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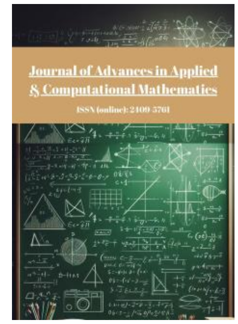

Equivalence Analysis of Two Different Methods for Solving Fuzzy Linear Systems

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ABSTRACT

In this paper, the distinction and connection between a new approach introduced in [29] and the traditional method presented in [1] for some semi-fuzzy linear systems are discussed. Firstly, the consistency of some primary algebraic operations between the α-center and α–radius of a fuzzy number $\tilde{x}$ and the α-levels of a fuzzy number $\tilde{x}$ is analyzed. Secondly, the equivalence property of the computing model and the strong fuzzy solution in recent paper [29] by discussing the proposed method twenty-eight years ago [1] are considered. Then, the dual fuzzy linear systems are also investigated in a similar way. Finally, two classical and well known examples are given to show the validity of the new method in which the idea and approach can be applied to simplifying calculation of any semi-fuzzy linear systems.

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1. Introduction

In the past decades fuzzy linear systems have been paid more attention by some scholars. In 1998, Friedman *et al*. [1] proposed a general model for solving an $n \times n$ fuzzy linear systems based on triangular fuzzy numbers by an embedding approach [2]. Substantial work has been done on handling various advanced fuzzy linear systems, such as dual fuzzy linear systems (DFLS), general fuzzy linear systems (GFLS), full fuzzy linear systems (FFLS), dual full fuzzy linear systems (DFFLS) and general dual fuzzy linear systems (GDFLS) [3-10]. Recently, new theories and methods for fuzzy linear systems and fuzzy numbers matrix appeared in the literature [11-26]. Meanwhile, fuzzy integro-differential equations also play an important role in fields such as control theory, electrical circuits, and signal processing, and significant progress has been made in their numerical solution methods, such as the reproductive kernel algorithm [27, 28].

In this paper, some distinctions and connections between a new approach introduced in [29] and the traditional method presented in [1] for solving two classes of semi-fuzzy linear systems are discussed. We analyze some primary algebraic operations between the α -center and α -radius of a fuzzy number $\tilde{x}$ and the α -levels of a fuzzy number $\tilde{x}$ at first. Based on this, we investigate the equivalence of the computing model and the strong fuzzy solution in the recent paper [29] by discussing the proposed method twenty-eight years ago [1]. Then, we consider the dual fuzzy linear systems in a similar way. Finally, we show the validity of the new method by solving two classical and well known examples in which the idea and approach can be applied to simplifying calculation of any semi-fuzzy linear systems.

In Section 2, some definitions and results on fuzzy numbers and fuzzy linear systems are presented. In Section 3, the equivalence property of the computing model and the strong fuzzy solution in recent paper [29] by discussing the proposed method twenty-four years ago [1] are considered. The dual fuzzy linear systems are also considered in a similar way in Section 4. Some illustrating numerical examples are given in Section 5. Finally, the conclusion and future research directions are presented in Section 6.

2. Preliminaries

There are several basic definitions for the concept of fuzzy numbers (see [30-32]).

Definition 2.1. A fuzzy number is a fuzzy set like $u: R \to I = [0,1]$ which satisfies:

(1) u is upper semi-continuous,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0,1]$,

(3) u is normal, i.e. there exists $x_0 \in R$ such that $u(x_0) = 1$,

(4) supp$u = \{x \in R | u(x) > 0\}$ is the support of the u, and its closure cl(supp$u$) is compact.

Let E^1 be the set of all fuzzy numbers on R.

Definition 2.2. A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfies the requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing right continuous function,

(3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For example, the fuzzy number $(2+r, 5-2r)$ is shown in Fig. (**1**). A crisp number x is simply represented by $(\underline{u}(r), \overline{u}(r)) = (x, x)$, $0 \leq r \leq 1$.

