leq \alpha(x_1,x_3)\zeta(x_1,x_3) + \alpha(x_3,a)\zeta(x_3,a),\ for\ \ all\ \ x_1,x_2,a \in X. \tag{1}$$

Finding the infimum on both sides of (1) over A yields;

$$\inf_{a\in A} \zeta(x_1,a) \leq \alpha(x_1,x_3)\zeta(x_1,x_3) + \inf_{a\in A}\alpha(x_3,a)\inf_{a\in A}\zeta(x_3,a).$$ From $\alpha(x_2,A) = \inf_{a\in A}\alpha(x_2,a)$.

we have that

$$\zeta(x_1,A) \leq \alpha(x_1,x_3)\zeta(x_1,x_3) + \alpha(x_3,A)\zeta(x_3,A).$$

We now define a Hausdorff controlled partial metric space by replacing the partial metric space in Definition 1.4 in [10] by a controlled partial metric space.

Definition 16

Let (X,ζ) be a controlled partial metric space. Then for $A,B \in CB^{\zeta}(X)$, we define the function $H_{\zeta}: CB^{\zeta}(X) \times CB^{\zeta}(X) \to [0,\infty)$ as the Hausdorff controlled partial metric space given by;

$$H_{\zeta}(A,B) = \begin{cases} max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,A)\right\}, & if\ the\ maximum\ exists; \\ \infty, & otherwise. \end{cases}$$

Remark 17

One can easily check that the Hausdorff controlled partial metric space enjoys the same properties as the Huasdorff partial metric space, except for the triangular inequality which in Hausdorff controlled partial metric spaces has the following form:

$$H_{\zeta}(A,C) \leq max\left\{\sup_{a\in A}\alpha(a,B), \alpha(b,A)\right\}H_{\alpha}(A,B) +$$

$$max\left\{\alpha(b,C), \sup_{c\in C}\alpha(c,B)\right\}H_{\alpha}(B,C)\text{ for all } A,B,C \subset CB^{\zeta}(X).$$

Throughout this section, we shall denote the Hausdorff controlled partial metric space by $HCPM - space$.

Proposition 18

Let (X,ζ) be a controlled partial metric space and let H_ζ be defined as in Definition 16. Then for all $A,B,C \in CB^\zeta(X)$, the function H_ζ satisfies the following properties:

1. $H_\zeta(A,A) \le H_\zeta(A,B)$, (small self-distance)
2. $H_\zeta(A,B) = H_\zeta(B,A)$, (symmetry)
3. $H_\zeta(A,C) \le \max\{\sup_{a\epsilon A} \alpha(a,B), \sup_{b\epsilon B}\alpha(b,A)\}\, H_\zeta(A,B) + \max\{\sup_{b\epsilon B} \alpha(b,C), \sup_{c\epsilon C}\alpha(c,B)\}\, H_\zeta(B,C)$. (controlled triangular inequality)

Hence, $(CB^\zeta(X), H_\zeta)$ forms a Hausdorff controlled partial metric space (HCPM-space).

Proof.

For any $A,B \in CB^\zeta(X)$, we have by definition

$$H_\zeta(A,B) = \max\{\sup_{a\in A}\zeta(a,B), \sup_{b\in B}\zeta(b,A)\}.$$

Since $\zeta(a,a) \le \zeta(a,b)$ for all $a,b \in X$ (Axiom 1 of the controlled partial metric), it follows that

$$H_\zeta(A,A) = \sup_{a\in A}\zeta(a,A) \le \sup_{a\in A}\zeta(a,B) \le H_\zeta(A,B).$$

The definition of H_ζ is symmetric in A and B; hence $H_\zeta(A,B) = H_\zeta(B,A)$.

Let $a \in A$ and $c \in C$. By Axiom (4) of the controlled partial metric ζ, we have for every $b \in B$:

$$\zeta(a,c) \le \alpha(a,b)\zeta(a,b) + \alpha(b,c)\zeta(b,c).$$

Taking the infimum over $b \in B$ and then the supremum over $a \in A$ and $c \in C$, and applying Lemma 15, we obtain

$$\sup_{a\in A}\zeta(a,C) \le \max\{\sup_{a\in A}\alpha(a,B), \sup_{b\in B}\alpha(b,A)\}\sup_{a\in A}\zeta(a,B) + \max\{\sup_{b\in B}\alpha(b,C), \sup_{c\in C}\alpha(c,B)\}\sup_{b\in B}\zeta(b,C)$$

By the definition of H_ζ, the right-hand side equals the claimed bound in 3.

Thus, all three properties hold, and $(CB^\zeta(X), H_\zeta)$ forms a Hausdorff controlled partial metric space.

Example 19

Let $X = [0,1]$ be endowed with the controlled partial metric $\zeta: X \times X \to [0,\infty)$ defined by $\zeta(x,y) = max\{x,y\}$, where $x,y \in X$. Then

$$H_\zeta(X,X) = sup\{x: 0 \le x \le 1\} = 1$$

is a Hausdorff controlled partial metric space induced by ζ.

Theorem 20

Let (X,ζ) be a controlled partial metric space. Then the function

$$H_\zeta: CB^\zeta(X) \times CB^\zeta(X) \to [0,\infty)$$

is a generalised controlled partial metric space in $CB^\zeta(X)$.

Proof.

We first show that $A = B$ if and only if $H_\zeta(A,B) = H_\zeta(A,A) = H_\zeta(B,B)$.

Let $A = B$. Then $H_\zeta(A,B) = H_\zeta(A,A) = H_\zeta(B,B)$.

Conversely, let $H_\zeta(A,B) = H_\zeta(A,A) = H_\zeta(B,B)$. Then

$$max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,A)\right\} = max\left\{\sup_{a\in A}\zeta(a,A)\,,\sup_{b\in B}\zeta(b,A)\right\} = max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,B)\right\}.$$

$\Rightarrow$ $\boldsymbol{A = B}$.

