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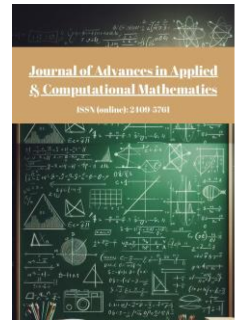

Existence, Uniqueness, and Ulam Stability of Conformable Integro-Dynamic Equations with Deviating Arguments on Time Scales

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ABSTRACT

In this paper, we investigate a class of nonlinear conformable integro-dynamic equations with deviating arguments on time scales. By transforming the considered problem into an equivalent integral equation, sufficient conditions for the existence and uniqueness of solutions are established using Schauder's and Banach's fixed point theorems. Furthermore, Hyers–Ulam and generalized Hyers–Ulam stability results are derived under suitable assumptions. The obtained results extend several existing studies on nonlinear integro-dynamic equations within the framework of conformable calculus on time scales.

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1. Introduction

The calculus of time scales, initiated by Hilger [1], was developed to unify continuous and discrete analysis within a single mathematical framework. This theory enables the simultaneous treatment of differential and difference equations and has become an effective tool in the investigation of hybrid dynamical systems arising in physics, engineering, economics, and biological sciences [2-6]. In many practical models, the present state of a system depends not only on instantaneous changes but also on accumulated past effects. Such memory and delay phenomena arise naturally in population dynamics, engineering systems, control theory, and biological processes, where the current evolution is influenced by previous states of the system. Consequently, dynamic equations with deviating arguments have attracted considerable attention in the qualitative theory of dynamic equations, particularly in the study of oscillation and asymptotic behavior. We refer the reader to the works of Bohner *et al*. on Fite–Hille–Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments [7] and Kamenev-type oscillation criteria for nonlinear damped dynamic equations [8], among other related contributions [9-13].

Over the last decades, qualitative properties of integro-dynamic equations have been extensively investigated. In particular, several authors studied existence and stability of solutions for nonlinear dynamic systems involving integral terms. Xing *et al*. [14] employed monotone iterative methods together with fixed-point techniques to analyze nonlinear integro-dynamic equations on time scales of the form

$$\boldsymbol{r}^{\Delta}(\theta) = F\left(\theta, \boldsymbol{r}(\theta), \int_{\theta_0}^{\theta} K(\theta,\tau)\boldsymbol{r}(\tau)\Delta\tau\right), \qquad \theta \in T$$

where the nonlinearities satisfy suitable continuity assumptions.

Subsequently, Liu *et al*. [15] studied solutions in L^p-spaces on time scales, while related developments concerning generalized integro-dynamic models can be found in [16]. Stability aspects have also received considerable attention. Using fixed-point arguments, Shen [17] established Hyer's-Ulam stability criteria for nonlinear integral equation on time scales. In another direction, motivated by the works in [18], Milcé [19] studied almost automorphic mild solutions for nonlinear dynamic systems in R^n.

Important contributions to nonlinear integro-dynamic equations on time scales were later obtained by Bohner *et al*. [20] and Khuddush *et al*. [21]. Their investigations demonstrated the effectiveness of time-scale methods in studying nonlinear systems with integral interactions.

Another important class of nonlinear problems involves equations with deviating or iterative arguments. Such equations appear naturally in population models, control theory, and feedback systems where the unknown function depends on transformed states of itself. In this direction, Lauran [22] investigated existence and uniqueness results for iterative differential equations involving terms such as

$$\boldsymbol{r}'(\theta) = H(\theta, \boldsymbol{r}(\theta), \boldsymbol{r}(\lambda\theta)),$$

as well as more general iterative structures containing compositions of the unknown function.

Recently, increasing attention has been devoted to conformable-type operators on time scales. Unlike classical fractional operators, which are nonlocal and memory-dependent, conformable derivatives preserve a local structure while maintaining compatibility with the calculus on time scales. Early conformable formulations lacked consistency with the Hilger framework and failed to recover several structural properties of dynamic equations. To overcome these difficulties, Anderson, Georgiev, Bayour, Hammoudi, Torres, and others developed a conformable framework compatible with the standard time-scale exponential function and variation-of-constants formulas [23, 24].

In recent years, there have been attempts to extend the time scale concept to include a fractional-type operators. However, the classical fractional calculus is inherently nonlocal, with convolution kernels and memory operators. Most of the early attempts at a fractional order conformable time scales were actually local and failed to capture memory effects. Specifically, the conformable derivative proposed by Khalil *et al*. [25] is local and fails to preserve the structural properties when extended to arbitrary time scales. For example, it does not naturally interpolate between the identity operator and the delta derivative in a way that is consistent with time scale calculus.

A structurally consistent conformable framework on time scales was later developed and refined in [23, 24]. Let T be a time scale with forward jump operator σ and delta derivative Δ, and let $\alpha \in [0,1]$. In this setting, the conformable derivative is defined by

$$D^{\alpha} f(\theta) = k_1(\alpha, \theta) f(\theta) + k_0(\alpha, \theta) f^{\Delta}(\theta),$$

where the coefficient functions satisfy the following structural assumption.