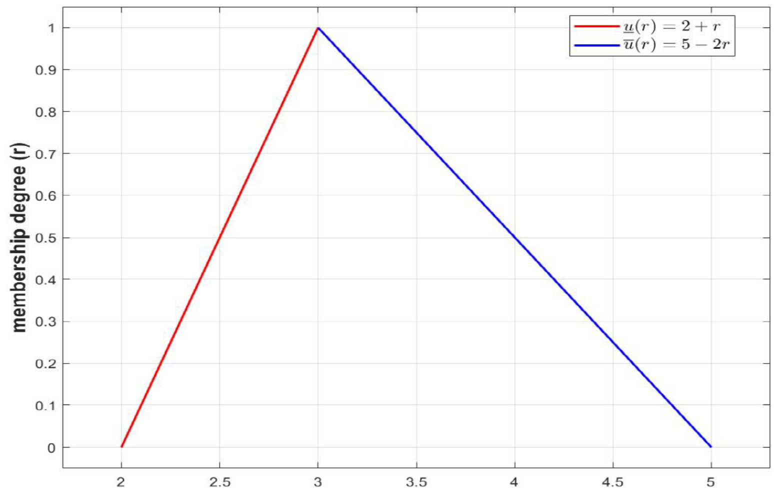

Figure 1: A fuzzy number.

Definition 2.3. Let $\tilde{x} = (\underline{x}(r), \overline{x}(r))$, $\tilde{y}\tilde{x} = (\underline{y}(r), \overline{y}(r)) \in E^1$, $0 \le r \le 1$ and $k \in R$. Then

(1) $\tilde{x} = \tilde{y}$ iff $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(2) $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(3) $\tilde{x} - \tilde{y} = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r))$,

(4) $k\tilde{x} = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Now we recall the following two concepts [29].

Definition 2.4. The r -center of the fuzzy number $\tilde{x}$ is denoted by $x^C(r)$ and it is defined as

$$x^C(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}, r \in [0,1]. \tag{2.1}$$

The r -radius of the fuzzy number $\tilde{x}$ is denoted by $x^R(r)$ and it is defined as

$$x^R(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2}, r \in [0,1]. \tag{2.2}$$

Remark 2.5. The r -center and r -radius of an arbitrary fuzzy number are crisp real functions of r. Also

$$[\tilde{x}]_r = [\underline{x}(r), \overline{x}(r)] = [x^C(r) - x^R(r), x^C(r) + x^R(r)], r \in [0,1]. \tag{2.3}$$

Obviously, two fuzzy numbers $\tilde{x}$ and $\tilde{y}$ are equal, if and only if $x^C(r) = y^C(r)$ and $x^R(r) = y^R(r)$, for every $r \in [0,1]$. For the fuzzy number $\tilde{x}$, if for any $r \in [0,1]$, $x^R(r) = 0$, then it can be easily concluded that $\tilde{x}$ is a crisp real number.

Remark 2.6. Let $\tilde{x}$ and $\tilde{y}$ to be two fuzzy numbers. Since

$$\tilde{x} + \tilde{y} = [\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{x}(r)] = [x^C(r) + y^C(r) - x^R(r) - y^R(r), x^C(r) + y^C(r) + x^R(r) + y^R(r)], r \in [0,1],$$

the r -center and the r -radius of the sun $\widetilde{m} = \tilde{x} + \tilde{y}$ are

$$m^C(r) = x^C(r) + y^C(r), m^R(r) = x^R(r) + y^R(r), r \in [0,1]. \tag{2.4}$$

Remark 2.7. Let λ to be a crisp real number. Since

$$[\lambda\tilde{x}]_r = \lambda[\tilde{x}]_r = \begin{cases} [\lambda\underline{x}(r), \lambda\overline{x}(r)] = [\lambda x^C(r) - \lambda x^R(r), \lambda x^C(r) + \lambda x^R(r)], & \lambda \ge 0, \\ [\lambda\overline{x}(r), \lambda\underline{x}(r)] = [\lambda x^C(r) - (-\lambda x^R(r)), \lambda x^C(r) + (-\lambda x^R(r))], & \lambda < 0, \end{cases}$$

the r -center and the r -radius of the product $\tilde{n} = \lambda\tilde{x}$ are

$$\begin{cases} n^C(r) = \lambda x^C(r), & n^R(r) = \lambda x^R(r), & \lambda \geq 0, \\ n^C(r) = \lambda x^C(r), & n^R(r) = |\lambda| x^R(r), & \lambda < 0. \end{cases} \tag{2.5}$$

Definition 2.8. Suppose that $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n \in E^1$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in R$ and also $\tilde{u} = \sum_{i=1}^{n} \lambda_i \tilde{x}_i$, then

$$u^C(r) = \sum_{i=1}^{n} \lambda_i x_i^C(r), u^R(r) = \sum_{i=1}^{n} |\lambda_i| x_i^R(r). \tag{2.6}$$