Next, for all $A, B \in CB^\zeta(X)$, it is clear that $H_\zeta(A,B) = H_\zeta(B,A)$.

$$\text{i.e } H_\zeta(A,B) = max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,A)\right\} = max\left\{\sup_{b\in B}\zeta(b,A)\,,\sup_{a\in A}\zeta(a,B)\right\} = H_\zeta(B,A).$$

Now, $H_\zeta(A,A) = max\left\{\sup_{a\in A}\zeta(a,A)\,,\sup_{b\in B}\zeta(b,A)\right\} \leq max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,A)\right\} = H_\zeta(A,B)$

i.e $H_\zeta(A,A) \leq H_\zeta(A,B)$

Lastly, let $A, B, C \in CB^\zeta(X)$ and assume that $H_\zeta(A,B)$ and $H_\zeta(B,C)$ are finite. Then from Lemma 3.1.1 above we have for $a \in A$ and $b \in B$ that

$$\zeta(a,C) \leq \alpha(a,b)\zeta(a,b) + \alpha(b,C)\zeta(b,C)$$

$$\leq \alpha(a,b)\zeta(a,b) + \alpha(b,C)H_\zeta(B,C) \text{ since } \zeta(b,C) \leq H_\zeta(B,C)$$

$$\leq \alpha(a,b)H_\zeta(a,B) + \alpha(b,C)H_\zeta(B,C)$$

$$\leq \alpha(a,b)H_\zeta(A,B) + \alpha(b,C)H_\zeta(B,C) \text{ since } \zeta(a,B) \leq H_\zeta(A,B).$$

By taking supremum over $a \in A$, we have that

$$\sup_{a\in A}\zeta(a,C) \leq \sup_{a\in A}\alpha(a,b)H_\zeta(A,B) + \alpha(b,C)H_\zeta(B,C). \tag{1}$$

Similarly, taking supremum over $c \in C$, yields

$$\sup_{c\in C}\zeta(c,A) \leq \alpha(b,A)H_\zeta(A,B) + \sup_{c\in C}\alpha(c,b)H_\zeta(B,C). \tag{2}$$

From (1) and (2), we have that

$$max\left\{\sup_{a\in A}\zeta(a,C)\,,\sup_{c\in C}\zeta(c,A)\right\} \leq max\left\{\sup_{a\in A}\alpha(a,b)\,,\alpha(b,A)\right\}H_\zeta(A,B) + max\left\{\alpha(b,C)\,,\sup_{c\in C}\alpha(c,b)\right\}H_\zeta(B,C). \tag{3}$$

Since Definition 16 shows that

$$H_\zeta(A,B) = \begin{cases} max\left\{\sup_{a\in A}\zeta(a,B)\,,\sup_{b\in B}\zeta(b,A)\right\}, & \textit{if the maximum exists}; \\ \infty, & \textit{otherwise}. \end{cases}$$

it follows from (3) therefore that;

$$H_\zeta(A,C) \leq max\left\{\sup_{a\in A}\zeta(a,b)\,,\alpha(b,A)\right\}H_\zeta(A,B) + max\left\{\alpha(b,C)\,,\sup_{c\in C}\alpha(c,b)\right\}H_\zeta(B,C).$$

This completes the proof.

Remark 21

The use of controlled partial metrics and generalized metric properties in our framework is essential for several reasons. First, they allow us to capture the self-distance feature of points (via partial metrics), accommodate flexible scaling (via controlled metrics), and incorporate relational dependencies (via the graph structure). These features are not achievable in standard metric spaces.

Advantages: This approach provides a unified and flexible framework for establishing fixed point results for multivalued mappings and solving nonlinear integral equations under non-monotonic control conditions. It generalizes many existing results and allows for greater modeling versatility in complex systems.

Limitations: The framework is more abstract and may require additional assumptions for specific applications, such as completeness of the GHCPM space and boundedness of the subsets involved. Also, some classical fixed point results in simpler metric spaces cannot be directly recovered without additional modifications.

We have defined $HCPM - space$, a generalised Hausdorff metric on the class of nonempty closed and bounded subsets of controlled partial metric spaces, denoted by $H_\zeta(A,B)$, where $A,B \in CB^\zeta(X)$. By similar argument as in Theorem 2.4 in [36], we have the following notions on the completeness.

3.2. Completeness of the HCPM-Space

Let (X,ζ) be a controlled partial metric space, and denote by $CB^\zeta(X)$ the collection of its nonempty closed and bounded subsets, equipped with the Hausdorff-type metric

$$H_\zeta(A,B) = \max\left\{\sup_{a\in A}\zeta(a,B), \sup_{b\in B}\zeta(b,A)\right\}, \quad A,B \in CB^\zeta(X).$$

Theorem 22

If $(\boldsymbol{X},\boldsymbol{\zeta})$ is complete, then $(\boldsymbol{CB^\zeta}(\boldsymbol{X}),\boldsymbol{H_\zeta})$ is also complete.

Proof.

Let $\{A_n\}$ be a Cauchy sequence in $(CB^\zeta(X),H_\zeta)$. Then $\{A_n\}$ is uniformly Cauchy under ζ, so by completeness of X, one can show that the "limit set" $A = limsup\, A_n$ exists in $CB^\zeta(X)$, and $A_n \to A$ in H_ζ. The main steps mirror the classical Hausdorff metric case but adapted using properties of ζ. A thorough proof can be constructed via successive extraction of elements and using Cauchy convergence in (X,ζ).

Theorem 23

Let $H_\zeta(A,B)$ be the Pompeiu–Hausdorff partial controlled metric defined on the family $CB^\zeta(X)$ of nonempty closed and bounded subsets of a complete partial controlled metric space (X,ζ). Then $(CB^\zeta(X),H_\zeta)$ is complete.

Proof.

Suppose that (X,ζ) is a complete partial controlled metric space and that $CB^\zeta(X)$ denotes the collection of all nonempty closed and bounded subsets of X. Recall that the Pompeiu–Hausdorff partial controlled metric $H_\zeta: CB^\zeta(X) \times CB^\zeta(X) \to R^+$ is defined by

$$H_\zeta(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\zeta(a,b), \quad \sup_{b\in B}\inf_{a\in A}\zeta(b,a)\right\}.$$

To prove completeness, let $\{A_n\}$ be a Cauchy sequence in $(CB^\zeta(X),H_\zeta)$. We want to show that there exists a set $A \in CB^\zeta(X)$ such that

$$\lim_{n\to\infty} H_\zeta(A_n,A) = 0.$$

Step 1: Show existence of limit set $\boldsymbol{A}$. *Define*

$$A := \left\{x \in X \mid \exists x_n \in A_n \; such\; that\; \lim_{n\to\infty}\zeta(x_n,x) = \zeta(x,x)\right\}$$

This means each point x in A is the limit (with respect to ζ) of some sequence of points $x_n \in A_n$.

