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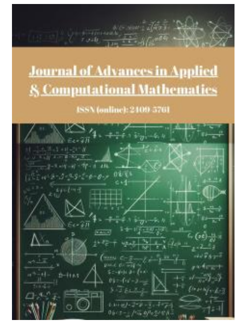

Normalized Solutions for Biharmonic Equation with Combined Pure-power and Saturable Nonlinearities

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ABSTRACT

We investigate the existence of normalized solutions to the biharmonic equation with combined pure-power and saturable nonlinearities:

$$\begin{cases} \Delta^2 u + \lambda u = |u|^{p-2}u + \mu \dfrac{g+|u|^2}{1+g+|u|^2}u \quad \text{in } \mathrm{R}^N, \\ \displaystyle\int_{R^N} |u|^2 dx = c\,, \end{cases}$$

where $5 \le N \le 7$, $2 < p < 4^* := \frac{2N}{N-4}$, $\mu > 0$ is a parameter, $\lambda \in R$ arises as a Lagrange multiplier associated with the L^2-constraint, and $-1 < g < 0$ is a constant. By employing variational methods and analyzing the problem on the Pohozaev manifold, we establish the existence of ground state solutions in the L^2-subcritical regime and mountain-pass type solutions in the L^2-supercritical regime.

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1. Introduction

In this paper, we consider the existence of normalized solutions for the following biharmonic problem with a saturable perturbation

$$\begin{cases} \Delta^2 u + \lambda u = |u|^{p-2}u + \mu \frac{g+|u|^2}{1+g+|u|^2} u \quad \text{in } \mathrm{R}^N, \\ \int_{R^N} |u|^2 dx = c\,, \end{cases} \tag{1.1}$$

where $5 \le N \le 7$, $2 < p < 4^*$, $\mu > 0$ is a parameter, $\lambda \in R$ appears as a Lagrange multiplier associated with the L^2-constraint, and $-1 < g < 0$ is a constant. This equation originates from the field of nonlinear optics, where it models the propagation of laser beams in bulk media with higher-order dispersion effects. The biharmonic operator Δ^2 accounts for fourth-order dispersion, which corrects the paraxial approximation error in traditional nonlinear Schrödinger equation (NLS) and suppresses the finite-time blow-up of solutions, see [1, 2]. The L^2-norm constraint $\int_{R^N} |u|^2 dx = c$ corresponds to the conservation of beam power, a key physical quantity in optical propagation, making the study of "normalized solutions" (solutions satisfying this constraint) physically meaningful. The saturable nonlinear term $\mu \frac{g+|u|^2}{1+g+|u|^2} u$ accurately describes the refractive index response of real optical materials, as the induced change saturates at high intensities, thereby preventing unphysical blow-up. Unlike power-type nonlinearities, its growth is bounded as the field intensity increases, avoiding unphysical infinite refractive index variations [3].

When the saturable term is omitted ($\mu = 0$), problem (1.1) reduces to the biharmonic Schrödinger equation with pure-power nonlinearity:

$$\begin{cases} \Delta^2 u + \lambda u = |u|^{p-2}u, \quad x \in R^N, \\ \int_{R^N} |u|^2 dx = c. \end{cases} \tag{1.2}$$

This simplified model has been extensively studied in recent years, with progress being made across different regimes of the exponent p. In the L^2-subcritical case ($2 < p < 2 + \frac{8}{N}$), Bellazzini and Visciglia [4] established foundational results. Under appropriate conditions on the spatial boundedness of the nonlinear coefficient, they proved the existence of ground state normalized solutions and demonstrated the orbital stability of minimizers. Their work utilized variational methods in the context of constrained minimization problems. For the L^2-critical case ($p = 2 + \frac{8}{N}$), Phan [5] extended these results by incorporating an external potential, showing that ground state solutions exist when the attraction strength parameter lies within a specific interval. This work highlighted the delicate balance required for existence in the critical case and the significant influence of external potentials on the solution structure. The L^2-supercritical case ($2 + \frac{8}{N} < p < 4^*$) presents additional challenges due to the lack of compactness and the unboundedness of the energy functional from below on the constraint manifold. Liu and Zhang [6] made substantial progress in this direction by verifying the Palais-Smale condition at the mountain-pass level and proving that for sufficiently large μ, radially symmetric non-negative normalized solutions exist. Their analysis also revealed that the energy of these solutions tends to zero as $c \to +\infty$, providing important qualitative information about the solution behavior. Further generalizations were pursued by Zhang *et al*. [7], who considered non-autonomous cases with spatially varying power nonlinearities of the form $h(\varepsilon x)|u|^{p-2}u$. They demonstrated that when the scaling parameter ε is sufficiently small, the number of normalized solutions is at least equal to the number of global maxima of the function $h(x)$. This result established an interesting connection between the spatial structure of the nonlinearity and the multiplicity of solutions. Although these works collectively provide a comprehensive theoretical framework for biharmonic equation with pure-power nonlinearities, they inherently cannot capture the saturation effects that are characteristic of real optical materials.

Before going to our concerns, we mention some results related to saturable nonlinearity due to its physical relevance and mathematical interest. In 2017, Lin *et al*. [8] proved the existence of normalized ground state solutions for the following problem

$$\begin{cases} -\Delta u + \lambda u = \mu \frac{I(x)+|u|^2}{1+I(x)+|u|^2} u, \quad x \in R^N, \\ \int_{R^N} |u|^2 dx = c, \end{cases} \tag{1.3}$$

when $I(x) \neq 0$ and $\mu > 0$ is sufficiently large. In 2020, Wang and Wang [9] proved that, if $I(x)$ is a radially symmetric function, there exist multiple bump normalized solutions for problem (1.3), which are concentrated at the maximal points of $I(x)$. The investigation of fourth order equations with saturable nonlinearities was undertaken by Han [3]. More precisely, for $N \geq 2$ and sufficiently large μ, he not only established radial ground state normalized solutions to the following problem

$$\begin{cases} \Delta^2 u - \Delta u + \lambda u = \mu \frac{I(x)+u^2}{1+I(x)+u^2} u, \\ \int_{R^N} |u|^2 dx = c, \end{cases} \tag{1.4}$$

but also derived explicit bounds for both the ground state energy and the Lagrange multiplier. However, it is worth pointing out that problem (1.4) dose not take into account the influence of the pure-power nonlinearities on the existence of normalized solutions.

Moreover, recent advances in approximation theory and iterative algorithms, such as generalized Stancu-Schurer operators [10], viscosity-based iterative methods for nonlinear analysis [11], and fractional integral-type operators preserving shape properties [12], provide valuable insights and potential numerical tools for future discretization and computational studies of the problem considered here. Although the present work focuses on theoretical existence results, these approximation techniques offer promising avenues for subsequent numerical investigations.

From a physical perspective, the exponent p in the pure-power term $|u|^{p-2}u$ governs the strength of the nonlinear Kerr effect. The different regimes of p correspond to distinct physical scenarios in beam propagation. L^2-subcritical case ($2 < p < 2 + \frac{8}{N}$): the nonlinearity is relatively weak compared to the dispersive effects. Ground state solutions obtained via minimization typically correspond to stable, low-power beam profiles. L^2-critical case ($p = 2 + \frac{8}{N}$): this represents a threshold where the focusing nonlinearity and the dispersion are in a precise balance. The existence of solutions becomes delicate and often depends sensitively on parameters like the prescribed power c. L^2-supercritical case ($2 + 8/N < p < 4^*$): the nonlinearity is dominant. This regime is associated with higher beam powers where strong focusing can lead to complex beam structures and instability, necessitating more advanced mathematical tools like the mountain-pass theorem on constraint manifolds to find solutions.

Motivated by the above discussion, the purpose of this paper is to present reasonable assumptions on c, p and g to guarantee the existence of normalized solutions of problem (1.1). As usual, solutions of problem (1.1) can be obtained as critical points of the energy functional $J: H^2(R^N) \to R$ defined by

$$J(u) := \frac{1}{2}\int_{R^N} |\Delta u|^2 dx - \frac{1}{p}\int_{R^N} |u|^p dx - \frac{\mu}{2}\int_{R^N} \left(|u|^2 - ln\left(1 + \frac{|u|^2}{1+g}\right)\right) dx \tag{1.5}$$

on the constraint

$$S(c) := \left\{u \in H^2(R^N) \middle| \int_{R^N} |u|^2 dx = c\right\}. \tag{1.6}$$

Here, $H^2(R^N)$ is the Sobolev space

$$H^2(R^N) = \{u \in L^2(R^N) : |\Delta u \in L^2(R^N)\},$$

which is equipped with the following norm

$$\|u\|_{H^2(R^N)} := \left(\int_{R^N} |u|^2 dx + \int_{R^N} |\Delta u|^2 dx\right)^{\frac{1}{2}}.$$

Note that J is a well-defined and C^1 functional on $S(c)$ with its Fréchet derivative

$$\langle J'(u), v\rangle = \int_{R^N} \Delta u \cdot \Delta v dx - \int_{R^N} |u|^{p-2} uv dx - \mu \int_{R^N} \frac{g+|u|^2}{1+g+|u|^2} uv dx$$

for any $u, v \in H^2(R^N)$. Thus, one may consider the following minimization problem

$$\sigma(c) := \inf_{u \in S(c)} J(u) \tag{1.7}$$

to get normalized ground states of problem (1.1). Here, we say that u is a ground state of problem (1.1) if it is a solution to problem (1.1) having minimal energy among all the solutions, namely,

$$J|_{S(c)}{}'(u) = 0 \quad and \quad J(u) = inf\{J(v) | J|_{S(c)}{}'(v) = 0 \ and \ v \in S(c)\}.$$

Theorem 1 Let $5 \le N \le 7$, $\mu > 0$ and $-1 < g < 0$. The following statements hold:

- If $2 < p < \overline{p} := 2 + \frac{8}{N}$ and $c > 0$, the infimum defined in (1.7) is achieved by some $w \in S(c)$, which is a ground state of problem (1.1) with the associated Lagrange multiplier $\bar{\lambda} > \frac{\mu g}{1+g}$.
- If $\overline{p} < p < 4^*$, there exists $\bar{c} > 0$ such that for every $c < \bar{c}$, problem (1.1) has a solution$(w, \bar{\lambda}) \in H_r^2(R^N) \times R^+$. In particular, we have

$$\bar{\lambda} > K_2 c^{\frac{8-4p}{Np-2N-8}} - \frac{\mu}{1+g}\left(\frac{p}{p-2} - g\right),$$

where

$$K_2 := \left(\frac{2N-p(N-4)}{Np-2N}\right)\left(\frac{8(N+4)(1+g)^{\frac{\bar{p}}{2}} - 2\mu \boldsymbol{B}_{\bar{p}} N^2 C_{N,\bar{p}}^{\bar{p}} c^{\frac{4}{N}})}{(1+g)^{\frac{\bar{p}}{2}} N(Np-2N)}\right)^{\frac{8}{Np-2N-8}} > 0.$$

This paper is organized as follows. In section 2. we present some preliminary results used to prove our main results. In section 3, we give the detailed proof of Theorem 1.1, divided into two parts: the subcritical case $2 < p < \bar{p}$ and the supercritical case $\bar{p} < p < 4^*$, where variational methods and minimax techniques are employed to eatablish the existence of normalized solutions.

2. Preliminaries

In this section, we recall and present several important inequalities and results that will be frequently used throughout the paper.

Lemma 2 (**Gagliardo-Nirenberg inequality** [13, Section 3]) For every $N \ge 5$, there exists a constant $C_{N,t}$ depending on N and t such that

$$\|u\|_t \le C_{N,t} \|\Delta u\|_2^{\gamma_t} \|u\|_2^{1-\gamma_t}, \quad \forall u \in H^2(R^N), \tag{2.1}$$

where $t \in (2, 4^*]$ and $\gamma_t := \frac{N}{2}(\frac{1}{2} - \frac{1}{t})$.