Further, suppose $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^T \in E^n$ is a fuzzy numbers vector, and is a real matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$, then the r -center and the r -radius of the fuzzy numbers vector $\tilde{v} = A\tilde{x}$ are given by

$$v^C(r) = Ax^C(r), v^R(r) = |A| x^R(r). \tag{2.7}$$

Definition 2.9. The $n \times n$ linear system:

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n & = \tilde{b}_1, \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n & = \tilde{b}_2, \\ & \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n & = \tilde{b}_n, \end{cases} \tag{2.8}$$

where the coefficient matrix $A = (a_{ij})_{n\times n}$ ($1 \leq i \leq n, 1 \leq j \leq n$) is a real $n \times n$ matrix; $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)^T$ is a column vector of fuzzy numbers; $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is a vector of fuzzy unknowns. Such a system is called a fuzzy linear system. Using matrix notation,we have

$$A\tilde{x} = \tilde{b}. \tag{2.9}$$

In 1998, Friedman *et al*. [1] presented a computing model:

$$SX(r) = Y(r), 0 \leq r \leq 1, \tag{2.10}$$

i.e.,

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} = \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix},$$

where

$$\tilde{x} = [\underline{x}(r), \overline{x}(r)], \quad A = A^+ + A^-, \quad B = A^+, \quad C = -A^-, \quad |A| = B + C. \tag{2.11}$$

And the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Matrix S is invertible if and only if both matrices $A = B - C$ and $|A| = B + C$ are nonsingular. Under this condition, they derived the solution to model (2.9) as follows:

$$\begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}^{-1} \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix} = \begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix}, \tag{2.12}$$

where

$$D = \frac{1}{2}[|A|^{-1} + A^{-1}], E = \frac{1}{2}[|A|^{-1} - A^{-1}]. \tag{2.13}$$

3. The Equivalence Property Analysis of the Fuzzy Linear Systems

3.1. The Equivalence Property of the Computing Model

In this subsection, we aim at discussion on the another computing form of the model (2.9).

In fact, the model (2.9) is equivalent to the following equation

$$\begin{cases} A^+\underline{x}(r) + A^-\overline{x}(r) = \underline{b}(r), \\ A^-\underline{x}(r) + A^+\overline{x}(r) = \overline{b}(r). \end{cases} \tag{3.1}$$

Adding the two equations of (3.1), we have

$$(A^+ + A^-)\underline{x}(r) + (A^+ + A^-)\overline{x}(r) = \underline{b}(r) + \overline{b}(r),$$

i.e.,

$$A(\underline{x}(r) + \overline{x}(r)) = \underline{b}(r) + \overline{b}(r). \tag{3.2}$$

Subtracting the first equation from the second equation of (3.1), we have

$$(A^- - A^+)\underline{x}(r) + (A^+ - A^-)\overline{x}(r) = \overline{b}(r) - \underline{b}(r),$$

i.e.,

$$|A|(\overline{x}(r) - \underline{x}(r)) = \overline{b}(r) - \underline{b}(r). \tag{3.3}$$

So we have

$$\begin{cases} A\frac{1}{2}(\underline{x}(r) + \overline{x}(r)) = \frac{1}{2}(\underline{b}(r) + \overline{b}(r)), \\ |A|\frac{1}{2}(\overline{x}(r) - \underline{x}(r)) = \frac{1}{2}(\overline{b}(r) - \underline{b}(r)). \end{cases} \tag{3.4}$$

Using the definitions of the r -center and r -radius for the fuzzy number vector $\tilde{x}$, Eqs.(3.4) are reduced to

$$\begin{cases} Ax^C(r) = b^C(r), \\ |A|x^R(r) = b^R(r). \end{cases} \tag{3.5}$$

From the above analysis, we can easily get the following Theorem 3.1.

Theorem 3.1. Under the assumptions of definitions 2.1 and 2.2, the fuzzy linear systems (2.9) can be extended into a crisp linear system as follows:

$$Gu(r) = v(r), 0 \le r \le 1, \tag{3.6}$$

where

$$G = \begin{pmatrix} A & O \\ O & |A| \end{pmatrix}, u(r) = \begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix}, v(r) = \begin{pmatrix} b^C(r) \\ b^R(r) \end{pmatrix}, \tag{3.7}$$

and

$$x^C(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}, x^R(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2}, 0 \le r \le 1.$$

3.2. The Equivalence Property of the Solution of the Model

The equivalence property between the solution of the model (2.10) and the solution of the model (3.6) will be discussed in this subsection.