(A1) The functions

$$k_0, k_1 : [0,1] \times \mathbb{T} \to [0, \infty)$$

are continuous and verify

$$\lim_{\alpha \to 0^+} k_1(\alpha, \theta) = 1 \qquad \lim_{\alpha \to 1^-} k_1(\alpha, \theta) = 0,$$

$$\lim_{\alpha \to 0^+} k_0(\alpha, \theta) = 0 \qquad \lim_{\alpha \to 1^-} k_0(\alpha, \theta) = 1\,, \quad \theta \in \mathbb{T}$$

with

$$k_1(\alpha, \theta) \neq 0 \quad \text{for } \alpha \in [0,1), \qquad k_0(\alpha, \theta) \neq 0 \quad \text{for } \alpha \in (0,1].$$

Unlike classical fractional operators, D^{α} remains local and introduces no memory effects. It is fully compatible with the Hilger exponential and the standard operational structure of time scale calculus, so that variation-of-constants formulas and exponential-type solutions retain their classical form. It is important to emphasize that, despite the terminology often used in the literature, the conformable derivative on time scales is not fractional in the classical sense: it introduces no kernel-based memory and does not involve Mittag–Leffler functions. Instead, its solutions are governed by Hilger exponential functions, preserving structural consistency with dynamic equations on time scales.

The main objective of this study is to investigate the existence, uniqueness, and Hyers–Ulam stability of solutions to nonlinear conformable integro-dynamic equations on arbitrary time scales:

$$D^{\alpha} \boldsymbol{r}(\theta) + p(\theta) \boldsymbol{r}^{\sigma}(\theta) = \int_{\eta}^{\theta} \aleph(\theta, \tau) Q(\tau, \boldsymbol{r}(\tau), \boldsymbol{r}(\boldsymbol{r}(\tau))) \Delta\tau, \quad \theta \in \mathbb{T}^{\kappa}, \quad \lambda \in (0,1), \tag{1}$$

subject to

$$\mathbf{r}(\eta) = \mathbf{r}_{\eta} \in \mathbb{R}. \tag{2}$$

Here, $T = [\eta, \xi] \cap \mathbb{T}$. Under suitable assumptions on the functions $p, \aleph$, and Q, we obtain existence, uniqueness, and Hyers–Ulam stability results for the considered problem by means of fixed-point methods.

2. Preliminaries

This section is devoted to presenting several preliminary notions and auxiliary results from conformable dynamic calculus [23].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$. For each $\theta \in \mathbb{T}$, the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(\theta) = \inf\, \{s \in \mathbb{T}: s > \theta\},$$

while the associated graininess function $\mu: \mathbb{T} \to [0, \infty)$ is given by

$$\mu(\theta) = \sigma(\theta) - \theta.$$

A point $\theta \in \mathbb{T}$ is called right–dense whenever $\mu(\theta) = 0$, whereas it is said to be right–scattered if $\mu(\theta) > 0$.

A mapping $f: \mathbb{T} \to \mathbb{R}$ is termed *regulated* provided that the right limit $f(\theta^+)$ exists for every $\theta \in \mathbb{T}$, and the left limit $f(\theta^-)$ exists whenever $\theta > \inf \mathbb{T}$.

Let $\mathbb{T}$ be a time scale. A function $f: \mathbb{T} \to \mathbb{R}$ is said to be *rd–continuous* if it is continuous at all right–dense points of $\mathbb{T}$ and possesses finite left-sided limits at each left–dense point. The collection of all rd–continuous functions is denoted by

$$C_{\mathrm{rd}}(\mathbb{T}) = \{f: \mathbb{T} \to \mathbb{R} : f \text{ is rd} - \text{continuous}\}.$$

For a function $f: \mathbb{T} \to \mathbb{R}$, the *delta derivative* of f at $\theta \in \mathbb{T}^\kappa$ is the number $f^\Delta(\theta)$ (if it exists) satisfying the following property: for every $\varepsilon > 0$, there exists a neighborhood U of θ such that

$$|f(\sigma(\theta)) - f(s) - f^\Delta(\theta)(\sigma(\theta) - s)| \le \varepsilon|\sigma(\theta) - s|, \qquad s \in U.$$

Let p be a regressive function on $\mathbb{T}$. The exponential function associated with p, denoted by $e_p(\theta, s)$, is defined as the unique solution of the initial value problem

$$y^\Delta(\theta) = p(\theta)y(\theta), \qquad y(s) = 1,$$

for $\theta \in \mathbb{T}^\kappa$.