Lemma 3 For any $a > 0$, the following inequalities hold:

$$ln\left(1 + \frac{s}{a}\right) - \frac{s}{a+s} \ge 0, \quad \forall s \ge 0$$

and

$$ln(1 + s) < s, \quad \forall s > 0.$$

Proof. Define $h(s) := ln(1+\frac{s}{a}) - \frac{s}{a+s}$, $\forall s \geq 0$. Compute its derivative

$$h'(s) = \frac{1}{a+s} - \frac{a}{(a+s)^2} = \frac{s}{(a+s)^2} \geq 0, \forall s \geq 0,$$

which means that $h(s)$ is non-decreasing on $s \geq 0$. Since $h(0) = 0$, it gives that $h(s) \geq h(0) = 0$ for all $s \geq 0$. That is, the first inequality holds.

Define $g(s) := ln(1+s) - s$, $\forall s > 0$. Compute its derivative

$$g'(s) = \frac{1}{1+s} - 1 = -\frac{s}{1+s} < 0, \forall s > 0.$$

Since $g(0) = 0$ and $g(s)$ is strictly decreasing, it follows that $g(s) < g(0) = 0$ for all $s > 0$, which proves the second inequality.

Lemma 4 ([4, Proposition 5.2]) Let $\{u_n\}$ be a sequence bounded in $H^2(R^N)$ such that

$$\lim_{n\to\infty}\left(\sup_{y\in R^N}\int_{B_1(y)}|u_n|^2\,dx\right)=0,$$

where $B_1(y)$ denotes the ball of radius 1 centered at y. Then $u_n \to 0$ in $L^s(R^N)$ for $2 < s < 4^*$.

Lemma 5 (**Generalized Lebesgue Dominated Convergence Theorem** [14, Theorem 2.22]) Suppose Ω is a domain in R^N, $\{u_n\}_{n=1}^{\infty}$ and u are measurable functions in Ω such that $u_n \to u$ a.e. in Ω. Then $u_n \to u$ in $L^1(\Omega)$ if and only if $\{\phi_n\}_{n=1}^{\infty}, \phi \in L^1(\Omega)$ exist such that $\phi_n \to \phi$ a.e. in Ω, $|u_n| \leq \phi_n$ a.e. in Ω for each n, and $\phi_n \to \phi$ in $L^1(\Omega)$.

We now demonstrate two essential estimates on the saturable nonlinearity.

Lemma 6 If $N \geq 5$ and $2 < q < 4$. Then, the following inequality holds

$$s^2 - ln\left(1+\frac{s^2}{1+g}\right) \leq \frac{g}{1+g}s^2 + \frac{\boldsymbol{A}_q}{(1+g)^{\frac{q}{2}}}s^q, \quad \forall s \geq 0,$$

where $\boldsymbol{A}_q$ is given by

$$\boldsymbol{A}_q := \frac{(q-2)^{\frac{q-2}{2}}(4-q)^{\frac{4-q}{2}}}{q}. \tag{2.2}$$

Proof. Let

$$f(s) := \frac{\boldsymbol{A}_q}{(1+g)^{\frac{q}{2}}}s^q - \frac{1}{1+g}s^2 + ln\left(1+\frac{s^2}{1+g}\right), \quad \forall s \geq 0.$$

Clearly, $f(0) = 0$ and a direct calculation shows that

$$\begin{aligned} f'(s) = & \frac{q\boldsymbol{A}_q}{(1+g)^{\frac{q}{2}}}s^{q-1} - \frac{2}{1+g}s + \frac{2s}{1+g+s^2} \\ = & s\left(\frac{q\boldsymbol{A}_q}{(1+g)^{\frac{q}{2}}}s^{q-2} + \frac{2}{1+g+s^2} - \frac{2}{1+g}\right), \quad \forall s \geq 0. \end{aligned}$$

To obtain the desired inequality, it is sufficient to show that $f'(s) \geq 0$ for all $s \geq 0$.

For this purpose, set $t := \frac{s^2}{1+g}$. Then, $f'(s) \geq 0$ is equivalent to

$$\boldsymbol{A}_q \geq \frac{1}{q}\frac{2t}{1+t}t^{\frac{2-q}{2}}, \quad \forall t \geq 0.$$

Utilizing the monotonicity of $\frac{1}{q}\frac{2t}{1+t}t^{\frac{2-q}{2}}$ with respect to $t \geq 0$, we see that its maximum value is exactly $\boldsymbol{A}_q$. That is, we reach the conclusion.

Lemma 7 If $N \geq 5$ and $2 < q < 4$. Then, the following inequality holds

$$\frac{g+s^2}{1+g+s^2}s^2 \leq \frac{g}{1+g}s^2 + \frac{\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}s^q, \quad \forall s \geq 0,$$

where $\boldsymbol{B}_q$ is given by

$$\boldsymbol{B}_q := \frac{2^{\frac{q+6}{2}}(q-2)^{\frac{q-2}{4}}(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{q(\sqrt{q+14}-\sqrt{q-2})^3}. \tag{2.3}$$

Proof. Let

$$k(s) := \frac{\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}s^q - \frac{1}{1+g}s^2 + \frac{s^2}{1+g+s^2}, \quad \forall s \geq 0.$$

Clearly, $k(0) = 0$ and a direct calculation shows that

$$\begin{aligned} k'(s) = &\ \frac{q\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}s^{q-1} - \frac{2}{1+g}s + \frac{2s(1+g)}{(1+g+s^2)^2} \\ = &\ s\left(\frac{q\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}s^{q-2} + \frac{2(1+g)}{(1+g+s^2)^2} - \frac{2}{1+g}\right), \quad \forall s \geq 0. \end{aligned}$$

We aim to show that $k'(s) \geq 0$ for all $s \geq 0$, which implies $k(s) \geq 0$, and hence the desired inequality.

Set $t := \frac{s^2}{1+g}$. Then, $f^{'}(s) \geq 0$ is equivalent to

$$\boldsymbol{B}_q \geq \frac{1}{q}\frac{2(t+2)t^{\frac{4-q}{2}}}{(1+t)^2}, \quad \forall t \geq 0.$$

Based on some direct analysis, we know that the maximum value of $\frac{2(t+2)t^{\frac{4-q}{2}}}{(1+t)^2}$ is precisely $\boldsymbol{B}_q$. As a consequence, the lemma is proved.

Lemma 8 ([8, Lemma 5.2]) Suppose that $h(x) = x - ln(1+x)$, $x \in [0, +\infty)$. For given $\alpha > 0$, $\beta > 0$, and $t \in (0,1)$, if $|x-y| \geq \alpha$ for $0 \leq x \leq \beta$ and $0 \leq y < +\infty$, then there exists $\xi > 0$ such that

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) - \xi.$$

Lemma 9 ([15, Theorem 1.2]) Let $u \in H^2(R^N)$ be a weak solution to the equation

$$\Delta^2 u + \lambda u = |u|^{p-2}u + \mu\frac{g+|u|^2}{1+g+|u|^2}u. \tag{2.4}$$

Then, u satisfies the Pohozaev identity

$$\frac{N-4}{2}\int_{R^N}|\Delta u|^2dx + \frac{N(\lambda-\mu)}{2}\int_{R^N}|u|^2dx = \frac{N}{p}\int_{R^N}|u|^pdx - \frac{\mu N}{2}\int_{R^N}ln\left(1+\frac{|u|^2}{1+g}\right)dx.$$

Furthermore, it holds that

$$\int_{R^N}|\Delta u|^2dx \quad -\frac{Np-2N}{4p}\int_{R^N}|u|^pdx = \frac{\mu N}{4}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{|u|^2}{1+g+|u|^2}\right)dx. \tag{2.5}$$

Following the idea of Soave [16], we introduce a constraint manifold $M(c)$ that contains all the critical points of the functional J restricted to $S(c)$. For each $u \in H^2(R^N)\backslash\{0\}$ and $t>0$, denote by

$$u^t(x):=t^{\frac{N}{2}}u(tx), \forall x \in R^N.$$

A direct calculation gives that

$$\|u^t\|_2^2=\|u\|_2^2, \quad \int_{R^N}|\Delta u^t|^2dx=t^4\int_{R^N}|\Delta u|^2dx\,, \quad \int_{R^N}|u^t|^pdx=t^{\frac{Np-2N}{2}}\int_{R^N}|u|^pdx$$

and

$$\int_{R^N} ln\left(1+\frac{|u^t|^2}{1+g}\right)dx=\frac{1}{t^N}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right)dx.$$

Define the fibering map $t\in(0,\infty)\to f_u(t):=J(u^t)$ as follows

$$f_u(t)=\frac{t^4}{2}\int_{R^N}|\Delta u|^2dx-\frac{t^{\frac{Np-2N}{2}}}{p}\int_{R^N}|u|^pdx-\frac{\mu}{2}\int_{R^N}|u|^2dx+\frac{\mu}{2t^N}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right)dx.$$

Calculating its first and second derivatives, we have

$$\begin{aligned} f_u'(t)= \quad & 2t^3\int_{R^N}|\Delta u|^2dx-\frac{Np-2N}{2p}t^{\frac{Np-2N}{2}-1}\int_{R^N}|u|^pdx \\ & -\frac{\mu N}{2t^{N+1}}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right)dx+\frac{\mu N}{2t}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} f_u''(t)= \quad & 6t^2\int_{R^N}|\Delta u|^2dx-\frac{(Np-2N)(Np-2N-2)}{4p}t^{\frac{Np-2N-4}{2}}\int_{R^N}|u|^pdx \\ & +\frac{\mu N(N+1)}{2t^{N+2}}\int_{R^N} ln(1+\frac{t^N|u|^2}{1+g})dx-\frac{\mu N^2}{2t^2}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx \\ & -\frac{\mu N}{2t^2}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx-\frac{\mu N^2t^{N-2}}{2}\int_{R^N}\frac{|u|^4}{(1+g+t^N|u|^2)^2}dx. \end{aligned}$$

Meanwhile, considering the Pohozaev functional

$$Q(u):= \quad \int_{R^N}|\Delta u|^2dx-\frac{Np-2N}{4p}\int_{R^N}|u|^pdx-\frac{\mu N}{4}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx,$$

we see that

$$\begin{aligned} Q(u^t)= \quad & \int_{R^N}|\Delta u^t|^2dx-\frac{Np-2N}{4p}\int_{R^N}|u^t|^pdx-\frac{\mu N}{4}\int_{R^N}\left(ln\left(1+\frac{|u^t|^2}{1+g}\right)-\frac{|u^t|^2}{1+g+|u^t|^2}\right)dx \\ = \quad & t^4\int_{R^N}|\Delta u|^2dx-\frac{Np-2N}{4p}t^{\frac{Np-2N}{2}}\int_{R^N}|u|^pdx-\frac{\mu N}{4t^N}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right)dx+\frac{\mu N}{4}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx\,. \end{aligned} \tag{2.7}$$

Obviously, (2.6) and (2.7) state that

$$\frac{2Q(u^t)}{t}=f_u'(t)=\frac{d}{dt}J(u^t). \tag{2.8}$$

In particular, $Q(u)=0$ corresponds to the Pohozaev identity (2.5). Hence, we can introduce the following subset of $S(c)$

$$M(c):=\{u\in S(c): Q(u)=0\}=\{u\in S(c): f_u'(1)=0\}.$$

Moreover, from (2.8), we also recognize that, for any $u\in S(c)$, $u^t(x):=t^{\frac{N}{2}}u(tx)$ belongs to $M(c)$ if and only if $t\in R^+$ is a critical point of the fibering map $f_u(t)$, namely $f_u'(t)=0$. To proceed furthermore, we should split $M(c)$ into three parts corresponding to local maxima, local minima and points of inflection, that is,

$$M^+(c) := \{u \in S(c) | f_u'(1) = 0, f_u''(1) > 0\},$$
$$M^0(c) := \{u \in S(c) | f_u'(1) = 0, f_u''(1) = 0\},$$
$$M^-(c) := \{u \in S(c) | f_u'(1) = 0, f_u''(1) < 0\}.$$