From the Theorem 3.1, we can obtain the solution of the model (3.6) as

$$\begin{cases} x^C(r) = A^{-1}b^C(r), \\ x^R(r) = |A|^{-1}b^R(r), \end{cases} \tag{3.8}$$

when matrices $A = B - C$ and $|A| = B + C$ are invertible.

Substituting back to the Initial assumption, we have

$$\underline{x}(r) = x^C(r) - x^R(r) = A^{-1}\frac{\underline{b}(r) + \overline{b}(r)}{2} - |A|^{-1}\frac{\overline{b}(r) - \underline{b}(r)}{2}, r \in [0,1],$$

$$\overline{x}(r) = x^C(r) + x^R(r) = A^{-1}\frac{\underline{b}(r) + \overline{b}(r)}{2} + |A|^{-1}\frac{\overline{b}(r) - \underline{b}(r)}{2}, r \in [0,1],$$

i.e.,

$$\begin{cases} \underline{x}(r) = \frac{1}{2}(A^{-1} + |A|^{-1})\underline{b}(r) + \frac{1}{2}(A^{-1} - |A|^{-1})\overline{b}(r), \\ \overline{x}(r) = \frac{1}{2}(A^{-1} - |A|^{-1})\underline{b}(r) + \frac{1}{2}(A^{-1} + |A|^{-1})\overline{b}(r). \end{cases} \tag{3.9}$$

Eqs.(3.9) can be rewritten as

$$\begin{cases} \underline{x}(r) = \frac{1}{2}(A^{-1} + |A|^{-1})\underline{b}(r) + \frac{1}{2}(|A|^{-1} - A^{-1})(-\overline{b}(r)), \\ -\overline{x}(r) = \frac{1}{2}(|A|^{-1} - A^{-1})\underline{b}(r) + \frac{1}{2}(A^{-1} + |A|^{-1})(-\overline{b}(r)). \end{cases} \tag{3.10}$$

Expressing the formula (3.10) in matrix form, we have

$$\begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(A^{-1} + |A|^{-1}) & \frac{1}{2}(|A|^{-1} - A^{-1}) \\ \frac{1}{2}(|A|^{-1} - A^{-1}) & \frac{1}{2}(A^{-1} + |A|^{-1}) \end{pmatrix} \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix} = \begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} \underline{b}(r) \\ -\overline{b}(r) \end{pmatrix}, \tag{3.11}$$

where

$$D = \frac{1}{2}[|A|^{-1} + A^{-1}], E = \frac{1}{2}[|A|^{-1} - A^{-1}].$$

The result is completely consistent with Eqs. (2.13) of the proposed method by Friedman *et al*. [1]

3.3. The Equivalence Property of the Strong Fuzzy Solution

Theorem 3.2. When $x^R(r) \geq 0, r \in [0,1]$, the fuzzy linear systems (2.9) exists the strong fuzzy solution as

$$[\underline{x}(r), \overline{x}(r)] = [x^C(r) - x^R(r), x^C(r) + x^R(r)], r \in [0,1], \tag{3.12}$$

where

$$x^C(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}, x^R(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2}, 0 \leq r \leq 1.$$

Proof. Since r -center and r -radius of a fuzzy numbers vector are crisp real-valued continuous functions of r, the functions $\underline{x}(r) = x^C(r) - x^R(r)$ and $\overline{x}(r) = x^C(r) + x^R(r)$ are , of course, continuous functions of $r, r \in [0,1]$. The condition $x^R(r) \geq 0, r \in [0,1]$ means that $\underline{x}(r) \leq \overline{x}(r), \forall r \in [0,1]$. So, $[\underline{x}(r), \overline{x}(r)]$ is a strong fuzzy solution of fuzzy linear systems (2.9).

4. The Equivalence Property Analysis of the Dual Fuzzy Linear Systems

For the dual fuzzy linear systems [29, 33]

$$A\tilde{x} + \tilde{y} = B\tilde{x} + \tilde{z}, \tag{4.1}$$

where the coefficient matrices $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ are two $n \times n$ crisp real matrices, and the vectors $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)^T$, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)^T$, and also the unknown vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ are fuzzy number vectors.