Moreover, the time-scale exponential admits the representation

$$e_p(\theta, s) = exp\left(\int_s^\theta \Psi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

where the cylinder transformation Ψ_h is defined by

$$\Psi_h(z) = \begin{cases} \frac{1}{h} log(1 + zh), & h > 0, \\ z, & h = 0. \end{cases} \tag{3}$$

In particular, when $\mathbb{T} = \mathbb{R}$, the above expression reduces to

$$exp\left(\int_s^t p(\tau)d\tau\right),$$

whereas for $\mathbb{T} = \mathbb{Z}$, one obtains

$$\prod_{k=s}^{t-1}(1 + p(k)).$$

Definition 2.1 [23] A function $p: \mathbb{T}^\kappa \to \mathbb{R}$ is called conformably regressive if

$$1 + \mu(\theta)h_p(\theta) \neq 0, \qquad \theta \in \mathbb{T}^\kappa,$$

where

$$h_p(\theta) = \frac{p(\theta) - k_0(\alpha, \theta)}{k_1(\alpha, \theta)}.$$

The family of all conformably regressive functions is denoted by

$$R_c = \{p: \mathbb{T}^\kappa \to \mathbb{R}: 1 + \mu(\theta)h_p(\theta) \neq 0\}.$$

Definition 2.2 [23] For $f, g \in R_c$, the conformable circle plus operation $\oplus_c$ is defined by

$$(f \oplus_c g)(\theta) := \frac{(f(\theta) + g(\theta) - k_1(\alpha,\theta))k_0(\alpha,\theta) + \mu(\theta)(f(\theta) - k_1(\alpha,\theta))(g(\theta) - k_1(\alpha,\theta))}{k_0(\alpha,\theta)},$$

where $\theta \in \mathbb{T}$ and $\alpha \in (0,1]$.

Definition 2.3 [23] Let $f \in R_c$. The conformable addition inverse of f is

$$\ominus_c f := -\frac{k_0(\alpha,\theta)(f(\theta) - k_1(\alpha,\theta))}{k_0(\alpha,\theta) + \mu(\theta)(f(\theta) - k_1(\alpha,\theta))} + k_1(\alpha,\theta).$$

Definition 2.4 [23] For $f, g \in R_c$, the conformable circle minus operation $!_c$ is given by

$$f \ominus_c g := f \oplus_c (\ominus_c g).$$

Definition 2.5 [23] Suppose that $\alpha \in (0,1]$ and $p \in R_c$. For $\theta, \theta_0 \in \mathbb{T}$, the conformable exponential function is defined by

$$E_p(\theta,\theta_0) = e_{\frac{p - k_1(\alpha,\cdot)}{k_0(\alpha,\cdot)}}(\theta,\theta_0).$$

We have

$$D^\alpha E_p(\theta,\theta_0) = p(\theta)E_p(\theta,\theta_0), \qquad \theta \in \mathbb{T}^\kappa.$$

Definition 2.6 [23] The conformable Δ -integral of f over $[a,b]$ is defined by

$$\int_a^\theta f(\tau)\Delta_{\alpha,\theta}\tau \; = \; \int_a^\theta f(\tau)\frac{E_0(\theta,\sigma(\tau))}{k_0(\alpha,\tau)}\Delta\tau, \qquad \theta \in [a,b]_{\mathbb{T}}.$$

Lemma 2.1 [23] Let $\alpha \in (0,1]$, $f \in C_{\mathrm{rd}}(\mathbb{T})$, and assume that $k_0(\alpha,\theta) - \mu(\theta)k_1(\alpha,\theta) \neq 0, \theta \in \mathbb{T}$ holds. Then

$$\int_a^\theta D^\alpha f(\tau)\Delta_{\alpha,\theta}\tau = f(\theta) - f(a)E_0(\theta,a), \qquad \theta \in [a,b]_{\mathbb{T}}^\kappa.$$

Lemma 2.2 [23] Consider the initial value problem

$$D^\alpha y(\theta) = p(t)y^\sigma(\theta) + q(\theta), \qquad \theta \in \mathbb{T}^\kappa, \tag{4}$$

subject to the initial condition

$$y(\theta_0) = y_0, \tag{5}$$

where $\theta_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, and $p, q \in C_{rd}(\mathbb{T})$. Assume that

$$k_0(\alpha,\theta) - \mu(t)p(\theta) \neq 0, \qquad t \in \mathbb{T}, \quad \alpha \in (0,1].$$

Then the unique solution of (4)-(5) is given by

$$y(\theta) = y_0 E_g(\theta,\theta_0) + \int_{\theta_0}^\theta q(s)E_{-p+k_1}(s,\theta)\Delta_{\alpha,\theta}s, \qquad \theta \in \mathbb{T}^\kappa,$$

where

$$g = \frac{(-p + k_1 - k_1)(k_0 - \mu k_1)}{k_0 + \mu(-p + k_1 - k_1)} = \frac{p(k_0 - \mu k_1)}{k_0 - \mu p}.$$

Lemma 2.3 Let $p \in R(\mathfrak{T}, \mathbb{R})$, $\eta \in \mathbb{T}$, $\boldsymbol{r}_\eta \in \mathbb{R}$, $\aleph \in C_{rd}(\mathfrak{T} \times \mathfrak{T})$ and $\mathcal{Q} \in C_{rd}(\mathfrak{T} \times \mathbb{R}, \mathbb{R})$. Assume that

$$k_0(\alpha, \theta) - \mu(\theta)p(\theta) \neq 0, \qquad \theta \in \mathbb{T}^\kappa.$$