Actually, for each $u \in M(c)$, we know that

$$\begin{aligned}
f_u''(1) = \;& 6\int_{R^N} |\Delta u|^2 dx - \frac{(Np-2N)(Np-2N-2)}{4p}\int_{R^N} |u|^p dx + \frac{\mu N(N+1)}{2}\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx \\
& - \frac{\mu N(N+1)}{2}\int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& 6\left(\frac{Np-2N}{4p}\int_{R^N} |u|^p dx + \frac{\mu N}{4}\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{\mu N}{4}\int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx\right) \\
& - \frac{(Np-2N)(Np-2N-2)}{4p}\int_{R^N} |u|^p dx + \frac{\mu N(N+1)}{2}\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx \\
& - \frac{\mu N(N+1)}{2}\int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& -\frac{(Np-2N)(Np-2N-8)}{4p}\int_{R^N} |u|^p dx + \frac{\mu N(N+4)}{2}\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx \\
& - \frac{\mu N(N+4)}{2}\int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& -(Np-2N-8)\int_{R^N} |\Delta u|^2 dx + \frac{\mu N(Np-2N-8)}{4}\left(\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx - \int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx\right) \\
& + \frac{\mu N(N+4)}{2}\left(\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx - \int_{R^N} \frac{|u|^2}{1+g+|u|^2} dx\right) - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& -(Np-2N-8)\int_{R^N} |\Delta u|^2 dx + \frac{\mu N^2 p}{4}\int_{R^N} \left(ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{|u|^2}{1+g+|u|^2}\right) dx \\
& - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& (2N+8)\int_{R^N} |\Delta u|^2 dx - Np\int_{R^N} |\Delta u|^2 dx + \frac{\mu N^2 p}{4}\int_{R^N} \left(ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{|u|^2}{1+g+|u|^2}\right) dx \\
& - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& (2N+8)\int_{R^N} |\Delta u|^2 dx - Np\left(\int_{R^N} |\Delta u|^2 dx - \frac{\mu N}{4}\int_{R^N} \left(ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{|u|^2}{1+g+|u|^2}\right) dx\right) \\
& - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx \\
= \;& (2N+8)\int_{R^N} |\Delta u|^2 dx - \frac{N(Np-2N)}{4}\int_{R^N} |u|^p dx - \frac{\mu N^2}{2}\int_{R^N} \frac{|u|^4}{(1+g+|u|^2)^2} dx.
\end{aligned} \tag{2.9}$$

3. Proof of Theorem 1

3.1. $2 < p < \bar{p}$

Lemma 10 Assume that $2 < p < \bar{p}$, $\mu > 0$, $g > -1$ and $c > 0$. Then, J is bounded from below and coercive on $S(c)$.

Proof. For any $u \in S(c)$, in view of (2.1), we see that

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{R^N} |\Delta u|^2 dx - \frac{1}{p}\int_{R^N} |u|^p dx - \frac{\mu}{2}\int_{R^N} |u|^2 dx + \frac{\mu}{2}\int_{R^N} ln\left(1+\frac{|u|^2}{1+g}\right) dx \\
&\geq \frac{1}{2}\int_{R^N} |\Delta u|^2 dx - \frac{1}{p} C_{N,p}^p \left(\int_{R^N} |\Delta u|^2 dx\right)^{\frac{N(p-2)}{8}} c^{\frac{2N-p(N-4)}{8}} - \frac{\mu c}{2}.
\end{aligned}$$

When $2 < p < \bar{p}$, we have $\frac{N(p-2)}{8} < 1$, which implies that J is coercive and bounded from below on $S(c)$.

Now we are ready to prove Theorem 1 (i).

Let $\{u_n\} \subset S(c)$ be a minimizing sequence for $\sigma(c)$. Then, $\{u_n\}$ is bounded in $H^2(R^N)$ by Lemma 10. First of all, we claim that

$$\eta := \lim_{n\to\infty} \sup_{y\in R^N} \int_{B_1(y)} |u_n|^2\, dx > 0. \tag{3.1}$$

Assume on the contrary that $\eta = 0$. Lemma 4 infers that $\|u_n\|_s \to 0$ as $n \to \infty$ for $2 < s < 4^*$. Together with Lemma 6 with $2 < q < 4$, it yields that

$$\begin{aligned}\sigma(c) + o(1) &= J(u_n)\\ &= \frac{1}{2}\int_{R^N} |\Delta u_n|^2 dx - \frac{1}{p}\int_{R^N} |u_n|^p dx - \frac{\mu}{2}\int_{R^N}\left(|u_n|^2 - ln\left(1+\frac{|u_n|^2}{1+g(x)}\right)\right) dx\\ &= \frac{1}{2}\int_{R^N} |\Delta u_n|^2 dx - \frac{\mu g}{2(1+g)}\int_{R^N} |u_n|^2 dx - \frac{\mu A_q}{2(1+g)^{\frac{q}{2}}}\int_{R^N} |u_n|^q dx + o(1)\\ &\geq -\frac{\mu g c}{2(1+g)} + o(1).\end{aligned}$$

That is to say, $\sigma \geq -\frac{\mu g c}{2(1+g)}$.

Fixing $u \in S(c)$ and using Lemma 3, we have

$$\begin{aligned}\sigma(c) \leq J(u^t) &= \frac{t^4}{2}\int_{R^N} |\Delta u|^2 dx - \frac{t^{\frac{Np-2N}{2}}}{p}\int_{R^N} |u|^p dx - \frac{\mu c}{2} + \frac{\mu}{2t^N}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right) dx\\ &< \frac{t^4}{2}\int_{R^N} |\Delta u|^2 dx - \frac{t^{\frac{Np-2N}{2}}}{p}\int_{R^N} |u|^p dx - \frac{\mu c}{2} + \frac{\mu}{2}\int_{R^N}\frac{|u|^2}{1+g} dx\\ &= \frac{t^4}{2}\int_{R^N} |\Delta u|^2 dx - \frac{t^{\frac{Np-2N}{2}}}{p}\int_{R^N} |u|^p dx - \frac{\mu c}{2}\left(1-\frac{1}{1+g}\right).\end{aligned} \tag{3.2}$$

Since $2 < p < \bar{p}$, the above inequality implies that

$$\sigma(c) < -\frac{\mu g c}{2(1+g)}, \tag{3.3}$$

which leads to an obvious contradiction. Thus, (3.1) holds.

According to (3.1), we can choose $\{y_n\} \subset R^N$ to guarantee that

$$\int_{B_1(y_n)} |u_n|^2 dx \geq \frac{\eta}{2}. \tag{3.4}$$

Let $\{w_n(x) := u_n(x+y_n)\}$, it gives that

$$\int_{R^N} |\Delta w_n|^2 dx = \int_{R^N} |\Delta u_n|^2 dx, \quad \int_{R^N} |w_n|^p dx = \int_{R^N} |u_n|^p dx,$$

and

$$\int_{R^N}\left(|w_n|^2 - ln\left(1+\frac{|w_n|^2}{1+g}\right)\right) dx = \int_{R^N}\left(|u_n|^2 - ln\left(1+\frac{|u_n|^2}{1+g}\right)\right) dx.$$

Therefore, $\{w_n\}$ is also a bounded minimizing sequence for $\sigma(c)$ on $S(c)$ and

$$\lim_{n\to\infty}\int_{B_1(0)} |w_n|^2\, dx \geq \frac{\eta}{2}.$$

Hence, we can assume that $w_n \rightharpoonup w$ in $H^2(R^N)$, $w_n \to w \neq 0$ in $L^2(B_1(0))$ and $w_n(x) \to w(x)$ a.e. on $B_1(0)$. On the basis of Egoroff's theorem, we can find a constant $\delta > 0$ such that

$$w_n(x) \to w(x) \text{ uniformly in } E \text{ and } meas(E) > 0, \tag{3.5}$$

where $E \subset \{x: |w(x)| \geq \delta, x \in B_1(0)\} \subset B_1(0)$.

Next, we prove that $\|w\|_2^2 = c$. Assume on the contrary that $\rho := \|w\|_2^2 \in (0, c)$. Let

$$\widetilde{w} := \frac{w}{\sqrt{1+g}} \text{ and } \tilde{v}_n := \frac{w_n - w}{\sqrt{1+g}},$$

then, from (3.5), it brings that

$$\widetilde{w}^2 = \frac{w^2}{1+g} \geq \frac{\delta^2}{1+g} > 0 \text{ in } E, \tag{3.6}$$

and

$$\tilde{v}_n^2 = \frac{(w_n - w)^2}{1+g} \to 0 \text{ in } E. \tag{3.7}$$

As a direct application of Lemma 8, we can find a constant $\xi > 0$ such that

$$\int_E h\left(\frac{\rho}{c}\frac{(\sqrt{c}\widetilde{w})^2}{PwP_2^2} + \frac{c-\rho}{c}\frac{(\sqrt{c}\tilde{v}_n)^2}{Pw_n - wP_2^2}\right)dx \leq -\xi + \frac{\rho}{c}\int_E h\left(\frac{(\sqrt{c}\widetilde{w})^2}{PwP_2^2}\right)dx + \frac{c-\rho}{c}\int_E h\left(\frac{(\sqrt{c}\tilde{v}_n)^2}{Pw_n - wP_2^2}\right)dx. \tag{3.8}$$

Meanwhile, observe that

$$\|w_n\|_p^p = \|w_n - w\|_p^p + \|w\|_p^p + o_n(1) \tag{3.9}$$

and

$$\int_{R^N} \frac{|w_n|^2}{1+g} dx = \int_{R^N} |\widetilde{w}|^2 + |\tilde{v}_n|^2 dx + o(1),$$

there holds that

$$\begin{aligned}
\sigma(c) &= J(w_n) + o(1) \\
&= \frac{1}{2}\int_{R^N} |\Delta w_n|^2 dx - \frac{1}{p}\int_{R^N} |w_n|^p dx - \frac{\mu}{2}\int_{R^N}\left(|w_n|^2 - ln\left(1 + \frac{|w_n|^2}{1+g}\right)\right)dx + o(1) \\
&= \frac{\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^2 dx + \frac{c-\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^2 dx \\
&\quad - \frac{1}{p}\left(\frac{\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^p dx - \frac{1}{p}\left(\frac{c-\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^p dx \\
&\quad - \frac{\mu g}{2}\int_{R^N}\frac{|w_n|^2}{1+g}dx - \frac{\mu}{2}\int_{R^N}\left(\frac{|w_n|^2}{1+g} - ln\left(1 + \frac{|w_n|^2}{1+g}\right)\right)dx + o(1) \\
&= \frac{\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^2 dx + \frac{c-\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^2 dx \\
&\quad - \frac{1}{p}\left(\frac{\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^p dx - \frac{1}{p}\left(\frac{c-\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^p dx \\
&\quad - \frac{\mu g}{2}\int_{R^N}\frac{|w_n|^2}{1+g}dx - \frac{\mu}{2}\int_{R^N} h\left(\frac{|w_n|^2}{1+g}\right)dx + o(1) \\
&= \frac{\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^2 dx + \frac{c-\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^2 dx \\
&\quad - \frac{1}{p}\left(\frac{\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^p dx - \frac{1}{p}\left(\frac{c-\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^p dx \\
&\quad - \frac{\mu g}{2}\int_{R^N}\frac{|w_n|^2}{1+g}dx - \frac{\mu}{2}\int_{R^N} h\left(\frac{\rho}{c}\left(\frac{(\sqrt{c}|\widetilde{w}|)^2}{PwP_2^2}\right) + \frac{c-\rho}{c}\frac{(\sqrt{c}|\tilde{v}_n|)^2}{Pw_n - wP_2^2}\right)dx + o(1) \\
&= \frac{\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^2 dx + \frac{c-\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^2 dx \\
&\quad - \frac{1}{p}\left(\frac{\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^p dx - \frac{1}{p}\left(\frac{c-\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^p dx \\
&\quad - \frac{\mu}{2}\int_{R^N} h\left(\frac{\rho}{c}\frac{(\sqrt{c}|\widetilde{w}|)^2}{PwP_2^2} + \left(\frac{c-\rho}{c}\right)\frac{(\sqrt{c}|\tilde{v}_n|)^2}{Pw_n - wP_2^2}\right)dx \\
&\quad - \frac{\mu g}{2}\int_{R^N}(|\widetilde{w}|^2 + |\tilde{v}_n|^2)dx + o(1) \\
&\geq \frac{\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^2 dx + \frac{c-\rho}{2c}\int_{R^N}\left|\Delta\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^2 dx \\
&\quad - \frac{1}{p}\left(\frac{\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}w}{PwP_2}\right)\right|^p dx - \frac{1}{p}\left(\frac{c-\rho}{c}\right)^{\frac{p}{2}}\int_{R^N}\left|\left(\frac{\sqrt{c}(w_n-w)}{Pw_n - wP_2}\right)\right|^p dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu\xi}{2}-\frac{\mu}{2}\left(\frac{\rho}{c}\int_E h\left(\frac{(\sqrt{c}\widetilde{w})^2}{PwP_2^2}\right)dx+\frac{c-\rho}{c}\int_E h\left(\frac{(\sqrt{c}\tilde{v}_n)^2}{Pw_n-wP_2^2}\right)dx\right)\\
&-\frac{\mu g}{2}\int_{R^N}(|\widetilde{w}|^2+|\tilde{v}_n|^2)\,dx+o(1)\\
&\geq\frac{\rho}{c}J\left(\frac{\sqrt{c}w}{PwP_2}\right)+\frac{c-\rho}{c}J\left(\frac{\sqrt{c}(w_n-w)}{Pw_n-wP_2}\right)+\frac{\mu\xi}{2}+o(1)\\
&\geq\frac{\rho}{c}\sigma(c)+\frac{c-\rho}{c}\sigma(c)+\frac{\mu\xi}{2}+o(1)\\
&\geq\sigma(c)+\frac{\mu\xi}{2}+o(1),
\end{aligned}
$$