Step 2: Show that A is nonempty

Since the sets A_n are nonempty and bounded, for some fixed $x_0 \in X$, there exists $M > 0$ such that for all n,

$$\sup_{a \in A_n} \zeta(a, x_0) \leq M.$$

This uniform boundedness ensures we can extract a sequence $a_n \in A_n$ such that $\zeta(a_n, x_0) \leq M$ for all n. By completeness of (X, ζ), the sequence $\{a_n\}$ has a convergent subsequence (w.r.t. ζ) whose limit lies in X. Hence, A is nonempty.

Step 3: Show that A is closed

Let $\{x_k\}$ be a sequence in A converging to some $x \in X$. For each $x_k \in A$, there is a sequence $\{x_n^{(k)}\} \subset A_n$ such that

$$\lim_{n\to\infty} \zeta\left(x_n^{(k)}, x_k\right) = \zeta(x_k, x_k).$$

Using the triangle inequality and properties of ζ, and the fact that $x_k \to x$, one can show

$$\lim_{n\to\infty} \zeta\left(x_n^{(k)}, x\right) = \zeta(x, x),$$

which implies $x \in A$. Thus A is closed.

Step 4: Show convergence of $\{A_n\}$ to A

Since $\{A_n\}$ is a Cauchy sequence in $(CB^\zeta(X), H_\zeta)$, for every $\varepsilon > 0$, there exists N such that for all $m, n \geq N$,

$$H_\zeta(A_n, A_m) < \varepsilon.$$

By definition of H_ζ, this implies that for any $a_n \in A_n$, there exists $a_m \in A_m$ such that $\zeta(a_n, a_m) < \varepsilon$, and vice versa. Using a diagonalization argument, construct a sequence $a_n \in A_n$ which is Cauchy in (X, ζ). By completeness of (X, ζ), this sequence converges to some $a \in X$. By Step 1, this $a \in A$.

Similarly, for any $a \in A$, by definition of A, there exist sequences $a_n \in A_n$ converging to a, so

$$\limsup_{n\to\infty} \inf_{b \in A_n} \zeta(a, b) = \zeta(a, a).$$

Therefore,

$$\lim_{n\to\infty} H_\zeta(A_n, A) = 0,$$

which proves that $\{A_n\}$ converges to A in $(CB^\zeta(X), H_\zeta)$.

Step 5: Show boundedness of A.

Since each A_n is bounded, for some fixed $x_0 \in X$, there is $M > 0$ with

$$\sup_{a_n \in A_n} \zeta(a_n, x_0) \leq M.$$

Passing to the limit in n, for $a \in A$ and corresponding sequence $a_n \to a$,

$$\zeta(a, x_0) \leq \liminf_{n\to\infty} \zeta(a_n, x_0) \leq M,$$

so A is bounded.

Hence, the limit set A belongs to $CB^\zeta(X)$, and $(CB^\zeta(X), H_\zeta)$ is complete.

3.3. Hausdorff Controlled Partial Metric Space Endowed with a Digraph Structure

We now present $HCPM - space$ endowed with a digraph structure.

Definition 24

Consider a Hausdorff controlled partial metric space($HCPM - space$) defined by (X, ζ). Let $G = (V(G), E(G))$ be a directed graph such that the vertex set $V(G)$ coincides with X and the edge set $E(G) \subseteq X \times X$, with $\Delta \subseteq E(G)$, where Δ is the diagonal of $X \times X$. Let the graph G be allowed to have loops but without parallel edges between distinct pairs of vertices. Let us further consider that the graph G is weighted. Then the Hausdorff controlled partial metric space satisfying the above is said to be endowed with a digraph structure.

We shall consider the graph G to be connected if there is a directed path between any two vertices in G. The undirected graph obtained from G by ignoring the directions of the edges shall be denoted by $\tilde{G}$. We consider G to be weekly connected if $\tilde{G}$ is connected. The set of the edges of G are considered to be symmetric, so that $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

Let $\tilde{G_x}$ be the component of G, consisting of all edges and vertices which are contained in some path in G beginning at x. With $E(G)$ assumed to be symmetric, the equaivalence class $[x]_G$ defined on $V(G)$ by the rule R is such that $V(G_x) = [x]_G$.

Here and in sequel, let us denote the Hausdorff controlled partial metric space endowed with a digraph structure G defined above by $GHCPM - space$.

Example 25

Let $G = (V\ , E)$, $X = \{0,1,2,3\} = V(G)$ and let

$E = \{(0,0), (1,1), (2,2), (3,3), (0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$. Let the

metric $\zeta(x, y) = \frac{1}{2}|x - y| + \frac{1}{2}max\{x, y\}$ endowed with the digraph G be defined by

$\zeta(0,0) = 0$, $\zeta(1,1) = \frac{1}{2}$, $\zeta(0,1) = \zeta(2,2) = 1$, $\zeta(1,2) = \zeta(3,3) = \frac{3}{2}$,

$\zeta(2,3) = 2$ and $\zeta(0,3) = 3$.

Then (X, ζ) is a controlled partial metric space with, say, $\alpha = x + y$.

Proof.