Using Friedman's method, it can be extended to the following crisp linear system of higher dimension:

$$SX(r) + Y(r) = TX(r) + Z(r), 0 \leq r \leq 1, \tag{4.2}$$

i.e.,

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}\begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} + \begin{pmatrix} \underline{y}(r) \\ -\overline{y}(r) \end{pmatrix} = \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}\begin{pmatrix} \underline{x}(r) \\ -\overline{x}(r) \end{pmatrix} + \begin{pmatrix} \underline{z}(r) \\ -\overline{z}(r) \end{pmatrix}, \tag{4.3}$$

where

$$\tilde{x} = [\underline{x}(r), \overline{x}(r)], A = A^+ + A^-, B = B^+ + B^-. \tag{4.4}$$

And the elements b_{ij}^+ of matrix B^+ and b_{ij}^- of matrix B^- are determined by this way: if $b_{ij} \geq 0, b_{ij}^+ = b_{ij}$ else $b_{ij}^+ = 0, 1 \leq i, j \leq n$; if $b_{ij} < 0, b_{ij}^- = b_{ij}$ else $b_{ij}^- = 0, 1 \leq i, j \leq n$. The elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by the same way.

The model (4.2) can be presented in detail as

$$\begin{cases} A^+\underline{x}(r) - A^-(-\overline{x}(r)) + \underline{y}(r) = B^+\underline{x}(r) - B^-(-\overline{x}(r)) + \underline{z}(r), \\ -A^-\underline{x}(r) + A^+(-\overline{x}(r)) - \overline{y}(r) = -B^-\underline{x}(r) + B^+(-\overline{x}(r)) - \overline{z}(r), \end{cases}$$

i.e.,

$$\begin{cases} A^+\underline{x}(r) + A^-\overline{x}(r) + \underline{y}(r) = B^+\underline{x}(r) + B^-\overline{x}(r) + \underline{z}(r), \\ A^-\underline{x}(r) + A^+\overline{x}(r) + \overline{y}(r) = B^-\underline{x}(r) + B^+\overline{x}(r) + \overline{z}(r). \end{cases} \tag{4.5}$$

By adding the two equations in (4.5), we have

$$(A^+ + A^-)\underline{x}(r) + (A^+ + A^-)\overline{x}(r) + \underline{y}(r) + \overline{y}(r) = (B^+ + B^-)\underline{x}(r) + (B^+ + B^-)\overline{x}(r) + \underline{z}(r) + \overline{z}(r).$$

Subtracting the first equation from the second equation of (4.5), we have

$$(A^- - A^+)\underline{x}(r) + (A^+ - A^-)\overline{x}(r) + \overline{y}(r) - \underline{y}(r) = (B^- - B^+)\underline{x}(r) + (B^+ - B^-)\overline{x}(r) + \overline{z}(r) - \underline{z}(r).$$

So we have

$$\begin{cases} A\frac{1}{2}(\underline{x}(r) + \overline{x}(r)) + \frac{1}{2}(\underline{y}(r) + \overline{y}(r)) = B\frac{1}{2}(\underline{x}(r) + \overline{x}(r)) + \frac{1}{2}(\underline{z}(r) + \overline{z}(r)), \\ |A|\frac{1}{2}(\overline{x}(r) - \underline{x}(r)) + \frac{1}{2}(\overline{y}(r) - \underline{y}(r)) = |B|\frac{1}{2}(\overline{x}(r) - \underline{x}(r)) + \frac{1}{2}(\overline{z}(r) - \underline{z}(r)). \end{cases} \tag{4.6}$$

Using the definitions of the r -center and r -radius for the fuzzy number vectors $\tilde{y}$ and $\tilde{x}$, Eqs.(4.6) are reduced to

$$\begin{cases} Ax^C(r) + y^C(r) = Bx^C(r) + z^C(r), \\ |A|x^R(r) + y^R(r)) = |B|x^R(r) + z^R(r). \end{cases} \tag{4.7}$$