which is a contradiction. That is, $w_n \to w$ in $L^2(R^N)$. Hence, combining with Lemma 2 and Lemma 5, we derive that

$$
\lim_{n\to\infty}\int_{\mathrm{R}^N}\left(|w_n|^2-\ln\left(1+\frac{|w_n|^2}{1+g}\right)\right)dx=\int_{\mathrm{R}^N}\left(|w|^2-\ln\left(1+\frac{|w|^2}{1+g}\right)\right)dx \tag{3.10}
$$

and

$$
\lim_{n\to\infty}\int_{\mathrm{R}^N}|w_n|^p\,dx=\int_{\mathrm{R}^N}|w|^p\,dx \tag{3.11}
$$

for $2 < p \leq \bar{p}$. Moreover, due to $w_n \rightharpoonup w$ in $H^2(R^N)$, we see that

$$
\int_{\mathrm{R}^N}|\Delta w|^2\,dx\leq\liminf_{n\to\infty}\int_{\mathrm{R}^N}|\Delta w_n|^2\,dx. \tag{3.12}
$$

Consequently, it follows from (3.10)-(3.12) that

$$
\begin{aligned}
\sigma(c)&=\lim_{n\to\infty}J(w_n)\\
&=\lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathrm{R}^N}|\Delta w_n|^2\,dx-\frac{1}{p}\int_{\mathrm{R}^N}|w_n|^p\,dx-\frac{\mu}{2}\int_{\mathrm{R}^N}\left(|w_n|^2-\ln\left(1+\frac{|w_n|^2}{1+g}\right)\right)dx\right)\\
&\geq\frac{1}{2}\int_{\mathrm{R}^N}|\Delta w|^2\,dx-\frac{1}{p}\int_{\mathrm{R}^N}|w|^p\,dx-\frac{\mu}{2}\int_{\mathrm{R}^N}\left(|w|^2-\ln\left(1+\frac{|w|^2}{1+g}\right)\right)dx\\
&\geq\sigma(c),
\end{aligned}
$$

which indicates that $\sigma(c)$ is achieved at $w \neq 0$ and $\|w_n - w\|_{H^2} \to 0$ as $n \to \infty$.

Since w is a minimizer of J restricted to $S(c)$, there exists a Lagrange multiplier $\bar{\lambda} \in R$ such that

$$
\begin{aligned}
\bar{\lambda}c=&\ -\int_{R^N}|\Delta w|^2dx+\int_{R^N}|w|^pdx+\mu\int_{R^N}\frac{g+|w|^2}{1+g+|w|^2}|w|^2dx\\
=&\ -2\sigma(c)-\frac{2}{p}\int_{R^N}|w|^pdx-\mu\int_{R^N}\left(|w|^2-ln\left(1+\frac{|w|^2}{1+g}\right)\right)dx+\int_{R^N}|w|^pdx+\mu\int_{R^N}\frac{g+|w|^2}{1+g+|w|^2}|w|^2dx\\
=&\ -2\sigma(c)-\frac{2}{p}\int_{R^N}|w|^pdx-\mu\int_{R^N}\left(|w|^2-ln\left(1+\frac{|w|^2}{1+g}\right)\right)dx+\int_{R^N}|w|^pdx\\
&+\mu\int_{R^N}|w|^2dx-\mu\int_{R^N}\frac{|w|^2}{1+g+|w|^2}dx\\
=&\ -2\sigma(c)+\frac{p-2}{p}\int_{R^N}|w|^pdx+\mu\int_{R^N}\left(ln\left(1+\frac{|w|^2}{1+g}\right)-\frac{|w|^2}{1+g+|w|^2}\right)dx\\
>&\ -2\sigma(c),
\end{aligned}
$$

where we have used Lemma 3 in the last inequality. This indicates that $\bar{\lambda} > \frac{\mu g}{1+g}$ by (3.3). The proof is completed.

3.2. $\bar{p} < p < 4^*$

In this subsection, we consider the case of $\bar{p} < p < 4^*$. For this situation, J is unbounded from below on $S(c)$, and it is impossible to look for a global minimizer on $S(c)$. To achieve our purpose, we shall use the Pohozaev manifold $M(c)$ defined in Section 2 to find critical points of J.

Next, we firstly give a general minimax theorem to establish the existence of a Palais-Smale sequence.

Definition 11 ([17, Definition 3.1]) Let Θ be a closed subset of X. We shall say that a class F of compact subsets of X is a homotopy-stable family with closed boundary Θ provided

- every set in F contains Θ;
- for any set $H \in F$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(s,x) = x$ for all $(s,x) \in (\{0\} \times X) \cup ([0,1] \times \Theta)$, we have that $\eta(\{1\} \times H) \in F$.

Lemma 12 ([17, Theorem 3.2]) Let ϕ be a C^1-functional on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy stable family F of compact subsets of X with a closed boundary Θ. Set

$$c := c(\varphi, \mathsf{F}) := \inf_{H \in \mathsf{F}} \max_{u \in H} \varphi(u),$$

and suppose that

$$sup\, \phi\, (\Theta) < c.$$

Then, for any sequence of sets $\{H_n\}$ in F such that $\lim_{n\to\infty} \sup_{H_n} \varphi = c$, there exists a sequence $\{u_n\}$ in X such that

- $\lim_{n\to\infty} \varphi(u_n) = c$;
- $\lim_{n\to\infty} \|d\varphi(u_n)\| = 0$;
- $\lim_{n\to\infty} dist(u_n, H_n) = 0$.

Moreover, if dϕ is uniformly continuous, then u_n can be chosen to be in H_n for each n.

Lemma 13 Assume that $\bar{p} < p < 4^*$, $-1 < g < 0$ and let $\{u_n\} \subset M^-(c) \cap H_r^2(R^N)$ be a bounded Palais-Smale sequence for J restricted to $S(c)$ at level β. In addition, denote by

$$c_0 := \left(\frac{4(N+4)(1+g)^{\frac{\bar{p}}{2}}}{\mu \boldsymbol{B}_{\bar{p}} N^2 C_{N,\bar{p}}^{\bar{p}}} \right)^{\frac{N}{4}}, \tag{3.13}$$

$$\Lambda_c := \left(\frac{2(4(N+4)(1+g)^{\frac{\bar{p}}{2}} - \mu \boldsymbol{B}_{\bar{p}} N^2 C_{N,\bar{p}}^{\bar{p}} c^{\frac{4}{N}})}{c^{\frac{2N-p(N-4)}{8}} C_{N,p}^{p} (1+g)^{\frac{\bar{p}}{2}} N(Np-2N)} \right)^{\frac{8}{Np-2N-8}}, \tag{3.14}$$

$$c_1 := \frac{(1+g)(2N-p(N-4))\Lambda_c}{2\mu N - \mu g(Np-2N)}$$

and suppose that the following conditions hold

$$\beta > \frac{\mu |g| c}{2(1+g)} \quad and \quad 0 < c < min\{c_0, c_1\}.$$

Then, up to a subsequence, $u_n \to u_0$ strongly in $H^2(R^N)$ and $u_0 \in S(c)$ is a solution of problem (1.1) for some $\bar{\lambda} > 0$.

Proof. Since $\{u_n\} \subset M^-(c)$ is bounded and the embedding $H_r^2(R^N) \hookrightarrow L^s(R^N)$ $(N \geq 5)$ is compact for $s \in (2, 4^*)$, there exists $u_0 \in H_r^2(R^N)$ such that $u_n \rightharpoonup u_0$ weakly in $H_r^2(R^N)$, $u_n \to u_0$ strongly in $L^s(R^N)$ for $s \in (2, 4^*)$, and a.e. in R^N. By the Lagrange multiplier rule, there exists $\lambda_n \in R$ such that for every $\phi \in H^2(R^N)$,

$$\int_{R^N}(\Delta u_n\Delta\phi+\lambda_n u_n\phi)dx-\int_{R^N}|u_n|^{p-2}u_n\phi dx-\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n\phi dx=o(1)\|\phi\|, \tag{3.15}$$

where $o(1)\to 0$ as $n\to\infty$. In other words, u_n solves

$$\Delta^2 u_n+\lambda_n u_n=|u_n|^{p-2}u_n+\mu\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n+o(1). \tag{3.16}$$

In particular, one has

$$\lambda_n c=-\int_{R^N}|\Delta u_n|^2dx+\int_{R^N}|u_n|^p dx+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1). \tag{3.17}$$

Then, noting that

$$|\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx|\le|\mu\int_{R^N}|u_n|^2dx|=|\mu|c \tag{3.18}$$

and using the Gagliardo-Nirenberg inequality, we see that $\{\lambda_n\}$ is bounded, since $\{u_n\}\subset M^-(c)$ is bounded. So, we are able to assume that $\lambda_n\to\bar{\lambda}\in R$ as $n\to\infty$.