The sets $\{0\}$, $\{0,1\}$, $\{0,2\}$ and $\{0,3\}$ are closed. We show this as follows;

$x \in \overline{\{0\}} \Leftrightarrow \zeta(x, \{0\}) = \zeta(x, x)$

$\Leftrightarrow \frac{1}{2}|x| + \frac{1}{2}max\{x, 0\} = \frac{1}{2}x$

$$\Leftrightarrow \quad x \in \{0\}$$

Hence, $\{0\}$ is a closed set with respect to ζ.

$$x \in \overline{\{0,1\}} \Leftrightarrow \zeta(x, \{0,1\}) = \zeta(x, x)$$

$$\Leftrightarrow \quad min\left\{\frac{1}{2}|x| + \frac{1}{2}max\{x, 0\}, \frac{1}{2}|x - 1| + \frac{1}{2}max\{x, 1\} = \frac{1}{2}x\right\}$$

$$\Leftrightarrow \quad x \in \{0,1\}$$

Hence, $\{0,1\}$ is closed with respect to ζ.

$$x \in \overline{\{0,2\}} \Leftrightarrow \zeta(x, \{0,2\}) = \zeta(x, x)$$

$$\Leftrightarrow \ min\left\{\frac{1}{2}|x| + \frac{1}{2}max\{x,0\}, \frac{1}{2}|x-2| + \frac{1}{2}max\{x,2\} = \frac{1}{2}x\right\}$$

$$\Leftrightarrow \ x \in \{0,2\}$$

Thus, $\{0,2\}$ is closed with respect to ζ.

$$x \in \overline{\{0,3\}} \Leftrightarrow \zeta(x,\{0,3\}) = \zeta(x,x)$$

$$\Leftrightarrow \ min\left\{\frac{1}{2}|x| + \frac{1}{2}max\{x,0\}, \frac{1}{2}|x-3| + \frac{1}{2}max\{x,3\} = \frac{1}{2}x\right\}$$

$$\Leftrightarrow \ x \in \{0,3\}$$

Similarly, $\{0,3\}$ is closed with respect to ζ.

Clearly, the sets $\{0\}$, $\{0,1\}$, $\{0,2\}$ and $\{0,3\}$ are bounded. Therefore, we have that $CB^{\zeta}(X) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, where $\overline{0} = \{0\}$, $\overline{1} = \{0,1\}$, $\overline{2} = \{0,2\}$, and $\overline{3} = \{0,3\}$.

Hence,

$$H_{\zeta}(\Xi_1, \Xi_2) = \begin{cases} 0, & if \quad \Xi_1 = \overline{0} \\ \frac{1}{2}, & if \quad \Xi_1 = \overline{1} \ and \ \Xi_1 = \Xi_2\overline{2} \\ 1, & if \quad \Xi_1 = \overline{0} \ and \ \Xi_2 = \overline{1}, and \ \Xi_1 = \Xi_2 = \overline{0} \\ \rightleftarrows \frac{3}{2}, & if \quad \Xi_1 = \overline{1} \ and \ \Xi_2 = \overline{2}, and \Xi_1 = \Xi_2\overline{3} \\ 2, & if \quad \Xi_1 = \overline{2} \ and \ \Xi_2 = \overline{3}, \Xi_1 = \overline{0} and \Xi_2 = \overline{2} \\ 3, & if \quad \Xi_1 = \overline{3} \\ \frac{5}{2}, & if \quad \Xi_1 = \overline{1} and \Xi_2 = \overline{3} \end{cases}$$

is a Hausdorff controlled partial metric space endowed with a digraph G. This function $H_{\zeta}(\Xi_1, \Xi_2)$ defines the Hausdorff controlled partial metric on the family of nonempty closed and bounded subsets of X. It explicitly quantifies the "distance" between subsets according to the underlying controlled partial metric ζ and the digraph G. In this way, the example illustrates how the abstract GHCPM construction operates concretely, showing how distances between sets are computed in practice. Consequently, $(CB^{\zeta}(X), f)$ forms a Hausdorff controlled partial metric space endowed with the digraph G.

Figure **1** below is the digraph of H_{ζ} for Example 25 above.

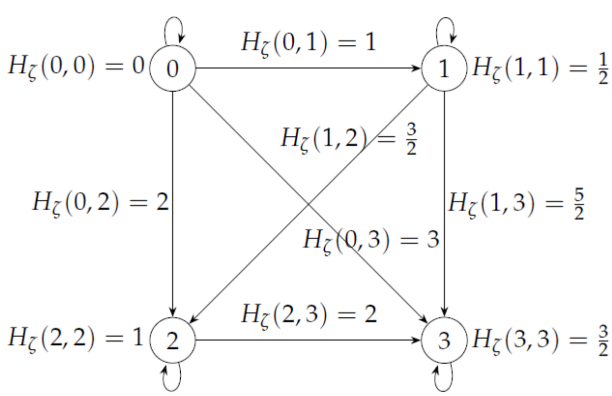

Figure 1: Digraph of H_{ζ}

3.4. Generalised Graph ϕ-contractions on GHCPM - Spaces

We now define generalised graph ϕ - contraction on $GHCPM - spaces$ by modifying Definition 7 above which comes from Definitions 1.6 and 1.7 in [10], as follows:

Definition 26

Let $T_\zeta: CB^\zeta(X) \to CB^\zeta(X)$ be a mapping from $CB^\zeta(X)$ into itself. Then T_ζ is called a generalised graph ϕ -contraction on $GHCPM - space$ if the following properties are satisfied:

1. e_{T_ζ} is an edge that links $T_\zeta(\Xi_1)$ to $T_\zeta(\Xi_2)$ whenever e is the preceding edge that links Ξ_1 and Ξ_2.
2. W_{T_ζ} is a path from $T_\zeta(\Xi_1)$ to $T_\zeta(\Xi_2)$ whenever W is a path from Ξ_1 to Ξ_2.
3. Take a function ϕ defined from R^+ into itself with the properties that: ϕ is upper semicontinuous, non - decreasing satisfying the inequality, $\phi(t) < t$ for every $t > 0$ such that if e is an edge from Ξ_1 to Ξ_2 we infer that

$H_\zeta(T_\zeta(\Xi_1), T_\zeta(\Xi_2)) \leq \phi(M_\zeta(\Xi_1, \Xi_2))$, where

$M_\zeta(\Xi_1, \Xi_2) = max\left\{H_\zeta(\Xi_1, \Xi_2), H_\zeta(\Xi_1, T_\zeta(\Xi_1)), H_\zeta(\Xi_2, T_\zeta(\Xi_2)), \frac{H_\zeta(\Xi_1, T_\zeta(\Xi_2)) + H_\zeta(\Xi_2, T_\zeta(\Xi_1))}{2}\right\}$.

Lemma 27

If $A, B \in CB^\zeta(X)$ and $y \in B$, then $\forall\ \varepsilon > 0$, there exists $x \in A$ such that

$$\zeta(x, y) \leq H_\zeta(A, B) + \varepsilon.$$

Proof.