Then, $\boldsymbol{r}$ satisfies (1)-(2) if and only if $\boldsymbol{r}$ satisfies the integral equation

$$\boldsymbol{r}(\theta) = \boldsymbol{r}_\eta E_g(\theta, \eta) + \int_\eta^\theta E_{p+k_1}(\tau, \theta) \left(\int_\eta^\tau \aleph(\tau, \mathfrak{z}) Q(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\tau}\mathfrak{z} \right) \Delta_{\alpha,\theta}\tau, \tag{6}$$

where

$$g(\theta) = \frac{-p(\theta)(k_0(\alpha, \theta) - \mu(\theta)k_1(\alpha, \theta))}{k_0(\alpha, \theta) + \mu(\theta)p(\theta)}.$$

Proof. Suppose that $\boldsymbol{r}$ satisfies (1) and (2). Define

$$q(\theta) = \int_\eta^\theta \aleph(\theta, \mathfrak{z}) \mathcal{Q}(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\theta}\mathfrak{z}.$$

Then equation (1) becomes

$$D^\alpha \boldsymbol{r}(\theta) + p(\theta)\boldsymbol{r}^\sigma(\theta) = q(\theta).$$

Equivalently,

$$D^\alpha \boldsymbol{r}(\theta) = -p(\theta)\boldsymbol{r}^\sigma(\theta) + q(\theta).$$

By applying Lemma 2.2 with

$$y = \boldsymbol{r}, \qquad p \mapsto -p, \qquad q \mapsto q,$$

we obtain

$$\boldsymbol{r}(\theta) = \boldsymbol{r}_\eta E_g(\theta, \eta) + \int_\eta^\theta q(\tau) E_{p+k_1}(\tau, \theta) \Delta_{\alpha,\theta}\tau,$$

where

$$g(\theta) = \frac{-p(\theta)(k_0(\alpha, \theta) - \mu(\theta)k_1(\alpha, \theta))}{k_0(\alpha, \theta) + \mu(\theta)p(\theta)}.$$

Substituting the expression of q into the above equation, we derive

$$\boldsymbol{r}(\theta) = \boldsymbol{r}_\eta E_g(\theta, \eta) + \int_\eta^\theta E_{p+k_1}(\tau, \theta) \left(\int_\eta^\tau \aleph(\tau, \mathfrak{z}) Q(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\tau}\mathfrak{z} \right) \Delta_{\alpha,\theta}\tau,$$

which is precisely (6). Conversely, suppose that (6) holds. Setting $\theta = \eta$ in (6), we obtain

$$\boldsymbol{r}(\eta) = \boldsymbol{r}_\eta,$$

which verifies the initial condition (2). Next, applying the conformable derivative operator D^α to both sides of (6) and using Lemma 2.2 together with the fundamental theorem of conformable calculus on time scales, we obtain

$$D^\alpha \boldsymbol{r}(\theta) = -p(\theta)\boldsymbol{r}^\sigma(\theta) + \int_\eta^\theta \aleph(\theta, \mathfrak{z})\, \mathcal{Q}(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\tau}\mathfrak{z}.$$

Hence,

$$D^\alpha \boldsymbol{r}(\theta) + p(\theta)\boldsymbol{r}^\sigma(\theta) = \int_\eta^\theta \aleph(\theta, \mathfrak{z}) \mathcal{Q}(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\tau}\mathfrak{z},$$

which shows that (1) holds. Therefore, $\boldsymbol{r}$ satisfies (1)-(2).

3. Existence and Uniqueness of Solutions

In this section, we establish sufficient conditions for the existence and uniqueness of solutions for the conformable integro-dynamic problem (1)-(2). To this end, we consider the Banach space

$$C[\eta,\xi]=\{\boldsymbol{r}:\mathbb{T}\to\mathbb{R}:\boldsymbol{r}\text{ is continuous on }[\eta,\xi]_{\mathbb{T}}\},$$

equipped with the supremum norm

$$\|\boldsymbol{r}\|=\sup_{\theta\in[\eta,\xi]_{\mathbb{T}}}|\boldsymbol{r}(\theta)|$$

For a given constant $N>0$, define the subset

$$\mathbb{F}_N=\{\boldsymbol{r}\in C[\eta,\xi]:\|\boldsymbol{r}\|\le N\}.$$

Clearly, $\mathbb{F}_N$ is a closed subset of $C[\eta,\xi]$. Next, for a fixed constant $\varepsilon>0$, we introduce the set

$$\mathbb{F}_N[\varepsilon]=\{\boldsymbol{r}\in\mathbb{F}_N:|\boldsymbol{r}(\theta_1)-\boldsymbol{r}(\theta_2)|\le\varepsilon|\theta_1-\theta_2|,\quad\forall\theta_1,\theta_2\in[\eta,\xi]_{\mathbb{T}}\}.$$

It follows that $F_N[\varepsilon]$ is bounded, closed, and convex in $C[\eta,\xi]$. We now define the operator

$$\mathcal{T}:\mathbb{F}_N[\varepsilon]\to\mathbb{F}_N$$

by

$$(\mathcal{T}\boldsymbol{r})(\theta)=\boldsymbol{r}_\eta E_g(\theta,\eta)+\int_\eta^\theta E_{p+k_1}(\mathcal{T},\theta)\left(\int_\eta^{\mathcal{T}}\aleph(\mathcal{T},\mathfrak{z})\mathcal{Q}(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta\mathcal{T}} \tag{7}$$

for $\theta\in\mathbb{T}$. A function $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$ is said to be a solution of problem (1)-(2) whenever it satisfies $\mathcal{T}\boldsymbol{r}=\boldsymbol{r}$. Accordingly, the analysis of the considered conformable integro-dynamic problem can be reduced to the study of fixed points of the operator $\mathcal{T}$.