In the following, we shall determine the sign of $\bar{\lambda}$. In fact, Lemma 7 brings that

$$\begin{aligned}\frac{s^4}{(1+g+s^2)^2}&\le\frac{s^4}{(1+g+s^2)(1+g)}\\&=\frac{g+s^2}{1+g+s^2}s^2-\frac{g}{1+g}s^2\\&\le\frac{\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}s^q,\quad\forall s\ge 0,\end{aligned} \tag{3.19}$$

where $2<q<4$. Then, for $\{u_n\}\subset M^-(c)\cap H_r^2(R^N)$, thanks to (2.9), (3.19) and Lemma 2, we infer that

$$\begin{aligned}\int_{R^N}|\Delta u_n|^2dx\le\;&\frac{N(Np-2N)}{8(N+4)}\int_{R^N}|u_n|^p dx+\frac{\mu N^2}{4(N+4)}\int_{R^N}\frac{|u_n|^4}{(1+g+|u_n|^2)^2}dx\\\le\;&\frac{N(Np-2N)}{8(N+4)}\int_{R^N}|u_n|^p dx+\frac{\mu N^2}{4(N+4)}\int_{R^N}\frac{\boldsymbol{B}_q}{(1+g)^{\frac{q}{2}}}|u_n|^q dx\\<\;&\frac{C_{N,p}^p N(Np-2N)}{8(N+4)}c^{\frac{2N-p(N-4)}{8}}\left(\int_{R^N}|\Delta u_n|^2\,dx\right)^{\frac{Np-2N}{8}}\\&+\frac{\mu\boldsymbol{B}_q N^2C_{N,q}^q}{4(N+4)(1+g)^{\frac{q}{2}}}c^{\frac{4q-N(q-2)}{8}}\left(\int_{R^N}|\Delta u_n|^2\,dx\right)^{\frac{Nq-2N}{8}}.\end{aligned}$$

In the sequel, choose $q=2+\frac{8}{N}=\bar{p}$. Then, the above inequality becomes

$$\int_{R^N}|\Delta u_n|^2dx<\frac{C_{N,p}^p N(Np-2N)}{8(N+4)}c^{\frac{2N-p(N-4)}{8}}\left(\int_{R^N}|\Delta u_n|^2dx\right)^{\frac{Np-2N}{8}}+\frac{\mu\boldsymbol{B}_{\bar{p}}N^2C_{N,\bar{p}}^{\bar{p}}}{4(N+4)(1+g)^{\frac{\bar{p}}{2}}}c^{\frac{4}{N}}\int_{R^N}|\Delta u_n|^2dx.$$

On the assumption of $0<c<c_0$, it immediately signifies that

$$\int_{R^N}|\Delta u_n|^2dx>\Lambda_c>0. \tag{3.20}$$

Therefore, taking into account that $Q(u_n)=0$, Lemma 3 and (3.20), we deduce that

$$
\begin{aligned}
\lambda_n c = &\ -\int_{R^N}|\Delta u_n|^2dx+\int_{R^N}|u_n|^pdx+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1)\\
= &\ -\int_{R^N}|\Delta u_n|^2dx+\frac{4p}{Np-2N}\left(\int_{R^N}|\Delta u_n|^2dx-\frac{\mu N}{4}\int_{R^N}\left(ln\left(1+\frac{|u_n|^2}{1+g}\right)-\frac{|u_n|^2}{1+g+|u_n|^2}\right)dx\right)\\
&+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|u_n|^2}{1+g}\right)-\frac{|u_n|^2}{1+g+|u_n|^2}\right)dx\\
&+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1)\\
> &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\left(\frac{|u_n|^2}{1+g}-\frac{|u_n|^2}{1+g+|u_n|^2}\right)dx\\
&+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx\\
&+\mu\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}|u_n|^2dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx\\
&+\mu\int_{R^N}\frac{(1+g)(g|u_n|^2+|u_n|^4)}{(1+g)(1+g+|u_n|^2)}dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx\\
&+\mu\int_{R^N}\frac{g|u_n|^2+|u_n|^4+g^2|u_n|^2+g|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx\\
&+\mu\int_{R^N}\frac{g|u_n|^2(1+g+|u_n|^2)+|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx-\frac{\mu Np}{Np-2N}\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx\\
&+\frac{\mu g}{1+g}\int_{R^N}|u_n|^2dx+\mu\int_{R^N}\frac{|u_n|^4}{(1+g)(1+g+|u_n|^2)}dx+o(1)\\
= &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta u_n|^2dx+\frac{\mu g}{1+g}\int_{R^N}|u_n|^2dx\\
&-\frac{2\mu N}{(1+g)(Np-2N)}\int_{R^N}\frac{|u_n|^4}{1+g+|u_n|^2}dx+o(1)\\
\geq &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\frac{\mu g}{1+g}c-\frac{2\mu N}{(1+g)(Np-2N)}c+o(1),
\end{aligned}
$$

which implies that

$$\bar{\lambda}\geq\frac{2N-p(N-4)}{c(Np-2N)}\Lambda_c+\frac{\mu g}{1+g}-\frac{2\mu N}{(1+g)(Np-2N)}.$$

Furthermore, taking advantage of the assumption $0<c<c_1$, there holds that

$$\bar{\lambda}\geq\frac{2N-p(N-4)}{c(Np-2N)}\Lambda_c+\frac{\mu g}{1+g}-\frac{2\mu N}{(1+g)(Np-2N)}>0.$$

Next, we claim that $u_0\neq 0$. Assume on the contrary. Then, by compact embedding of $H_r^2(R^N)\hookrightarrow L^s(R^N)$ with $2<s<4^*$ ($N\geq 5$), we have $\int_{R^N}|u_n|^pdx=o(1)$. Subsequently, with the help of $Q(u_n)=0$ and Lemma 3, it yields that

$$
\begin{aligned}
\beta + o(1) = \;& J(u_n) \\
= \;& \frac{1}{2}\int_{R^N} |\Delta u_n|^2 dx - \frac{1}{p}\int_{R^N} |u_n|^p dx - \frac{\mu}{2}\int_{R^N}(|u_n|^2 - ln(1+\frac{|u_n|^2}{1+g}))dx \\
= \;& \frac{1}{2}(\frac{Np-2N}{4p}\int_{R^N} |u_n|^p dx + \frac{\mu N}{4}\int_{R^N}\left(ln\left(1+\frac{|u_n|^2}{1+g}\right) - \frac{|u_n|^2}{1+g+|u_n|^2}\right)dx) \\
& -\frac{1}{p}\int_{R^N} |u_n|^p dx - \frac{\mu}{2}\int_{R^N}\left(|u_n|^2 - ln\left(1+\frac{|u_n|^2}{1+g}\right)\right)dx \\
= \;& \frac{Np-2N-8}{8p}\int_{R^N} |u_n|^p dx + \frac{\mu N}{8}\int_{R^N}\left(ln\left(1+\frac{|u_n|^2}{1+g}\right) - \frac{|u_n|^2}{1+g+|u_n|^2}\right)dx \\
& -\frac{\mu}{2}\int_{R^N}\left(|u_n|^2 - ln\left(1+\frac{|u_n|^2}{1+g}\right)\right)dx \\
\le \;& -\frac{\mu}{2}\int_{R^N}\left(|u_n|^2 - \frac{|u_n|^2}{1+g}\right)dx + \frac{\mu N}{8}\int_{R^N}\left(\frac{1}{1+g} - \frac{1}{1+g+|u_n|^2}\right)|u_n|^2 dx + o(1) \\
= \;& -\frac{\mu}{2}\int_{R^N}\left(|u_n|^2 - \frac{|u_n|^2}{1+g}\right)dx + \frac{\mu N}{8}\int_{R^N}\left(\frac{1+g-g}{1+g} - \frac{1+g+|u_n|^2-(g+|u_n|^2)}{1+g+|u_n|^2}\right)|u_n|^2 dx + o(1) \\
= \;& -\frac{\mu g c}{2(1+g)} + \frac{\mu N}{8}\int_{R^N}\left(\frac{g+|u_n|^2}{1+g+|u_n|^2} - \frac{g}{1+g}\right)|u_n|^2 dx + o(1) \\
\le \;& -\frac{\mu g c}{2(1+g)} + \frac{\mu \boldsymbol{B}_q N}{8(1+g)^{\frac{q}{2}}}\int_{R^N} |u_n|^q dx + o(1) \\
= \;& -\frac{\mu g c}{2(1+g)} + o(1),
\end{aligned}
$$

where we have used Lemma 7 with $2 < q < 4$. Clearly, this leads to a contradiction with $\beta > \frac{\mu|g|c}{2(1+g)}$, and so $u_0 \neq 0$.

Finally, let us prove that $u_n \to u_0$ in $H^2(R^N)$. Since $u_n \to u_0$ in $H_r^2(R^N)$ and $\lambda_n \to \bar{\lambda} > 0$ as $n \to \infty$, by (3.15), one has

$$\int_{R^N}(\Delta u_0 \Delta\phi + \bar{\lambda} u_0 \phi)dx - \int_{R^N} |u_0|^{p-2} u_0 \phi dx - \mu\int_{R^N} \frac{g+|u_0|^2}{1+g+|u_0|^2} u_0 \phi dx = 0, \quad \forall \phi \in H_r^2(R^N). \tag{3.21}$$

Taking $\phi := u_n - u_0$ in (3.15) and (3.21), and subtracting, we arrive at

$$
\begin{aligned}
o(1) = \;& \int_{R^N}(|\Delta(u_n-u_0)|^2 + \bar{\lambda}|u_n-u_0|^2)dx - \int_{R^N}(|u_n|^{p-2}|u_n| - |u_0|^{p-2}|u_0|)(u_n-u_0)dx \\
& -\mu\int_{R^N}\left(\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n - \frac{g+|u_0|^2}{1+g+|u_0|^2}u_0\right)(u_n-u_0)dx.
\end{aligned} \tag{3.22}
$$

Observe that

$$\int_{R^N}(|u_n|^{p-2}|u_n| - |u_0|^{p-2}|u_0|)(u_n-u_0)dx = o(1), \tag{3.23}$$

to finish the proof, it suffices to demonstrate that

$$\int_{R^N}\left(\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n - \frac{g+|u_0|^2}{1+g+|u_0|^2}u_0\right)(u_n-u_0)dx = o(1). \tag{3.24}$$

In fact, by the Hölder's inequality, we have

$$
\begin{aligned}
& \left|\int_{R^N}\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n(u_n-u_0)dx\right| \\
& \le \left(\int_{R^N}|u_n-u_0|^p dx\right)^{\frac{1}{p}}\left(\int_{R^N}|\frac{g+|u_n|^2}{1+g+|u_n|^2}u_n|^q dx\right)^{\frac{1}{q}} \\
& \le \left(\int_{R^N}|u_n-u_0|^p dx\right)^{\frac{1}{p}}\left(\int_{R^N}|\frac{g+|u_n|^2}{1+g+|u_n|^2}|^q|u_n|^q dx\right)^{\frac{1}{q}} \\
& \le \frac{1}{1+g}\left(\int_{R^N}|u_n-u_0|^p dx\right)^{\frac{1}{p}}\left(\int_{R^N}|u_n|^{2q}|u_n|^q dx\right)^{\frac{1}{q}} \\
& = \frac{1}{1+g}\left(\int_{R^N}|u_n-u_0|^p dx\right)^{\frac{1}{p}}\left(\int_{R^N}|u_n|^{3q} dx\right)^{\frac{1}{q}} \\
& \le \tilde{C}\left(\int_{R^N}|u_n-u_0|^p dx\right)^{\frac{1}{p}} \\
& \to 0,
\end{aligned} \tag{3.25}
$$

where $max\{2,\frac{2N}{12-N}\} < p < 4^*$. Similarly, we also have

$$\begin{aligned}
&\left|\int_{R^N}\frac{g+|u_0|^2}{1+g+|u_0|^2}u_n\,(u_n-u_0)dx\right| \\
&\le \left(\int_{R^N}|u_n-u_0|^p\,dx\right)^{\frac{1}{p}}\left(\int_{R^N}|\frac{g+|u_0|^2}{1+g+|u_0|^2}u_n|^q\,dx\right)^{\frac{1}{q}} \\
&\le \left(\int_{R^N}|u_n-u_0|^p\,dx\right)^{\frac{1}{p}}\left(\int_{R^N}|\frac{g+|u_0|^2}{1+g+|u_0|^2}|^q|u_n|^q\,dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{1+g}\left(\int_{R^N}|u_n-u_0|^p\,dx\right)^{\frac{1}{p}}\left(\int_{R^N}|u_0|^{2q}|u_n|^q\,dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{1+g}\left(\int_{R^N}|u_n-u_0|^p\,dx\right)^{\frac{1}{p}}\left(\int_{R^N}|u_0|^{3q}d\,x\right)^{\frac{2}{3q}}\left(\int_{R^N}|u_n|^3\,qdx\right)^{\frac{1}{3q}} \\
&\le \tilde{C}\left(\int_{R^N}|u_n-u_0|^p\,dx\right)^{\frac{1}{p}} \\
&\to 0,
\end{aligned} \tag{3.26}$$

where $max\{2,\frac{2N}{12-N}\} < p < 4^*$. Together with (3.25) and (3.26) guarantee that (3.24 holds. As a consequence, it follows from (3.22)-(3.24) that

$$\int_{R^N}(|\Delta(u_n-u_0)|^2+\bar{\lambda}|u_n-u_0|^2)dx = o(1),$$

which implies that $u_n \to u_0$ in $H^2(R^N)$, since $\bar{\lambda} > 0$. This concludes the proof.