Let $A, B \in CB^\zeta(X)$ and $y \in B$. Then from Definition 16, we have that

$$\zeta(A, y) \leq H_\zeta(A, B).$$

By definition of infimum, we assume that a sequence x_n in A is such that

$$\zeta(y, x_n) \leq \zeta(y, A) + \varepsilon, \text{ where } \varepsilon > 0.$$

Since $A \in CB^\zeta(X)$, it is closed. Thus, there exists $x \in A$ such that $x_n \to x$. Therefore, we have that

$$\zeta(x, y) \leq \zeta(y, x_n) < \zeta(A, y) + \varepsilon \leq H_\zeta(A, B) + \varepsilon,$$

i.e $\zeta(x, y) \leq H_\zeta(A, B) + \varepsilon$ as required.

We now define the fixed point of T_ζ on $GHCPM - space$

Definition 28

Let (X, ζ) be a Hausdorff controlled partial metric space endowed with a digraph G, and let $T_\zeta: CB^\zeta(X) \to CB^\zeta(X)$ be a generalised graph ϕ -contraction from $CB^\zeta(X)$ into itself. Then a fixed point of T_ζ is a set $\Xi \in CB^\zeta(X)$ whenever $T_\zeta(\Xi) = \Xi$. Here, the mapping T_ζ generates the set of fixed points $F(T_\zeta) = \{U \in CB^\zeta(X): T_\zeta(U) = U\}$.

Lemma 29

Let (X, ζ) be a $GHCPM - space$ and let $T_\zeta: CB^\zeta \to CB^\zeta$ be a generalised graph ϕ -contraction. If $x \in X_{T_\zeta} = \{x \in X: (x, T_\zeta x) \in E(G)\ or\ (T_\zeta x, x) \in E(G)\}$ then

$$H_\zeta(T_\zeta^n x, T_\zeta^{n+1} x) \leq \phi^n \zeta(x, T_\zeta x)\ \forall n \in N,$$

where ϕ is a nondecreasing function and $\{\phi^n(t)\}_{n \in N}$ converges to 0 for all $t > 0$.

Proof.

Let $x \in X_{T_\zeta}$. Then by definition of X_{T_ζ}, there is no loss of generate in assuming that $(x, T_\zeta x) \in E(G)$. Then

for $n = 1$ $(T_\zeta x, T_\zeta^2 x) \in E(G)$

for $n = 2$ $(T_\zeta^2 x, T_\zeta^3 x) \in E(G)$

$(T_\zeta^n x, T_\zeta^{n+1} x) \in E(G)$, $\forall \ n \in N$ (by induction).

Thus, $H_\zeta(T_\zeta^n x, T_\zeta^{n+1} x) \leq \phi\zeta(T_\zeta^{n-1} x, T_\zeta^n x)$

$$\leq \phi^2\zeta(T_\zeta^{n-2} x, T_\zeta^{n-1} x)$$

$$\leq \phi^3\zeta(T_\zeta^{n-3} x, T_\zeta^{n-2} x)$$

$$\leq \phi^{n-1}\zeta(T_\zeta x, T_\zeta^2 x)$$

$$\leq \phi^n\zeta(x, T_\zeta x).$$

i.e $H_\zeta(T_\zeta^n x, T_\zeta^{n+1} x) \leq \phi^n\zeta(x, T_\zeta x)$ $\forall \ n \in N$, as required.

We adopt the definition of a recursive iterative sequence from Nazir *et al*.[10] as shown in the remark below:

Remark 30

$$u = u_0$$

$$T_\zeta(u_0) = u_1$$

$$T^2(u_0) = T_\zeta(u_1) = u_2$$

$$T_\zeta^3(u_0) = T_\zeta^2(u_1) = T_\zeta(u_2) = u_3$$

. . .

$$T_\zeta^n(u_0) = T_\zeta^{n-1}(u_1) = T_\zeta^{n-2}(u_2) = \ldots = T_\zeta(u_n)$$

where it is assumed that $u_{n+1} \neq u_n$ for all $n \in \{0, 1, 2, 3, \ldots\}$. In case $u_{k+1} = u_k$, then u_k is the fixed point of T_ζ, that is, $T_\zeta(u_{k+1}) = u_k = T_\zeta(u_k)$. Since $\tilde{G}$ is weakly connected, there exists a sequence $\{x_i\}_{i=1}$ for $x_0 = x$ and $x_n = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$, for $n = 1,2,3,\ldots,n$ such that $x_i \in u_i$ for $i = 1,2,3,\ldots,n$.

Remark 31

Let (X, ζ) be a $GHCPM - space$. For any sequence $\{x_n\}_{n\in N}$ in X, if $x_n \to x$

and $(x_n, x_{n+1}) \in E(G)$ then there is a subsequence $\{x_{k_n}\}_{n\in N}$ with

$(x_{k_n}, x) \in E(G)$. Furthermore, for every $x \in X$, we assume that $\lim_{n\to\infty}\zeta(T_\zeta^m x, T_\zeta^n x)$

$\forall \ m \geq 1$ and $\lim_{n\to\infty}\zeta(T_\zeta^n x, T_\zeta^{n+1} x)$ exist and are finite. Let us refer to this property as (P).

Theorem 32

Let $T_\zeta: CB^\zeta(X) \to CB^\zeta(X)$ be a generalised graph ϕ -contraction on $GHCPM - space$ and assume that property (P) holds. Let $x \in X_{T_\zeta} \neq \emptyset$ and $\tilde{G}$ endowed with property (P). Then $T_\zeta|\ |x|\ \tilde{G}$ has a fixed point.

Proof.

We first show that $\{T_\zeta^m(x)\}$ for $m, n \in N$ with $m > n$, is Cauchy.

Let $x \in X_{T_\zeta} \neq \emptyset$. Then, $CB^\zeta(X) \subseteq [x]_{\tilde{G}} P(X)$, where $P(X)$ is the nonempty power set on X. This is so because $x \in CB^\zeta(X)$ and $\tilde{G}$ is weakly connected. Since T_ζ is a self-map and $[x]_{\tilde{G}}$ satisfies the transitive property on $CB^\zeta(X)$, we have that $T_\zeta(x) \in [x]_{\tilde{G}}$. Similarly, we have that $T_\zeta(x_i) \in [x]_{\tilde{G}}$ for each $x_i \in [x]_{\tilde{G}}$.