For later use in the sequel, we introduce the following quantities:

$$\widehat{\aleph}:=\sup_{(\mathcal{T},\theta)\in\mathbb{K}\times\mathbb{K}}|\aleph(\mathcal{T},\theta)|\;>\;\mathbf{0},$$

$$\mathrm{E}_1:=\sup_{\theta\in\mathbb{K}}\int_\eta^\theta\left|\mathrm{E_g}(\theta,\eta)\right|,$$

and

$$\boldsymbol{E_3}:=\sup_{(\theta,t)\in\mathbb{K}\times\mathbb{K}}\left|\boldsymbol{E_{p+k_1}}(\theta,t)\right|\;>\;\mathbf{0},$$

We shall assume throughout this section that the following hypotheses are satisfied:

(A) $\alpha_1>0$ and $\alpha_2>0$ with

$$|\mathcal{Q}(\mathfrak{z},\boldsymbol{r}_1,\boldsymbol{r}_2)-\mathcal{Q}(\mathfrak{z},v_1,v_2)|\le\alpha_1|\boldsymbol{r}_1-v_1|+\alpha_2|\boldsymbol{r}_2-v_2|,$$

for all $\mathfrak{z}\in\mathbb{K}$ and $\boldsymbol{r}_1,\boldsymbol{r}_2,v_1,v_2\in\mathbb{R}$.

(B) There exists a constant $N>0$ such that

$$\frac{|\boldsymbol{r}_\eta|E_1+\widehat{\aleph}\mathcal{Q}_0(\xi-\eta)E_2}{1-\widehat{\aleph}(\xi-\eta)(\alpha_1+\alpha_2\lambda\varepsilon)E_2}\le N,$$

where

$$\mathcal{Q}_0=\max_{\mathfrak{z}\in\mathbb{K}}|\mathcal{Q}(\mathfrak{z},0,0)|,$$

and

$$\widehat{\aleph}(\xi-\eta)(\alpha_1+\alpha_2\lambda\varepsilon)E_2<1.$$

(C) The inequality

$$(|\boldsymbol{r}_\eta|+\widehat{\aleph}N(\xi-\eta)^2(\alpha_1+\alpha_2\lambda\varepsilon+1))E_3\le\varepsilon$$

holds.

(D)

$$\widehat{\aleph}(\xi-\eta)(\alpha_1+(\lambda\varepsilon+1)\alpha_2)E_2<1.$$

The following auxiliary results play an essential role in proving the main existence and uniqueness theorems.

Lemma 3.1 Suppose that assumption (A) holds true. Then the operator $\mathcal{T}$ is continuous and completely continuous on $\mathbb{F}_N[\varepsilon]$.

Proof. Let $\boldsymbol{r},v\in\mathbb{F}_N[\varepsilon]$ and choose an arbitrary $\theta\in\mathbb{T}$. Using hypothesis (A), we obtain

$$\begin{aligned}
\|\mathcal{T}\boldsymbol{r}-\mathcal{T}\mathbf{v}\| &= \sup_{\theta\in\mathbb{T}}|(\mathcal{T}\boldsymbol{r})(\theta)-(\mathcal{T}\mathbf{v})(\theta)|\\
&\le\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^\theta|\aleph(\mathcal{T},\mathfrak{z})|\,|\mathcal{Q}(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-\mathcal{Q}(\mathfrak{z},\mathbf{v}(\mathfrak{z}),\mathbf{v}(\lambda\mathbf{v}(\mathfrak{z})))|\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^\theta|\aleph(\mathcal{T},\mathfrak{z})|\,(\alpha_1|\boldsymbol{r}(\mathfrak{z})-\mathbf{v}(\mathfrak{z})|+\alpha_2|\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))-\mathbf{v}(\lambda\mathbf{v}(\mathfrak{z}))|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^{\mathcal{T}}|\aleph(\mathcal{T},\mathfrak{z})|\,(\alpha_1|\boldsymbol{r}(\mathfrak{z})-\mathbf{v}(\mathfrak{z})|+\alpha_2|\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))-\boldsymbol{r}(\lambda\mathbf{v}(\mathfrak{z}))|+\alpha_2|\boldsymbol{r}(\lambda\mathbf{v}(\mathfrak{z}))-\mathbf{v}(\lambda\mathbf{v}(\mathfrak{z}))|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^{\mathcal{T}}|\aleph(\mathcal{T},\mathfrak{z})|\,(\alpha_1|\boldsymbol{r}(\mathfrak{z})-\mathbf{v}(\mathfrak{z})|+\alpha_2\varepsilon\lambda|\boldsymbol{r}(\mathfrak{z})-\mathbf{v}(\mathfrak{z})|+\alpha_2|\boldsymbol{r}(\lambda\mathbf{v}(\mathfrak{z}))-\mathbf{v}(\lambda\mathbf{v}(\mathfrak{z}))|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^{\xi}|\aleph(\mathcal{T},\mathfrak{z})|\,(\alpha_1\|\boldsymbol{r}-\mathbf{v}\|+\alpha_2\varepsilon\lambda\|\boldsymbol{r}-\mathbf{v}\|+\alpha_2\|\boldsymbol{r}-\mathbf{v}\|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\widehat{\aleph}\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\int_\eta^{\xi}(\alpha_1+(\varepsilon\lambda+1)\alpha_2)\,\|\boldsymbol{r}-\mathbf{v}\|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\widehat{\aleph}\,(\xi-\eta\,)(\alpha_1+(\varepsilon\lambda+1)\alpha_2)\|\boldsymbol{r}-\mathbf{v}\|\int_\eta^\theta\left|E_{p+k_1}(\mathcal{T},\theta)\right|\Delta_{\alpha,\theta\mathcal{T}}\\
&\le\widehat{\aleph}\,(\xi-\eta\,)(\alpha_1+(\varepsilon\lambda+1)\alpha_2)E_2\|\boldsymbol{r}-\mathbf{v}\|.
\end{aligned}$$