Denoting Eqs.(4.7) in matrix form, we have

$$\begin{pmatrix} A & O \\ O & |A| \end{pmatrix}\begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} + \begin{pmatrix} y^C(r) \\ y^R(r) \end{pmatrix} = \begin{pmatrix} B & O \\ O & |B| \end{pmatrix}\begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} + \begin{pmatrix} z^C(r) \\ z^R(r) \end{pmatrix}, \tag{4.8}$$

where

$$x^C(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}, x^R(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2},$$

$$y^C(r) = \frac{\underline{y}(r) + \overline{y}(r)}{2}, y^R(r) = \frac{\overline{y}(r) - \underline{y}(r)}{2},$$

$$z^C(r) = \frac{\underline{z}(r) + \overline{z}(r)}{2}, z^R(r) = \frac{\overline{z}(r) - \underline{z}(r)}{2}, 0 \le r \le 1. \tag{4.9}$$

Theorem 4.1. Under the assumptions of definitions 2.1 and 2.2, the dual fuzzy linear systems (4.1) can be converted to a crisp linear system as follows:

$$Gu(r) = v(r), 0 \le r \le 1, \tag{4.10}$$

where

$$G = \begin{pmatrix} A - B & O \\ O & |A| - |B| \end{pmatrix}, u(r) = \begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix}, v(r) = \begin{pmatrix} z^C(r) - y^C(r) \\ z^R(r) - y^R(r) \end{pmatrix}. \tag{4.11}$$

Proof. The proof of Theorem 4.1. is straightforward.

When matrices $A - B$ and $|A| - |B|$ are invertible, we can obtain the solution of the model (4.10) is

$$\begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} = \begin{pmatrix} A - B & O \\ O & |A| - |B| \end{pmatrix}^{-1} \begin{pmatrix} z^C(r) - y^C(r) \\ z^R(r) - y^R(r) \end{pmatrix}$$

$$= \begin{pmatrix} (A - B)^{-1} & O \\ O & (|A| - |B|)^{-1} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(\begin{pmatrix} \underline{z}(r) \\ \overline{z}(r) \end{pmatrix} - \begin{pmatrix} \underline{y}(r) \\ \overline{y}(r) \end{pmatrix} \right)$$

$$= \frac{1}{2} \begin{pmatrix} (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) \\ (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \end{pmatrix}. \tag{4.12}$$

Thus we have

$$\begin{pmatrix} \underline{x}(r) \\ \overline{x}(r) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) \\ (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) - (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \\ (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) + (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \end{pmatrix}. \tag{4.13}$$

Theorem 4.2. When $x^R(r) \ge 0, r \in [0,1]$, the dual fuzzy linear systems (4.1) exists the strong fuzzy solution as

$$\begin{pmatrix} \underline{x}(r) \\ \overline{x}(r) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) - (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \\ (A - B)^{-1}((\overline{z}(r) - \overline{y}(r)) + (\underline{z}(r) - \underline{y}(r))) + (|A| - |B|)^{-1}((\overline{z}(r) - \overline{y}(r)) - (\underline{z}(r) - \underline{y}(r))) \end{pmatrix}, \tag{4.14}$$

where

$$x^C(r) = \frac{\underline{x}(r) + \overline{x}(r)}{2}, x^R(r) = \frac{\overline{x}(r) - \underline{x}(r)}{2}, 0 \le r \le 1.$$

Proof. From the analysis and results of Theorem 4.1, the proof of Theorem 4.2. is straightforward.

5. Numerical Examples

Example 5.1. [1] Consider the semi fuzzy linear system

$$\begin{cases} \tilde{x}_1 - \tilde{x}_2 = [r, 2 - r], \\ \tilde{x}_1 + 3\tilde{x}_2 = [4 + r, 7 - 2r]. \end{cases}$$

By the Theorem 3.1, the fuzzy linear systems can be extended into the crisp linear system as

$$\begin{pmatrix} A & O \\ O & |A| \end{pmatrix} \begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} = \begin{pmatrix} b^C(r) \\ b^R(r) \end{pmatrix},$$

where

$$b^C(r) = \frac{\underline{b}(r) + \overline{b}(r)}{2} = \begin{pmatrix} 1 \\ 5.5 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} r,$$

$$b^R(r) = \frac{\overline{b}(r) - \underline{b}(r)}{2} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} + \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} r.$$