Since $x \in X_{T_\zeta}$, there is an edge between x and $T_\zeta(x)$, and since T_ζ is a graph ϕ -contraction, we have that

$$(T_\zeta^n(x), T_\zeta^{n+1}(x)) \in E(\tilde{G}) \text{ for all } n \in N.$$

By Lemma 29 and owing to the graph ϕ - contraction T_ζ, we infer that

$$H_\zeta(T_\zeta^n(x), T_\zeta^{n+1}(x)) = H_\zeta(x_n, x_{n+1})$$

$= H_\zeta(T_\zeta(x_{n-1}), T_\zeta(x_n))$

$\leq \phi(M_\zeta(x_{n-1}, x_n))$, where

$$M_\zeta(x_{n-1}, x_n) = max\{H_\zeta(x_{n-1}, x_n), H_\zeta(x_{n-1}, T_\zeta(x_{n-1}), H_\zeta(x_n, T_\zeta(x_n)), \frac{H_\zeta(x_{n-1}, T_\zeta(x_{n-1})) + H_\zeta(x_n, T_\zeta(x_n))}{2}\}$$

$$= max\{H_\zeta(x_{n-1}, x_n), H_\zeta(x_{n-1}, x_n), H_\zeta(x_n, x_{n+1}), \frac{H_\zeta(x_{n-1}, x_n) + H_\zeta(x_n, x_{n+1})}{2}\}$$

$$= max\{H_\zeta(x_{n-1}, x_n), H_\zeta(x_n, x_{n+1})\}.$$

We have a contradiction if $M_\zeta(x_{n-1}, x_n) = H_\zeta(x_n, x_{n+1})$, since

$H_\zeta(x_n, x_{n+1}) \leq \phi H_\zeta(x_n, x_{x+1}) \leq H_\zeta(x_n, x_{n+1})$. So the only possibility is that

$$M_\zeta(x_{n-1}, x_n) = H_\zeta(x_{n-1}, x_n).$$

Thus, $H_\zeta(T_\zeta^n(x), T_\zeta^{n+1}(x)) = H_\zeta(x_n, x_{n+1})$

$$\begin{aligned}
&= H_\zeta(T_\zeta(x_{n-1}), T_\zeta(x_n)) \\
&\leq \phi(H_\zeta(x_{n-1}, x_n)) \\
&= \phi(H_\zeta(T_\zeta(x_{n-2}), T_\zeta(x_{n-1})) \\
&\leq \phi^2(H_\zeta(x_{n-2}, x_{n-1})) \\
&= \phi^2(H_\zeta(T_\zeta(x_{n-3}), T_\zeta(x_{n-2}))) \\
&\leq \phi^3(H_\zeta(x_{n-3}, x_{n-2}))
\end{aligned}$$

. . .

$$\begin{aligned}
&\leq \phi^n(H_\zeta(x_0, x_1)) \\
&= \phi^n(H_\zeta(x_0, T_\zeta(x))).
\end{aligned}$$

Let $m, n \in N$ with $m > n$. Then:

$$\begin{aligned}
H_\zeta(T_\zeta^n(x), T_\zeta^m(x)) &\leq H_\zeta(T_\zeta^n(x), T_\zeta^{n+1}(x)) + H_\zeta(T_\zeta^{n+1}(x), T_\zeta^{n+2}(x)) + \\
&\quad H_\zeta(T_\zeta^{n+2}(x), T_\zeta^{n+3}(x)) + \ldots + H_\zeta(T_\zeta^{m-1}(x), T_\zeta^m(x)) \\
&\leq \phi^n(H_\zeta(x, T_\zeta(x))) + \phi^{n+1}(H_\zeta(x, T_\zeta(x))) + \phi^{m-1}(H_\zeta(x, T_\zeta(x)))
\end{aligned}$$

$$= (\phi^n + \phi^{n+1} + \phi^{n+2} + ... + \phi^{m-1})(H_\zeta(x, T_\zeta(x))),$$

where, $\phi^n + \phi^{n+1} + \phi^{n+2} + ... + \phi^{m-1}$ is sum of semi-continuous functions and so is semi-continuous as well and $H_\zeta(x, T_\zeta(x))$ can be made a fixed non-zero value. Thus, $\{T_\zeta^m(x)\}$ is Cauchy.

We now show that $H_\zeta(x^*, x^*) = 0$ for some $x^* \in CB^\zeta(x)$. From the completeness of the space (X, ζ), there exists $x^* \in CB^\zeta(X)$ such that $T_\zeta^m(x) \to x^*$ as $m \to \infty$.

Thus, we have $\{T^n(x)\}$ such that $T_\zeta^m(x) \to x^*$ and we have

$$(T^m(x), T^{n+1}(x)) \in E(\tilde{G}) \ \forall\, n \in N.$$

Suppose by way of contradiction that $H_\zeta(x^*, x^*) \neq 0$. Then since T_ζ is a generalised graph ϕ -contraction, for $(x_{n-1}, x_n) \subseteq E(G)$, we have that

$$H_\zeta(T^n(x), T^{n+1}(x)) = H_\zeta(T_\zeta(x_{x-1}, x_n))$$

$$\leq \phi(M_\zeta(x_{n-1}, x_n)) \tag{3.1}$$

where $M_\zeta(x_{n-1}, x_n) = max\{H_\zeta(x_{n-1}, x_n), H_\zeta(x_{n-1}, T_\zeta(x_{n-1})), H_\zeta(x_n, T_\zeta(x_n))$, $\frac{H_\zeta(x_{n-1}, T_\zeta(x_n)) + H_\zeta(x_n, T_\zeta(x_{n-1}))}{2}\}$.

$$= max\{H_\zeta(x_{n-1}, x_n), H_\zeta(x_n, T_\zeta(x)), \frac{H_\zeta(x_{n-1}, x_{n+1}) + H_\zeta(x_n, x_n)}{2}\}.$$

Taking limits on both sides of 3.1 yields;

$$\lim_{n\to\infty} H_\zeta(T^n(x), T^{n+1}(x)) \leq \lim_{n\to\infty} \phi(M_\zeta(x_{n-1}, x_n))$$

$$0 \neq H_\zeta(x^*, x^*) \leq \phi(M_{T_\zeta}(x^*, x^*))$$

$$< H_\zeta(x^*, x^*).$$

Now the statement

$$0 \neq H_\zeta(x^*, x^*) < H_\zeta(x^*, x^*)$$

is a contradiction. Thus, $H_\zeta(x^*, x^*) = 0$.