Hence, the operator $\mathcal{T}$ is continuous on $\mathbb{F}_N[\varepsilon]$. Furthermore, since $\mathcal{T}(\mathbb{F}_N[\varepsilon])$ is uniformly bounded and equicontinuous, the Arzela –Ascoli theorem guarantees that $\mathcal{T}$ is completely continuous.

Lemma 3.2 Assume (A) – (B) hold. Then $|(\mathcal{T}\boldsymbol{r})(\theta)|\le N$ for all $\theta\in\mathbb{T}$ and $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$.

Proof. Let $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$ and $\theta\in\mathbb{T}$. Then

$$\begin{aligned}
|(\mathcal{T}\boldsymbol{r})(\theta)| &= |\boldsymbol{r}_\eta E_g(\theta,\eta)+\int_\eta^\theta E_{p+k_1}(\mathcal{T},\theta)\left(\int_\eta^{\mathcal{T}}\aleph(\mathcal{T},\mathfrak{z})\mathcal{Q}(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))\,\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}|\\
&\le|\boldsymbol{r}_\eta||E_g(\theta,\eta)|+\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\left(\int_\eta^\tau|\aleph(\mathcal{T},\mathfrak{z})|\,|\mathcal{Q}(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-\mathcal{Q}(\mathfrak{z},0,0)|\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}\\
&\qquad+\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\left(\int_\eta^\tau|\aleph(\mathcal{T},\mathfrak{z})|\,|\mathcal{Q}(\mathfrak{z},0,0)|\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}\\
&\le|\boldsymbol{r}_\eta|E_1+\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\left(\int_\eta^{\mathcal{T}}\widehat{\aleph}(\alpha_1|\boldsymbol{r}|+\alpha_2\lambda\varepsilon|\boldsymbol{r}|)\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}|+\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\left(\int_\eta^\tau\widehat{\aleph}\,\mathcal{Q}_0\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}\\
&\le|\boldsymbol{r}_\eta|E_1+\widehat{\aleph}N(\xi-\eta)(\alpha_1+\alpha_2\lambda\varepsilon)\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\Delta_{\alpha,\theta}\mathcal{T}+\widehat{\aleph}\mathcal{Q}_0(\xi-\eta)\int_\eta^\theta|E_{p+k_1}(\mathcal{T},\theta)|\Delta_{\alpha,\theta}\mathcal{T}\\
&\le|\boldsymbol{r}_\eta|E_1+\widehat{\aleph}N(\xi-\eta)(\alpha_1+\alpha_{2\lambda}\varepsilon)E_2+\widehat{\aleph}\mathcal{Q}_0(\xi-\eta)E_2\\
&\le N.
\end{aligned}$$

The proof is completed.

Lemma 3.3 Suppose that (A) holds. Assume further that there exist positive constants E_3, E_4, and M_α such that

$$|E_{p+k_1}(\mathcal{T},\theta_1)-E_{p+k_1}(\mathcal{T},\theta_2)|\le E_3|\theta_1-\theta_2|,$$

$$|E_g(\theta_1,\eta)-E_g(\theta_2,\eta)|\le E_3|\theta_1-\theta_2|,$$

$$|E_{p+k_1}(\tau,\theta)|\le E_4,$$

and

$$\int_\eta^{\mathcal{T}} 1\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\le M_\alpha,\qquad \forall\mathcal{T}\in\mathfrak{T}.$$

Then

$$|(\mathcal{T}\boldsymbol{r})(\theta_1)-(\mathcal{T}\boldsymbol{r})(\theta_2)|\le\varepsilon|\theta_1-\theta_2|$$

for all $\theta_1,\theta_2\in\mathbb{T}$ and $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$, provided

$$|\boldsymbol{r}_\eta|E_3+\widehat{\aleph}(\alpha_1N+\alpha_2N+Q)(E_3M_\alpha E_2+E_4M_\alpha)\le\varepsilon.$$