Now that the matrices A and $|A|$ are invertible, we obtain the solution of the model (3.6) is

$$\begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} = \begin{pmatrix} A & O \\ O & |A| \end{pmatrix}^{-1} \begin{pmatrix} b^C(r) \\ b^R(r) \end{pmatrix},$$

i.e.,

$$x^C(r) = A^{-1}b^C(r) = \begin{pmatrix} 0.75 & 0.25 \\ -0.25 & 0.25 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 5.5 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} r \right) = \begin{pmatrix} 2.125 - 0.125r \\ 1.125 - 0.125r \end{pmatrix},$$

$$x^R(r) = |A|^{-1}b^R(r) = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1.5 \end{pmatrix} + \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} r \right) = \begin{pmatrix} 0.750 - 0.750r \\ 0.250 - 0.250r \end{pmatrix}.$$

Thus we have

$$\underline{x}(r) = x^C(r) - x^R(r) = \begin{pmatrix} 2.125 - 0.125r \\ 1.125 - 0.125r \end{pmatrix} - \begin{pmatrix} 0.750 - 0.750r \\ 0.250 - 0.250r \end{pmatrix} = \begin{pmatrix} 1.375 + 0.625r \\ 0.875 + 0.125r \end{pmatrix},$$

$$\overline{x}(r) = x^C(r) + x^R(r) = \begin{pmatrix} 2.125 - 0.125r \\ 1.125 - 0.125r \end{pmatrix} + \begin{pmatrix} 0.750 - 0.750r \\ 0.250 - 0.250r \end{pmatrix} = \begin{pmatrix} 2.875 - 0.875r \\ 1.375 - 0.375r \end{pmatrix}.$$

It means that the fuzzy solution of the original fuzzy linear systems

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} \underline{x}_1(r) & \overline{x}_1(r) \\ \underline{x}_2(r) & \overline{x}_2(r) \end{pmatrix},$$

i.e.,

$$\begin{cases} \tilde{x}_1 = [1.375 + 0.625r, 2.875 - 0.875r], \\ \tilde{x}_2 = [0.875 + 0.125r, 1.375 - 0.375r]. \end{cases}$$

is a strong fuzzy solution since $x^R(r) \ge 0$ for all $r \in [0,1]$. A visual representation of this solution is shown in Fig. (**2**).

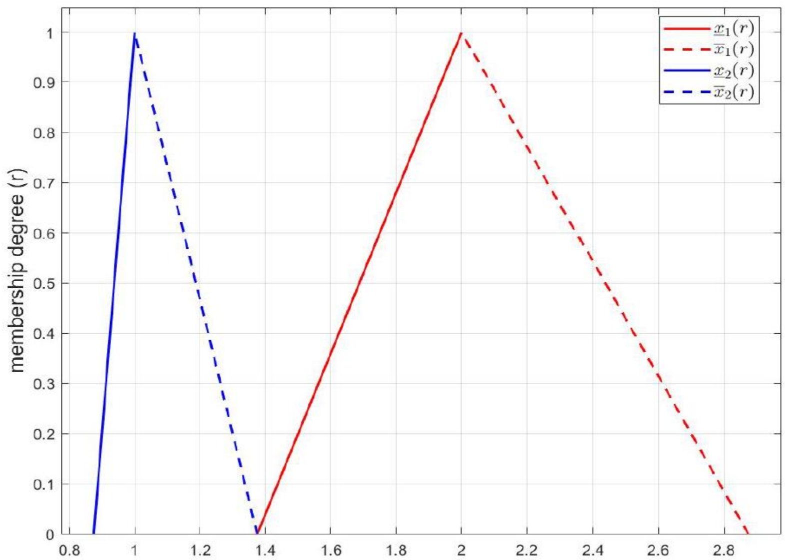

Figure 2: A strong fuzzy solution.

Example 5.2. [1] Consider the 3×3 dual semi fuzzy linear system

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{y}_1 = [0,1-r] \\ \tilde{y}_2 = [-1+r, 1-r] \\ \tilde{y}_3 = [-2,-1-r] \end{pmatrix}, \begin{pmatrix} \tilde{z}_1 = [2,2-r] \\ \tilde{z}_2 = [2+r, 3] \\ \tilde{z}_3 = [2+2r, 5-r] \end{pmatrix}.$$