By property (P), there exists a subsequence $\{T^{n_k}(x)\}_{\substack{k\geq 1\\ n\in N}}$ of $\{T^n(x)\}_{n\in N}$ such that $(T^{n_k}(x), x^*) \in E(G) \ \forall\ n \in N$, i.e $\{T^{n_k}(x)\}_{\substack{k\geq 1\\ n\in N}}$ provides us with an edge connecting $T^{n_k}(x)$ and $x^å$ for every $k \in N$.

It follows from Lemma 15 and property 3 of Definition 26 that,

$$H_\zeta(T_\zeta(x^*), x^*) \leq \alpha(T_\zeta(x^*), T^{n_k}(x)) H_\zeta(T(x^*), T^{n_k}(x)) + \alpha(T_\zeta^{n_k}(x), x^*) H_\zeta(T^{n_k}(x), x^*)$$

$$\leq \alpha(T_\zeta(x^*), T^{n_k}(x)) \phi M_\zeta(x^*, T_\zeta^{n_k - 1}(x)) + \alpha(T_\zeta^{n_k}(x), x^*) H_\zeta(T^{n_k}(x), x^*) \tag{3.2}$$

where

$$M_\zeta(x^*, T_\zeta^{n_k - 1}(x)) = max\{H_\zeta(x^*, T_\zeta^{n_k - 1}(x)), H_\zeta(x^*, T(x^*)), H_\zeta(T^{n_k - 1}(x), T_\zeta^{n_k}(x)),$$

$$\frac{H_\zeta(x^å, T(x^å)) + H_\zeta(T_\zeta^{n_k - 1}(x), T_\zeta^{n_k - 1}(x), T_\zeta^{n_k}(x)}{2}\}.$$

Since any subsequence of a convergent sequence also converges to the same limit due to the uniqueness of limits, we have that

$$\lim_{n_k\to\infty} M_\zeta(x^*, T_\zeta^{n_k-1}(x)) = max\{H_\zeta(x^*, x^*), H_\zeta(x^*, T_\zeta(x^*))\} = H_\zeta(x^*, T_\zeta(x^*)).$$

Hence, $\lim_{n_k\to\infty} T_\zeta^{n_k}(x) = x^*$ for any $k \to \infty$. Thus, from (3.2) above, we have that

$$H_\zeta(T_\zeta(x^*), x^*) \le \alpha(T_\zeta(x^*), x^*)\phi H_\zeta(x^*, x^*) + \alpha(x^*, x^*)H_\zeta(x^*, x^*) = 0 \text{ since } H_\zeta(x^*, x^*) = 0.$$

Therefore, $x^* \in T_\zeta(x^*)$, so that x^* is a fixed point of T_ζ, that is, $T_\zeta| \; |x| \; \tilde{G}$ has a fixed point. This completes the proof.

4. Application

4.1. Fredholm Integral Equation

Fixed point theorems have been widely applied to establish existence and uniqueness results for solutions of nonlinear integral equations under various conditions. In this section, we investigate the solution of a Fredholm-type integral equation of the form:

$$f(t) = \lambda(t) + \int_a^b K\,(t, s, f(s))ds, \tag{2}$$

where $K: [a, b] \times [a, b] \times R \to R$ is the integral kernel, and $\lambda: [a, b] \to R$ is a given continuous function.

Theorem 33

Let (X, ζ) be a $GHCPM$ -space, and let $T_\zeta: CB^\zeta(X) \to CB^\zeta(X)$ be a generalized graph ϕ -contraction on the $GHCPM$ -space, where property (P) holds.

Define the controlled partial metric $\zeta: X \times X \to [0, \infty)$ by

$$\zeta(f,g) = \sup_{t\in[a,b]}\left(\frac{1}{3}(|\,f(t)\,|+|\,g(t)\,|)\right),$$

and assume the following conditions hold for all $t, s \in [a, b]$ and $f, g \in X$:

1. $|K(t,s,f(s))|+|K(t,s,g(s))| \le \alpha(\sup_{t\in[a,b]}(|\,f(t)\,|+|\,g(t)\,|))\cdot(|\,f(s)\,|+|\,g(s)\,|)$, where $\alpha: R^+ \to [0,1)$;
2. $K(t, s,\cdot)$ is continuous, and the mapping $f \mapsto \int_a^b K\,(t, s, f(s))ds$ is well-defined for all $f \in C[a, b]$.

Then the integral equation (2) has a unique solution in $C[a, b]$.

Proof.

Let $X = C[a, b]$, the space of continuous real-valued functions on $[a, b]$. Define the operator $T_\zeta: X \to X$ by

$$(T_\zeta f)(t) = \lambda(t) + \int_a^b K\,(t, s, f(s))ds.$$

Let us estimate $\zeta(T_\zeta f, T_\zeta g)$ for $f, g \in X$:

$$\zeta(T_\zeta f, T_\zeta g) = \sup_{t\in[a,b]}\left(\frac{1}{3}\left|(T_\zeta f)(t)\right| + \frac{1}{3}\left|(T_\zeta g)(t)\right|\right)$$

$$= \sup_{t\in[a,b]}\frac{1}{3}\left(\left|\lambda(t)+\int_a^b K(t,s,f(s))ds\right|+\left|\lambda(t)+\int_a^b K(t,s,g(s))ds\right|\right)$$

$$\le \sup_{t\in[a,b]}\frac{1}{3}\left(|\,\lambda(t)\,|+\int_a^b |\,K(t,s,f(s))\,|\,ds+|\,\lambda(t)\,|+\int_a^b |\,K(t,s,g(s))\,|\,ds\right)$$

$$= \sup_{t\in[a,b]} \frac{2}{3}|\lambda(t)| + \frac{1}{3}\int_a^b \big(|K(t,s,f(s))| + |K(t,s,g(s))|\big)ds.$$

By assumption (1),

$$|K(t,s,f(s))| + |K(t,s,g(s))| \leq \alpha(M)\cdot(|f(s)| + |g(s)|),$$

where $M = \sup_{t\in[a,b]}(|f(t)| + |g(t)|)$.