Lemma 14 Assume that $\bar{p} < p < 4^*$. Then, J is coercive and bounded from below on $M(c)$ for all $c > 0$. Furthermore, there exists a constant $c_2 > 0$ such that for $0 < c < c_2$, J is bounded from below by a positive constant on $M^-(c)$.

Proof. For each $u \in M(c)$, taking advantage of Lemma 3, we see that

$$\begin{aligned}
J(u) =\ & \frac{1}{2}\int_{R^N}|\Delta u|^2dx-\frac{1}{p}\int_{R^N}|u|^pdx-\frac{\mu}{2}\int_{R^N}\left(|u|^2-ln\left(1+\frac{|u|^2}{1+g}\right)\right)dx \\
=\ & \frac{1}{2}\int_{R^N}|\Delta u|^2dx-\frac{4}{Np-2N}\int_{R^N}|\Delta u|^2dx+\frac{\mu N}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx \\
& -\frac{\mu}{2}\int_{R^N}\left(|u|^2-ln\left(1+\frac{|u|^2}{1+g}\right)\right)dx \\
=\ & \left(\frac{1}{2}-\frac{4}{Np-2N}\right)\int_{R^N}|\Delta u|^2dx+\frac{\mu N}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx \\
& -\frac{\mu}{2}\int_{R^N}\frac{g+|u|^2}{1+g+|u|^2}|u|^2dx-\frac{\mu}{2}\int_{R^N}\frac{1}{1+g+|u|^2}|u|^2dx+\frac{\mu}{2}\int_{R^N}ln\left(1+\frac{|u|^2}{1+g}\right)dx \\
=\ & \left(\frac{1}{2}-\frac{4}{Np-2N}\right)\int_{R^N}|\Delta u|^2dx+\frac{\mu N}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx \\
& +\frac{\mu}{2}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx-\frac{\mu}{2}\int_{R^N}\frac{g+|u|^2}{1+g+|u|^2}|u|^2\,dx \\
=\ & \left(\frac{1}{2}-\frac{4}{Np-2N}\right)\int_{R^N}|\Delta u|^2dx-\frac{\mu}{2}\int_{R^N}\frac{g+|u|^2}{1+g+|u|^2}|u|^2dx \\
& +\mu\left(\frac{1}{2}+\frac{N}{Np-2N}\right)\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right)-\frac{|u|^2}{1+g+|u|^2}\right)dx \\
\ge\ & \left(\frac{1}{2}-\frac{4}{Np-2N}\right)\int_{R^N}|\Delta u|^2dx-\frac{\mu}{2}\int_{R^N}|u|^2dx \\
=\ & \left(\frac{1}{2}-\frac{4}{Np-2N}\right)\int_{R^N}|\Delta u|^2dx-\frac{\mu c}{2},
\end{aligned} \tag{3.27}$$

which states that J is bounded from below and coercive on $M(c)$.

When $u \in M^-(c)$, following the argument in Lemma 13, we infer that

$$\int_{R^N}|\Delta u|^2dx > \Lambda_c > 0 \quad for\ 0 < c < c_0,$$

where Λ_c and c_0 are the same as in (3.20) and (3.13), respectively. Choose

$$c_2 := \left(\frac{4(N+4)}{C_{N,p}^{p} N(Np-2N)}\right)^{\frac{2}{p-2}} \left(\frac{\mu(Np-2N)}{Np-2N-8}\right)^{-\frac{Np-2N-8}{4(p-2)}},$$

a direct calculation brings that

$$J(u) > \frac{\mu|g|c}{2(1+g)} \quad for\ 0 < c < c_2. \tag{3.28}$$

Lemma 15 Assume that $\bar{p} < p < 4^*$. Then, $M^0(c) = \emptyset$ for $0 < c < c_0$.

Proof. Suppose on the contrary and fix some $u \in M^0(c)$. Similar to the argument of Lemma 13, we deduce that

$$\int_{R^N} |\Delta u|^2 dx \geq \left(\frac{2(4(N+4)(1+g)^{\frac{\bar{p}}{2}} - \mu \boldsymbol{B}_{\bar{p}} N^2 C_{N,\bar{p}}^{\bar{p}} c^{\frac{4}{N}})}{c^{\frac{2N-p(N-4)}{8}} C_{N,p}^{p} (1+g)^{\frac{\bar{p}}{2}} N(Np-2N)}\right)^{\frac{8}{Np-2N-8}} \to +\infty \quad \text{as } c \to 0. \tag{3.29}$$

Moreover, by (2.9) and Lemma 3, there holds that

$$\begin{aligned}
(Np-2N-8)\int_{R^N} |\Delta u|^2 dx &= \frac{\mu N^2 p}{4}\int_{R^N}\left(ln\left(1+\frac{|u|^2}{1+g}\right) - \frac{|u|^2}{1+g+|u|^2}\right)dx - \frac{\mu N^2}{2}\int_{R^N}\frac{|u|^4}{(1+g+|u|^2)^2}dx \\
&\leq \frac{\mu N^2 p}{4}\int_{R^N}\left(\frac{|u|^2}{1+g} - \frac{|u|^2}{1+g+|u|^2}\right)dx - \frac{\mu N^2}{2}\int_{R^N}\frac{|u|^4}{(1+g+|u|^2)^2}dx \\
&= \frac{\mu N^2 p}{4}\int_{R^N}\frac{|u|^4}{(1+g)(1+g+|u|^2})dx - \frac{\mu N^2}{2}\int_{R^N}\frac{|u|^4}{(1+g+|u|^2)^2}dx \\
&\leq \frac{\mu N^2 p}{4(1+g)}\int_{R^N}\frac{|u|^4}{1+g+|u|^2}dx \\
&\leq \frac{\mu N^2 pc}{4(1+g)},
\end{aligned}$$

which means that

$$\int_{R^N} |\Delta u|^2 dx \leq \frac{\mu N^2 pc}{4(Np-2N-8)(1+g)} \to 0 \quad as\ c \to 0. \tag{3.30}$$

Obviously, (3.29) and (3.30) lead to a contradiction. We finish the proof.

According to Lemma 15, it holds $M(c) = M^+(c) \cup M^-(c)$, which is a natural constraint manifold. Next, let us prove that the submanifold $M^-(c)$ is nonempty.

Lemma 16 Assume that $\bar{p} < p < 4^*$. Then, for any $u \in S(c)$, there exists a constant $t_u^- > 0$ such that $u^{t_u^-} \in M^-(c)$. In particular, t_u^- is a local maximum point of $f_u(t)$.

Proof. By Lemma 3, a direct calculation shows that

$$\begin{aligned}
f_{u'}(t) = &\ 2t^3\int_{R^N}|\Delta u|^2dx - \frac{Np-2N}{2p}t^{\frac{Np-2N-2}{2}}\int_{R^N}|u|^p dx - \frac{\mu N}{2t^{N+1}}\int_{R^N} ln\left(1+\frac{t^N|u|^2}{1+g}\right)dx \\
&+ \frac{\mu N}{2t}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx \\
\geq &\ 2t^3\int_{R^N}|\Delta u|^2dx - \frac{Np-2N}{2p}t^{\frac{Np-2N-2}{2}}\int_{R^N}|u|^p dx - \frac{\mu N}{2t}\int_{R^N}\frac{|u|^2}{1+g}dx + \frac{\mu N}{2t}\int_{R^N}\frac{|u|^2}{1+g+t^N|u|^2}dx \\
= &\ 2t^3\int_{R^N}|\Delta u|^2dx - \frac{Np-2N}{2p}t^{\frac{Np-2N-2}{2}}\int_{R^N}|u|^p dx - \frac{\mu N}{2t}\int_{R^N}\left(\frac{|u|^2}{1+g} - \frac{|u|^2}{1+g+t^N|u|^2}\right)dx \\
= &\ 2t^3\int_{R^N}|\Delta u|^2dx - \frac{Np-2N}{2p}t^{\frac{Np-2N-2}{2}}\int_{R^N}|u|^p dx - \frac{\mu N}{2}\int_{R^N}\frac{t^{N-1}|u|^4}{(1+g)(1+g+t^N|u|^2)}dx \\
\geq &\ 2t^3\int_{R^N}|\Delta u|^2dx - \frac{Np-2N}{2p}t^{\frac{Np-2N-2}{2}}\int_{R^N}|u|^p dx - \frac{\mu N t^{N-1}}{2(1+g)^2}\int_{R^N}|u|^4\,dx.
\end{aligned} \tag{3.31}$$

Since $p > \bar{p}$, it is clear that $f_{u'}(t) > 0$ for $t > 0$ small enough by (3.31).

In addition, it follows from Lemma 3 again that

$$\begin{aligned} f_{u'}(t) = &\ 2t^3 \int_{R^N} |\Delta u|^2 dx - \frac{Np-2N}{2p} t^{\frac{Np-2N-2}{2}} \int_{R^N} |u|^p dx \\ & - \frac{\mu N}{2t^{N+1}} \int_{R^N} ln\left(1 + \frac{t^N |u|^2}{1+g}\right) dx + \frac{\mu N}{2t} \int_{R^N} \frac{|u|^2}{1+g+t^N|u|^2} dx \\ = &\ 2t^3 \int_{R^N} |\Delta u|^2 dx - \frac{Np-2N}{2p} t^{\frac{Np-2N-2}{2}} \int_{R^N} |u|^p \, dx \\ & - \frac{\mu N}{2t^{N+1}} \int_{R^N} \left(ln\left(1 + \frac{t^N|u|^2}{1+g}\right) - \frac{t^N|u|^2}{1+g+t^N|u|^2}\right) dx \\ \leq &\ 2t^3 \int_{R^N} |\Delta u|^2 dx - \frac{Np-2N}{2p} t^{\frac{Np-2N-2}{2}} \int_{R^N} |u|^p \, dx. \end{aligned}$$

Since $p > \bar{p}$, the above inequality ensures that $f_u'(t) < 0$ for $t > 0$ large enough. Note that, for $u \in S(c)$ and $t > 0$, $u^t \in M(c)$ if and only if $f_{u'}(t) = 0$. Therefore, there exists a constant $t_u^- > 0$ such that $f_u'(t_u^-) = 0$ and $f_u''(t_u^-) < 0$, which means that $u^{t_u^-} \in M^-(c)$ and t_u^- is a local maximum point of $f_u(t)$.

From now on, we define

$$S_r(c) := S(c) \cap H_r^2(R^N), \quad M_r(c) := M(c) \cap H_r^2(R^N) \quad \text{and} \quad M_r^-(c) := M^-(c) \cap H_r^2(R^N). \tag{3.32}$$

By virtue of Lemmas 14 and 16, one has

$$m_r^-(c) := \inf_{u \in M_r^-(c)} J(u) \geq \inf_{u \in M^-(c)} J(u) > 0.$$

To apply Lemma 12 to construct a Palais-Smale sequence $\{u_n\} \subset M_r^-(c)$ for J restricted to $S(c)$, we introduce the following lemma.

Lemma 17 The map $u \in S_r(c) \mapsto t_u^- \in R$ is of class C^1.

Proof. Consider the C^1-function $\varphi: (0, \infty) \times S_r(c) \to R$ defined by $\varphi(t, u) = f_u'(t)$. Since $\varphi(t_u^-, u) = 0$, $\partial_t \varphi(t_u^-, u) = f''_u(t_u^-)$, the proof is completed by using the implicit function theorem.

Now we define the functional $G^-: S_r(c) \to R$ by $G^-(u) := J(u^{t_u^-})$. Clearly, it follows from Lemma 17 that the functional G^- is of class C^1. We also need the following result.