By the Theorem 4.1, the dual fuzzy linear systems can be extended into the crisp linear system as

$$Gu(r) = v(r), 0 \le r \le 1,$$

where

$$G = \begin{pmatrix} A-B & O \\ O & |A|-|B| \end{pmatrix}, u(r) = \begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix}, v(r) = \begin{pmatrix} z^C(r) - y^C(r) \\ z^R(r) - y^R(r) \end{pmatrix}$$

and

$$y^C(r) = \frac{\underline{y}(r) + \overline{y}(r)}{2} = \begin{pmatrix} 0.5-0.5r \\ 0 \\ -1.5-0.5r \end{pmatrix}, y^R(r) = \frac{\overline{y}(r) - \underline{y}(r)}{2} = \begin{pmatrix} 0.5-0.5r \\ 1-r \\ 0.5-0.5r \end{pmatrix},$$

$$z^C(r) = \frac{\underline{z}(r) + \overline{z}(r)}{2} = \begin{pmatrix} 2.0-0.5r \\ 2.5+0.5r \\ 3.5+0.5r \end{pmatrix}, z^R(r) = \frac{\overline{z}(r) - \underline{z}(r)}{2} = \begin{pmatrix} 0-0.5r \\ 0.5-0.5r \\ 1.5-1.5r \end{pmatrix}.$$

Now that the matrices $A-B$ and $|A|-|B|$ are invertible, we can obtain the solution of the above model is

$$\begin{pmatrix} x^C(r) \\ x^R(r) \end{pmatrix} = \begin{pmatrix} A-B & O \\ O & |A|-|B| \end{pmatrix}^{-1} \begin{pmatrix} z^C(r) - y^C(r) \\ z^R(r) - y^R(r) \end{pmatrix}$$

$$= \begin{pmatrix} (A-B)^{-1} & O \\ O & (|A|-|B|)^{-1} \end{pmatrix} \begin{pmatrix} z^C(r) - y^C(r) \\ z^R(r) - y^R(r) \end{pmatrix} = \begin{pmatrix} -0.8636 + 0.4545r \\ -0.6818 - 0.2727r \\ 1.0909 + 0.1363r \\ -0.5000 + 0.0000r \\ 1.5000 - 1.0000r \\ 1.0000 - 0.5000r \end{pmatrix}.$$

Thus we have

$$\underline{x}(r) = x^C(r) - x^R(r) = \begin{pmatrix} -0.8636 + 0.4545r \\ -0.6818 - 0.2727r \\ 1.0909 + 0.1363r \end{pmatrix} - \begin{pmatrix} -0.5000 + 0.0000r \\ 1.5000 - 1.0000r \\ 1.0000 - 0.5000r \end{pmatrix} = \begin{pmatrix} -0.3636 + 0.4545r \\ -2.1818 + 0.7273r \\ 0.0909 + 0.6363r \end{pmatrix},$$

$$\overline{x}(r) = x^C(r) + x^R(r) = \begin{pmatrix} -0.0476 - 0.2381r \\ -0.9286 - 0.1428r \\ 1.2619 + 0.3095r \end{pmatrix} + \begin{pmatrix} -0.5000 + 0.0000r \\ 1.5000 - 1.0000r \\ 1.0000 - 0.5000r \end{pmatrix} = \begin{pmatrix} -1.3636 + 0.4545r \\ 0.8182 - 1.2727r \\ 2.0909 - 0.3637r \end{pmatrix}.$$

So we know that the fuzzy solution of the original fuzzy linear systems is

$$\begin{cases} \tilde{x}_1 = [-0.3636 + 0.4545r, -1.3636 + 0.4545r], \\ \tilde{x}_2 = [-2.1818 + 0.7273r, 0.8182 - 1.2727r], \\ \tilde{x}_3 = [0.0909 + 0.6363r, 2.0909 - 0.3637r]. \end{cases}$$

and it admits a weak fuzzy solution since in which $\tilde{x}_1 = [-0.3636 + 0.4545r, -1.3636 + 0.4545r]$ is not an appropriate fuzzy number.

6. Conclusion

In this paper, the equivalence between the new method proposed in [29] and the conventional method in [1] for solving two types of fuzzy linear systems is analyzed from the perspective of the basic operations of fuzzy numbers. Firstly, through an in-depth analysis of the traditional method for solving the system $A\tilde{x} = \tilde{b}$, it is rigorously demonstrated that the computational model used in the new method and its solutions are mathematically fully equivalent to those of the traditional method. Subsequently, this analytical framework is extended to dual fuzzy linear systems, and similar equivalence conclusions are verified. Finally, two numerical examples are provided to intuitively illustrate the effectiveness of the new method in simplifying computations while maintaining solution consistency. In the future, we will further explore the deeper mathematical mechanisms underlying different algorithms.

Conflict of Interest

The authors declare no conflict of interest.

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