Thus,

$$\zeta(T_\zeta f, T_\zeta g) \leq \sup_{t\in[a,b]}\left(\frac{2}{3}|\lambda(t)| + \frac{1}{3}\alpha(M)\int_a^b(|f(s)| + |g(s)|)ds\right) \leq C + \alpha(M)\cdot\zeta(f,g),$$

for some constant C depending on λ.

Since $\alpha(M) < 1$, T_ζ is a ϕ -contraction in the sense of the $GHCPM$ structure. Therefore, by the main fixed point theorem proved earlier, T_ζ has a unique fixed point in $C[a,b]$, which is the unique solution of equation (2).

Example 34

Solve the Fredholm integral equation:

$$f(t) = \sin t + \int_0^\pi \frac{1}{6}(f(s) + t^2)ds.$$

Solution:

We define the operator T on the space $C[0,\pi]$ by:

$$Tf(t) = \sin t + \int_0^\pi \frac{1}{6}(f(s) + t^2)ds.$$

Let $\zeta(f,g) = \sup_{t\in[0,\pi]}\frac{1}{3}(|f(t)| + |g(t)|)$. This defines a controlled partial metric space.

We verify that T satisfies a contractive condition:

$$\begin{aligned}\zeta(Tf,Tg) &= \sup_{t\in[0,\pi]}\frac{1}{3}\left|\int_0^\pi \frac{1}{6}(f(s)-g(s))ds\right| \\ &= \frac{1}{18}\int_0^\pi |f(s)-g(s)|ds \\ &\leq \frac{\pi}{18}\|f-g\|_\infty.\end{aligned}$$

Also, for the metric $\zeta(f,g)$, we have:

$$\zeta(f,g) = \sup_{t\in[0,\pi]}\frac{1}{3}(|f(t)| + |g(t)|) \geq \frac{2}{3}\|f-g\|_\infty.$$

Thus:

$$\zeta(Tf,Tg) \leq \frac{\pi}{12}\zeta(f,g).$$

Since $\frac{\pi}{12} < 1$, T is a contraction with respect to ζ, and hence, by the Banach-type fixed point theorem on controlled partial metric spaces, T has a unique fixed point f^* in $C[0,\pi]$.

Therefore, the equation

$$f(t) = \sin t + \int_0^\pi \frac{1}{6}(f(s) + t^2)ds$$

has a unique continuous solution.

Remark 35

While the examples above are simple enough to handle analytically, for more complex kernels or nonlinear integral operators, one can employ symbolic computation methods using software such as *Mathematica*, *Maple*, or MATLAB's symbolic toolbox. These methods allow exact computation of integrals, simplification of algebraic expressions, and verification of contraction constants. Recent advancements in symbolic calculation methods [53-57], provide efficient algorithms that can significantly enhance the applicability of fixed point techniques to more complex integral and differential equations.

Remark 36 (Physical Interpretation of the Results)

The fixed points obtained within the proposed GHCPM framework correspond to equilibrium states of the underlying nonlinear integral operators, which, in many physical contexts, represent steady-state or self-consistent configurations of the system. For example, in diffusion-reaction or population models, such fixed points describe a balance between competing effects such as growth, decay, and diffusion. The convergence behavior illustrated in the figures reflects the stability of these equilibria with respect to variations in initial conditions or control parameters. As the control function ϕ and graph connectivity are varied, the resulting trajectories reveal how interaction strength and feedback mechanisms influence both the rate of convergence and the eventual equilibrium configuration.

Furthermore, these fixed point results have direct implications for nonlinear wave phenomena. The existence of a fixed point corresponds to a steady or self-sustaining wave profile in the associated nonlinear system, while convergence toward this fixed point indicates the stability or persistence of the wave under perturbations. Such behavior is relevant to models describing diffusion-reaction processes, optical pulse propagation, fluid flow dynamics, and signal transmission in nonlinear media. Hence, the presented graphical and numerical examples not only validate the theoretical framework but also provide qualitative physical insight into the stability and long-term behavior of nonlinear wave interactions.

5. Conclusion

In this paper, we have introduced a novel framework that integrates controlled partial metric spaces with digraph structures, resulting in what we term *Hausdorff-controlled partial metric spaces endowed with a graph* (GHCPM-spaces). Building upon the foundational work of Nazir *et al*. [10] and Alamgir *et al*. [36], we generalized the notion of ϕ -contractions to this setting, formulating and proving new fixed point theorems for multivalued mappings under graph-based contractive conditions.

Our contributions extend classical and recent results in fixed point theory by situating them within a richer structural context that accommodates both non-symmetric distance measures and directed relations between sets. Specifically, we generalized the Hausdorff metric to controlled partial metric spaces, developed a graph-theoretic contraction framework, and demonstrated its applicability to nonlinear integral equations of Fredholm type. These results not only broaden the scope of existing fixed point theorems but also open pathways for their use in analysis and applied mathematics, where such generalized distance and relational structures naturally occur.

Future work may explore more general classes of contractions (such as weakly contractive or cyclic-type mappings), stability and convergence properties of iterative processes in GHCPM-spaces, or further applications in areas such as computational models, domain theory, and the analysis of complex systems with inherent directionality or asymmetry. We note that the generalized metric framework employed here, combining controlled and partial metrics with graph structures, provides significant flexibility and unifies several existing approaches. Its advantages include handling non-monotonic dynamics and multivalued mappings, while limitations involve added abstraction and the need for certain completeness and boundedness conditions. Although our framework applies to a broad class of nonlinear integral and differential equations, it may be limited for (Nonlinear Partial Differential Equations)NLPDEs with highly singular, unbounded, or non-Lipschitz nonlinearities, where additional assumptions would be required.

Conflicts of Interest

The authors declare no conflict of interest.

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