Proof. Let $\theta_1,\theta_2\in\mathbb{T}$ with $\theta_1\le\theta_2$, and let $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$. From the definition of the operator $\mathcal{T}$, we have

$$\begin{aligned}
&(\mathcal{T}\boldsymbol{r})(\theta_1)-(\mathcal{T}\boldsymbol{r})(\theta_2)\\
=\;&\boldsymbol{r}_\eta(E_g(\theta_1,\eta)-E_g(\theta_2,\eta))\\
&+\int_\eta^{\theta_1}(E_{p+k_1}(\mathcal{T},\theta_1)-E_{p+k_1}(\mathcal{T},\theta_2))H(\mathcal{T})\Delta_{\alpha,\theta_1}\mathcal{T}\\
&-\int_{\theta_1}^{\theta_2}E_{p+k_1}(\mathcal{T},\theta_2)H(\mathcal{T})\Delta_{\alpha,\theta_2}\mathcal{T},
\end{aligned}$$

where

$$H(\mathcal{T})=\int_\eta^{\mathcal{T}}\aleph(\mathcal{T},\mathfrak{z})Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))\Delta_{\alpha,\mathcal{T}}\mathfrak{z}.$$

Taking absolute values and using the triangle inequality, we obtain

$$\begin{aligned}
&|(\mathcal{T}\boldsymbol{r})(\theta_1)-(\mathcal{T}\boldsymbol{r})(\theta_2)|\\
\le\;&|\boldsymbol{r}_\eta||E_g(\theta_1,\eta)-E_g(\theta_2,\eta)|\\
&+\int_\eta^{\theta_1}|E_{p+k_1}(\mathcal{T},\theta_1)-E_{p+k_1}(\mathcal{T},\theta_2)||H(\mathcal{T})|\Delta_{\alpha,\theta_1}\mathcal{T}\\
&+\int_{\theta_1}^{\theta_2}|E_{p+k_1}(\mathcal{T},\theta_2)||H(\mathcal{T})|\Delta_{\alpha,\theta_2}\mathcal{T}.
\end{aligned}$$

By assumption (A),

$$\begin{aligned}
&|Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))|\\
\le\;&|Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-U(\mathfrak{z},0,0)|+|U(\mathfrak{z},0,0)|\\
\le\;&\alpha_1|\boldsymbol{r}(\mathfrak{z})|+\alpha_2|\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))|+Q.
\end{aligned}$$

Since $\boldsymbol{r}\in\mathbb{F}_N[\varepsilon]$ and $\|\boldsymbol{r}\|\le N$, we obtain

$$|\boldsymbol{r}(\mathfrak{z})|\le N,\qquad |\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))|\le N.$$

Hence,

$$|Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))|\le\alpha_1N+\alpha_2N+Q.$$

Therefore,

$$|H(\mathcal{T})|\le\widehat{\aleph}(\alpha_1N+\alpha_2N+Q)\int_\eta^{\mathcal{T}}1\Delta_{\alpha,\mathcal{T}}\mathfrak{z}.$$

Using the definition of M_α, we get

$$|H(\mathcal{T})| \leq \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q).$$

Consequently,

$$\begin{aligned} |(\mathcal{T}\boldsymbol{r})(\theta_1) - (\mathcal{T}\boldsymbol{r})(\theta_2)| \leq \quad & |\boldsymbol{r}_\eta| E_3 |\theta_1 - \theta_2| + \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q) \\ & \times \int_\eta^{\theta_1} |E_{p+k_1}(\mathcal{T}, \theta_1) - E_{p+k_1}(\mathcal{T}, \theta_2)| \Delta_{\alpha,\theta_1} \mathcal{T} \\ & + \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q) \times \int_{\theta_1}^{\theta_2} |E_{p+k_1}(\mathcal{T}, \theta_2)| \Delta_{\alpha,\theta_2} \mathcal{T}. \end{aligned}$$

Using the hypotheses,

$$\begin{aligned} |(\mathcal{T}\boldsymbol{r})(\theta_1) - (\mathcal{T}\boldsymbol{r})(\theta_2)| \leq \quad & |\boldsymbol{r}_\eta| E_3 |\theta_1 - \theta_2| + \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q) E_3 E_2 |\theta_1 - \theta_2| \\ & + \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q) E_4 |\theta_1 - \theta_2|. \end{aligned}$$

Hence,

$$|(\mathcal{T}\boldsymbol{r})(\theta_1) - (\mathcal{T}\boldsymbol{r})(\theta_2)| \leq \varepsilon |\theta_1 - \theta_2|.$$

Thus the proof is complete.

Theorem 3.1 Assume that conditions (A)–(C) hold. Then the problem (1)–(2) possesses at least one solution in the set $\mathbb{F}_N[\varepsilon]$.