Lemma 18 The map $\Psi: T_u S_r(c) \to T_{u^{t_u^-}} S_r(c)$ defined by $\psi \to \psi^{t_u^-}$ is isomorphism, where $T_u S_r(c)$ denotes the tangent space to $S_r(c)$ at u.

Proof. Let $\psi \in T_u S_r(c)$. Then, we have

$$\int_{R^N} u^{t_u^-}(x) \psi^{t_u^-}(x) dx = \int_{R^N} (t_u^-)^{\frac{N}{2}} u(t_u^- x)(t_u^-)^{\frac{N}{2}} \psi(t_u^- x) dx = \int_{R^N} u(y) \psi(y) dy = 0,$$

which implies that $\psi^{t_u^-} \in T_{u^{t_u^-}} S_r(c)$. Thus, the map Ψ is well defined.

For $\forall \psi_1, \psi_2 \in T_u S_r(c)$ and $\forall k \in R$, it holds that

$$\Psi(\psi_1 + \psi_2) = (\psi_1 + \psi_2)^{t_u -} = (t_u^-)^{\frac{N}{2}} (\psi_1(t_u^- x) + \psi_2(t_u^- x)) = \psi_1^{t_u^-} + \psi_2^{t_u^-} = \Psi(\psi_1) + \Psi(\psi_2)$$

and

$$\Psi(k\psi_1) = (k\psi_1)^{t_u^-} = k\psi_1^{t_u^-} = k\Psi(\psi_1).$$

This shows that the map Ψ is linear. Finally, let us check that the map Ψ is a bijection. For $\forall \psi_1$, $\psi_2 \in T_u S_r(c)$ with $\psi_1 \neq \psi_2$, since $t_u^- > 0$, we see that

$$\Psi(\psi_1) = (t_u^-)^{\frac{N}{2}} \psi_1(t_u^- x) \neq (t_u^-)^{\frac{N}{2}} \psi_2(t_u^- x) = \Psi(\psi_2).$$

Moreover, let $\chi \in T_{u^{t_u^-}} S_r(c)$ and define

$$\psi(x) := (t_u^-)^{-\frac{N}{2}} \chi(\frac{x}{t_u}).$$

Then, it gives that

$$\int_{R^N} \psi(x)u(x)dx = \int_{R^N} (t_u^-)^{-\frac{N}{2}} \chi(\frac{x}{t_u^-}) u(x)dx = \int_{R^N} \chi(y)(t_u^-)^{\frac{N}{2}} u(t_u^- y)dy = \int_{R^N} \chi(y) u^{t_u^-}(y)dy = 0,$$

which means that $\psi \in T_u S_r(c)$. Moreover, $\Psi(\psi) = (t_u)^{\frac{N}{2}} \psi(t_u x) = (t_u)^{\frac{N}{2}} (t_u)^{-\frac{N}{2}} \chi(x) = \chi$. Hence, Ψ is a bijection.

Lemma 19 It holds that $(G^-)'(u)[\psi] = J'(u^{t_u^-})[\psi^{t_u^-}]$ for any u $\in S_r(c)$ and $\psi \in T_u S_r(c)$.

Proof. Let $u \in S_r(c)$ and $\psi \in T_u S_r(c)$. Recall that $G^-(u) = J(u^{t_u^-})$, where $t_u^- > 0$ is the constant guaranteed by Lemma 16 such that $u^{t_u^-} \in M_r^-(c)$, and the scaling transformation $v^t(x) = t^{\frac{N}{2}} v(tx)$ preserves the L^2-norm (i.e., $\|v^t\|_2^2 = \|v\|_2^2$ for all $t > 0$ and $v \in H^2(R^N)$. Moreover, by Lemma 17, the mapping $u \mapsto t_u^-$ is of class C^1 on $S_r(c)$.

For small $|h| > 0$, set $u_h = u + h\phi$ and $t_h^- := t_{u_h}^-$. Since t_u^- is continuous in u, we have $t_h^- \to t_u^-$ as $h \to 0$. The Gâteaux derivative of G^- at u along ψ is defined as

$$(G^-)'(u)(\phi) = \lim_{h\to 0} \frac{J(u_h^{t_h^-}) - J(u^{t_u^-})}{h}.$$

By the fact that t_u^- is the local maximum point of the function $J(u^t)$, we have

$$\begin{aligned}
&J((u+h\phi)^{t_h^-}) - J(u^{t_u^-}) \\
&\le J((u+h\phi)^{t_h^-}) - J(u^{t_h^-}) \\
&= \frac{1}{2}(t_h^-)^4 \int_{R^N} [|\Delta(u+h\phi)|^2 - |\Delta u|^2]\, dx \\
&\quad - \frac{(t_h^-)^{\frac{Np-2N}{2}}}{p} \int_{R^N} |u+h\phi|^p dx - \frac{\mu}{2}\int_{R^N} |u+h\phi|^2 dx + \frac{\mu}{2(t_h^-)^N} \int_{R^N} ln\left(1 + \frac{(t_h^-)^N |u+h\phi|^2}{1+g}\right) dx \\
&\quad - \left[\frac{(t_h^-)^{\frac{Np-2N}{2}}}{p} \int_{R^N} |u|^p dx - \frac{\mu}{2}\int_{R^N} |u|^2 dx + \frac{\mu}{2(t_h^-)^N} \int_{R^N} ln\left(1 + \frac{(t_h^-)^N |u|^2}{1+g}\right) dx\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&J((u+h\phi)^{t_h^-}) - J(u^{t_u^-}) \\
&\ge J((u+h\phi)^{t_u^-}) - J(u^{t_u^-}) \\
&= \frac{1}{2}(t_u^-)^4 \int_{R^N} [|\Delta(u+h\phi)|^2 - |\Delta u|^2]\, dx \\
&\quad - \frac{(t_u^-)^{\frac{Np-2N}{2}}}{p} \int_{R^N} |u+h\phi|^p dx - \frac{\mu}{2}\int_{R^N} |u+h\phi|^2 dx + \frac{\mu}{2(t_u^-)^N} \int_{R^N} ln\left(1 + \frac{(t_u^-)^N |u+h\phi|^2}{1+g}\right) dx \\
&\quad - \left[\frac{(t_u^-)^{\frac{Np-2N}{2}}}{p} \int_{R^N} |u|^p dx - \frac{\mu}{2}\int_{R^N} |u|^2 dx + \frac{\mu}{2(t_u^-)^N} \int_{R^N} ln\left(1 + \frac{(t_u^-)^N |u|^2}{1+g}\right) dx\right].
\end{aligned}$$

Since $\lim_{h\to 0} t_h^- = t_u^-$, from the two inequalities above and utilizing the mean value theorem, it follows that

$$\lim_{h\to 0}\frac{G^-(u+h\phi)-G^-(u)}{h}=(t_u^-)^4\int_{R^N}\Delta u\Delta\phi dx-(t_u^-)^{\frac{Np-2N}{2}}\int_{R^N}|u|^{p-2}u\phi dx-\mu\int_{R^N}\frac{g+|t_u^-|^N|u|^2}{1+g+|t_u^-|^N|u|^2}u\phi dx.$$

Note that the Gâteaux derivative of G^- is bounded linear in ϕ and continuous in u. Therefore, G^- is of class C^1. In particular, by changing variables in the integrals, we have

$$(G^-)'(u)[\phi]=\int_{R^N}\Delta u^{t_u^-}\Delta\phi^{t_u^-}dx-\int_{R^N}|u^{t_u^-}|^{p-2}u^{t_u^-}\phi^{t_u^-}dx-\mu\int_{R^N}\frac{g+|u^{t_u^-}|^2}{1+g+|u^{t_u^-}|^2}u^{t_u^-}\phi^{t_u^-}dx$$

$$=J'(u^{t_u^-})[\phi^{t_u^-}].$$

The proof is complete.

Lemma 20 Assume that $\bar{p}<p<4^*$ holds. Let F be a homotopy stable family of compact subsets of $S_r(c)$ with closed boundary Θ and let

$$e_F^-:=\inf_{H\in F}\max_{u\in H}G^-(u).$$

Suppose that Θ is contained in a connected component of $M_r^-(c)$ and that $max\{sup\,G^-(\Theta),0\}<e_F^-<\infty$. Then, there exists a Palais-Smale sequence $\{u_n\}\subset M_r^-(c)$ for J restricted to $S_r(c)$ at the level e_F^-.

Proof. First of all, take $\{D_n\}\subset F$ such that $\max_{u\in D_n}G^-(u)<e_F^-+\frac{1}{n}$ and define the map $\eta:[0,1]\times S(c)\to S(c)$ by $\eta(s,u)=u^{1-s+st_u^-}$. Since $t_u^-=1$ for any $u\in M_r^-(c)$ and $\Theta\subset M_r^-(c)$, we have

$$\eta(s,u)=u \text{ for } (s,u)\in(\{0\}\times S_r(c))\cup([0,1]\times\Theta).$$

By the definition of F, it follows that

$$\boldsymbol{A}_n:=\eta(\{1\}\times D_n)=\{u^{t_u^-}:u\in D_n\}\in F.$$

Clearly, $\boldsymbol{A}_n\subset M_r^-(c)$ for all $n\in N$. For any $v\in\boldsymbol{A}_n$, we have $v:=u^{t_u^-}$ for some $u\in D_n$. Then, $G^-(u)=J(u^{t_u^-})=J(v)=G^-(v)$, which shows that

$$\max_{u\in D_n}G^-(u)=\max_{u\in A_n}G^-(u).$$

Thus, $\{\boldsymbol{A}_n\}\subset M_r^-(c)$ is an another minimizing sequence of e_F^-. By Lemma 12, we obtain a Palais-Smale sequence $\{v_n\}$ for G^- on $S_r(c)$ at the level e_F^- satisfying $dist(v_n,\boldsymbol{A}_n)\to 0$ as $n\to\infty$. For each $v_n\in S_r(c)$, there exists a constant $t_{v_n}^->0$ such that $u_n:=v_n^{t_{v_n}^-}\in M_r^-(c)$.

Next, we claim that there exists a constant $C_0>0$ such that

$$\frac{1}{C_0}\le(t_{v_n}^-)^2\le C_0\quad\text{for } n\in\mathrm{N}. \tag{3.33}$$

Indeed, we observe that

$$(t_{v_n}^-)^4=\frac{\int_{R^N}|\Delta v_n^{t_{v_n}^-}|^2dx}{\int_{R^N}|\Delta v_n|^2dx}.$$

Since $J(v_n^{t_{v_n}^-})=G^-(v_n)\to e_F^-$, it follows from Lemma 14 that there exists a constant $M_0>0$ such that

$$\frac{1}{M_0}\le\int_{R^N}|\Delta v_n^{t_{v_n}^-}|^2\,dx\le M_0. \tag{3.34}$$

Moreover, since $\{\boldsymbol{A}_n\} \subset M_r^-(c)$ is a minimizing sequence for e_F^- and J is coercive on $M^-(c)$, we know that $\{\boldsymbol{A}_n\}$ is uniformly bounded in $H^2(R^N)$. Note that $dist(v_n, \boldsymbol{A}_n) \to 0$ as $n \to \infty$, so $sup_n \|v_n\|_{H^2} < \infty$. Meanwhile, since $\boldsymbol{A}_n$ is compact for each $n \in N$, there exists a $\bar{v}_n \in \boldsymbol{A}_n$ such that $dist(v_n, \boldsymbol{A}_n) = \|\bar{v}_n - v_n\|_{H^2}$. By Lemma 14, there exists a constant $\delta > 0$ such that $\int_{R^N} |\Delta \bar{v}_n|^2 dx \geq \delta$ for all n. Since $\|\bar{v}_n - v_n\|_{H^2} \to 0$, we have $\|\Delta \bar{v}_n - \Delta v_n\|_{L^2} \to 0$. Thus, for sufficiently large n,

$$\int_{R^N} |\Delta v_n|^2 dx \geq \int_{R^N} |\Delta \bar{v}_n|^2 dx - \int_{R^N} |\Delta (v_n - \bar{v}_n)|^2 dx \geq \frac{\delta}{2}. \tag{3.35}$$

Combining (3.34) and (3.35), we conclude that (3.33) holds.

In what follows, we show that $\{u_n\} \subset M_r^-(c)$ is a Palais-Smale sequence for J on $S_r(c)$ at the level e_F^-. Denote the norm on the tangent space $T_{u_n} S_r(c)$ by $\|\cdot\|$ and the dual norm on $T_{u_n}^* S_r(c)$ by $\|\cdot\|_*$. Then,

$$\|J'(u_n)\|_* = \sup_{\Psi \in T_{u_n} S_r(c) \|\Psi\| \leq 1} |\langle J'(u_n), \Psi \rangle| = \sup_{\Psi \in T_{u_n} S_r(c) \|\Psi\| \leq 1} \left| \langle J'(u_n), (\Psi^{-t_{\bar{v}_n}})^{t_{\bar{v}_n}} \rangle \right|. \tag{3.36}$$

By Lemma 18, the map $\Psi: T_{v_n} S_r(c) \to T_{v_n^{t_{\bar{v}_n}}} S_r(c)$ defined by $\psi \to \psi^{t_{\bar{v}_n}}$ is isomorphism. Moreover, Lemma 19 implies that $\langle (G^-)'(v_n), \psi^{-t_{\bar{v}_n}} \rangle = \langle J'(u_n), \psi \rangle$. Hence, we obtain from (3.36) that

$$\|J'(u_n)\|_* = \sup_{\Psi \in T_{u_n} S_r(c) \|\Psi\| \leq 1} |\langle J'(u_n), \Psi \rangle| = \sup_{\Psi \in T_{u_n} S_r(c) \|\Psi\| \leq 1} \left| \langle (G^-)', (v_n), \Psi^{-t_{\bar{v}_n}} \rangle \right|. \tag{3.37}$$

By (3.33), we know that $||\psi^{-t_{\bar{v}_n}}|| \leq C||\psi|| \leq C$ for some constant $C > 0$. Consequently, owing to $\|(G^-)'(v_n)\|_* \to 0$ as $n \to \infty$, it immediately follows from (3.37) that $\|J'(u_n)\|_* \to 0$. That is to say, $\{u_n\} \subset M_r^-(c)$ is a Palais-Smale sequence for J on $S_r(c)$ at the level e_F^-.

Lemma 21 Assume that $\bar{p} < p < 4^*$. Then, there exists a Palais-Smale sequence $\{u_n\} \subset M_r^-(c)$ for J restricted to $S_r(c)$ at the level $m_r^-(c) > \frac{\mu|g|c}{2(1+g)}$.

Proof. Based on Lemma 20, we choose the set F of all singletons belonging to $S_r(c)$ and $\Theta = \emptyset$, which is clearly a homotopy stable family of compact subsets of $S_r(c)$ (without boundary). Note that

$$e_{\mathsf{F}}^- = \inf_{H \in \mathsf{F}} \max_{u \in H} G^-(u) = \inf_{u \in S_r(c)} G^-(u) = \inf_{u \in \mathsf{M}_r^-(c)} J(u) = m_r^-(c),$$

the lemma follows directly from Lemma 20.

Now we are in the position to finish the proof of Theorem 1 (ii). By Lemma 21, there exists a Palais-Smale sequence $\{u_n\} \subset M_r^-(c)$ for J restricted to $S(c)$ at level $m_r^-(c) > \frac{\mu|g|c}{2(1+g)}$, which is bounded in $H_r^2(R^N)$ via Lemma 14. So, for $\bar{p} < p < 4^*$, according to Lemma 13, when

$$0 < c < \bar{c} := min\{c_0, c_1, c_2\},$$

problem (1.1) admits a radially symmetric solution w satisfying $J(w) = m_r^-(c) > \frac{\mu|g|c}{2(1+g)}$ for some $\bar{\lambda} > 0$.

Moreover, since $Q(w) = 0$, by (3.20) and Lemma 3, one has

$$\begin{aligned}
\bar{\lambda}c = &\ \frac{2N-p(N-4)}{Np-2N}\int_{R^N}|\Delta w|^2dx+\mu\int_{R^N}\frac{g+|w|^2}{1+g+|w|^2}|w|^2dx \\
&-\frac{\mu Np}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|w|^2}{1+g}\right)-\frac{|w|^2}{1+g+|w|^2}\right)dx \\
> &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\mu\int_{R^N}|w|^2dx-\mu\int_{R^N}\frac{1}{1+g+|w|^2}|w|^2\,dx \\
&-\frac{\mu Np}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|w|^2}{1+g}\right)-\frac{|w|^2}{1+g+|w|^2}\right)dx \\
> &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\mu c-\frac{\mu c}{1+g}-\frac{\mu Np}{Np-2N}\int_{R^N}\left(ln\left(1+\frac{|w|^2}{1+g}\right)-\frac{|w|^2}{1+g+|w|^2}\right)dx \\
> &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\frac{\mu gc}{1+g}-\frac{\mu Np}{Np-2N}\int_{R^N}\left(\frac{|w|^2}{1+g}-\frac{|w|^2}{1+g+|w|^2}\right)dx \\
= &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\frac{\mu gc}{1+g}-\frac{\mu Np}{(Np-2N)(1+g)}\int_{R^N}\frac{|w|^4}{1+g+|w|^2}dx \\
> &\ \frac{2N-p(N-4)}{Np-2N}\Lambda_c+\frac{\mu gc}{1+g}-\frac{\mu Npc}{(Np-2N)(1+g)} \\
= &\ K_2c^{-\frac{2N-p(N-4)}{Np-2N-8}}-\frac{\mu c}{1+g}\left(\frac{Np}{Np-2N}-g\right) \\
= &\ K_2c^{-\frac{2N-p(N-4)}{Np-2N-8}}-\frac{\mu c}{1+g}\left(\frac{p}{p-2}-g\right),
\end{aligned} \tag{3.38}$$

where

$$K_2=\left(\frac{2N-p(N-4)}{Np-2N}\right)\left(\frac{8(N+4)(1+g)^{\frac{\tilde{p}}{2}}-2\mu\boldsymbol{B}_{\tilde{p}}N^2C_{N,\tilde{p}}^{\tilde{p}}c^{\frac{4}{N}}}{(1+g)^{\frac{\tilde{p}}{2}}N(Np-2N)}\right)^{\frac{8}{Np-2N-8}}.$$

This indicates that

$$\bar{\lambda}>K_2c^{\frac{8-4p}{Np-2N-8}}-\frac{\mu}{1+g}\left(\frac{p}{p-2}-g\right).$$

4. Appendix

Derivation of $\boldsymbol{A_q}$: Let $g(t):=\frac{2t}{1+t}t^{\frac{2-q}{2}}$, it gives that

$$\begin{aligned}
ln\,g\,(t) = &\ ln(\,2t)+\frac{2-q}{2}ln\,t-ln(\,1+t), \\
\frac{g'(t)}{g(t)} = &\ \frac{1}{t}+\frac{2-q}{2t}-\frac{1}{1+t}=\frac{4-q}{2t}-\frac{1}{1+t}, \\
t_{max} = &\ \frac{4-q}{q-2}.
\end{aligned}$$

Hence, one has

$$g(t_{max})=\frac{2t}{1+t}t^{\frac{2-q}{2}}=\frac{\frac{2(4-q)}{q-2}}{1+\frac{4-q}{q-2}}\left(\frac{4-q}{q-2}\right)^{\frac{2-q}{2}}=(4-q)\left(\frac{4-q}{q-2}\right)^{\frac{2-q}{2}},$$

which brings that

$$\boldsymbol{A}_q=\frac{1}{q}max\,g\,(t)=\frac{4-q}{q}\left(\frac{4-q}{q-2}\right)^{\frac{2-q}{2}}=\frac{(4-q)^{\frac{4-q}{2}}}{q(q-2)^{\frac{2-q}{2}}}=\frac{(q-2)^{\frac{q-2}{2}}(4-q)^{\frac{4-q}{2}}}{q}.$$

Derivation of $\boldsymbol{B_q}$: Considering $h(t):=\frac{2(t+2)t^{\frac{4-q}{2}}}{(1+t)^2}$, we see that

$$\begin{aligned} \ln h(t) &= \ln 2(t+2) + \frac{4-q}{2}\ln t - 2\ln(1+t), \\ \frac{h'(t)}{h(t)} &= \frac{1}{2+t} + \frac{4-q}{2}\cdot\frac{1}{t} - \frac{2}{1+t}. \end{aligned}$$

Letting $h'(t) = 0$, we deduce that $(q-2)t^2 + 3(q-2)t + 2(q-4) = 0$, which yields that

$$\begin{aligned} t &= \frac{-3(q-2)\pm\sqrt{9(q-2)^2-8(q-2)(q-4)}}{2(q-2)} \\ &= \frac{-3(q-2)\pm\sqrt{(q-2)(9(q-2)-8(q-4))}}{2(q-2)} \\ &= \frac{-3\sqrt{q-2}\pm\sqrt{9(q-2)-8(q-4)}}{2\sqrt{q-2}} \\ &= \frac{-3\sqrt{q-2}\pm\sqrt{9q-18-8q+32}}{2\sqrt{q-2}} \\ &= \frac{-3\sqrt{q-2}\pm\sqrt{q+14}}{2\sqrt{q-2}}. \end{aligned}$$

Substituting $t_{max}\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}}$ into $h(t)$, there holds that

$$\begin{aligned} h_{max} &= \frac{(\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}}+2)(\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}})^{\frac{4-q}{2}}}{(1+\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}})^2} \\ &= \frac{(\frac{\sqrt{q+14}-3\sqrt{q-2}+4\sqrt{q-2}}{2\sqrt{q-2}})(\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}})^{\frac{4-q}{2}}}{(\frac{2\sqrt{q-2}+\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}})^2} \\ &= \frac{(\frac{\sqrt{q+14}+\sqrt{q-2}}{2\sqrt{q-2}})(\frac{\sqrt{q+14}-3\sqrt{q-2}}{2\sqrt{q-2}})^{\frac{4-q}{2}}}{(\frac{\sqrt{q+14}-\sqrt{q-2}}{2\sqrt{q-2}})^2} \\ &= \frac{(\sqrt{q+14}+\sqrt{q-2})(2\sqrt{q-2})^{-1}(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}(2\sqrt{q-2})^{-\frac{4-q}{2}}}{(\sqrt{q+14}-\sqrt{q-2})^2(2\sqrt{q-2})^{-2}} \\ &= \frac{(\sqrt{q+14}+\sqrt{q-2})(2\sqrt{q-2})^{\frac{q-2}{2}}(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{(\sqrt{q+14}-\sqrt{q-2})^2} \\ &= \frac{2^{\frac{q-2}{2}}(q-2)^{\frac{q-2}{4}}(\sqrt{q+14}+\sqrt{q-2})(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{(\sqrt{q+14}-\sqrt{q-2})^2} \\ &= \frac{2^{\frac{q-2}{2}}(q-2)^{\frac{q-2}{4}}(q+14-q+2)(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{(\sqrt{q+14}-\sqrt{q-2})^3} \\ &= \frac{2^{\frac{q+6}{2}}(q-2)^{\frac{q-2}{4}}(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{(\sqrt{q+14}-\sqrt{q-2})^3}. \end{aligned}$$

As a consequence, we obtain that

$$\boldsymbol{B}_q = \frac{1}{q}\max h(t) = \frac{2^{\frac{q+6}{2}}(q-2)^{\frac{q-2}{4}}(\sqrt{q+14}-3\sqrt{q-2})^{\frac{4-q}{2}}}{q(\sqrt{q+14}-\sqrt{q-2})^3}.$$

Conflict of Interest

The authors declare that there are no conflicts of interest, including any financial or personal relationships that could have influenced the work reported in this paper.

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