Proof. By definition the set $\mathbb{F}_N[\varepsilon]$ is nonempty, bounded, closed, and convex in $C[\eta, \xi]$. By Lemma 3.2,

$$\|\mathcal{T}\boldsymbol{r}\| \leq N, \text{ for all } \mathbf{r} \in \mathbb{F}_N[\varepsilon].$$

Hence,

$$\mathcal{T}(\mathbb{F}_N[\varepsilon]) \subseteq \mathbb{F}_N.$$

Further, by Lemma 3.3,

$$|(\mathcal{T}\boldsymbol{r})(\theta_1) - (\mathcal{T}\boldsymbol{r})(\theta_2)| \leq \varepsilon |\theta_1 - \theta_2|,$$

for all $\theta_1, \theta_2 \in \mathbb{T}$ and $\boldsymbol{r} \in \mathbb{F}_N[\varepsilon]$. Therefore, the family $\mathcal{T}(\mathbb{F}_N[\varepsilon])$ is equicontinuous. Since $\mathcal{T}(\mathbb{F}_N[\varepsilon])$ is uniformly bounded and equicontinuous, the Arzela –Ascoli theorem implies that $\mathcal{T}(\mathbb{F}_N[\varepsilon])$ is relatively compact in $C[\eta, \xi]$. Consequently, the operator $\mathcal{T}$ is compact. Moreover, by Lemma 3.1, the operator $\mathcal{T}$ is continuous on $\mathbb{F}_N[\varepsilon]$. Combining Lemmas 3.2 and 3.3, we conclude that

$$\mathcal{T}(F_N[\varepsilon]) \subseteq F_N[\varepsilon].$$

Hence, $\mathcal{T}$ is a continuous compact self-map on the nonempty closed convex subset $\mathbb{F}_N[\varepsilon]$ of the Banach space $C[\eta, \xi]$. Therefore, Schauder's fixed point theorem guarantees the existence of a fixed point $\boldsymbol{r}^* \in \mathbb{F}_N[\varepsilon]$ such that $\mathcal{T}\boldsymbol{r}^* = \boldsymbol{r}^*$. Thus, $\boldsymbol{r}^*$ is a solution of (1)–(2).

Corollary 3.1 Assume that condition (A) holds. Suppose that

$$\frac{|r_\eta| E_1 + \widehat{\aleph} Q_0 M_\alpha E_2}{1 - \widehat{\aleph} M_\alpha(\alpha_1 + \alpha_2) E_2} \leq N,$$

and

$$|\boldsymbol{r}_\eta| E_3 + \widehat{\aleph} M_\alpha(\alpha_1 N + \alpha_2 N + Q)(E_3 E_2 + E_4) \leq \varepsilon. \tag{8}$$

Then the problem (1)–(2) admits at least one solution in $\mathbb{F}_N[\varepsilon]$.

Proof. The first inequality guarantees that condition (B) holds, while inequality (8) ensures condition (C). Hence, assumptions (A)–(C) are satisfied. Therefore, all hypotheses of Theorem 3.1 hold. Consequently, Theorem 3.1 implies that the problem (1)–(2) possesses at least one solution in $\mathbb{F}_N[\varepsilon]$.

Theorem 3.2 Assume that conditions (A)–(D) hold. Further, suppose that

$$\widehat{\aleph} M_\alpha(\alpha_1+\alpha_2)E_2 < 1.$$

Then the boundary value problem (1)–(2) admits a unique solution in $\mathbb{F}_N[\varepsilon]$.

Proof. Let $\boldsymbol{r},\boldsymbol{v} \in \mathbb{F}_N[\varepsilon]$. For any $\theta \in \mathfrak{T}$, we have

$$\begin{aligned}|(\mathcal{T}\boldsymbol{r})(\theta)-(\mathcal{T}\boldsymbol{v})(\theta)| \leq \quad & \int_\eta^\theta |E_{p+k_1}(\mathcal{T},\theta)| \\ & \times \left(\int_\eta^\tau |\aleph(\mathcal{T},\mathfrak{z})|\,|Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-Q(\mathfrak{z},\boldsymbol{v}(\mathfrak{z}),\boldsymbol{v}(\lambda\boldsymbol{v}(\mathfrak{z})))|\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}.\end{aligned}$$

By assumption (A),

$$\begin{aligned}|Q(\mathfrak{z},\boldsymbol{r}(z),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-Q(\mathfrak{z},\boldsymbol{v}(\mathfrak{z}),\boldsymbol{v}(\lambda\boldsymbol{v}(\mathfrak{z})))| \leq \quad & \alpha_1|\boldsymbol{r}(\mathfrak{z})-\boldsymbol{v}(\mathfrak{z})| \\ & +\alpha_2|\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))-\boldsymbol{v}(\lambda\boldsymbol{v}(\mathfrak{z}))|.\end{aligned}$$

Using condition (D), we obtain

$$|\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z}))-\boldsymbol{v}(\lambda\boldsymbol{v}(\mathfrak{z}))| \leq \|\boldsymbol{r}-\boldsymbol{v}\|.$$

Hence,

$$|Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))-Q(\mathfrak{z},\boldsymbol{v}(\mathfrak{z}),\boldsymbol{v}(\lambda\boldsymbol{v}(\mathfrak{z})))| \leq \quad (\alpha_1+\alpha_2)\|\boldsymbol{r}-\boldsymbol{v}\|.$$

Therefore,

$$|(\mathcal{T}\boldsymbol{r})(\theta)-(\mathcal{T}\boldsymbol{v})(\theta)| \leq \quad \widehat{\aleph}(\alpha_1+\alpha_2)\|\boldsymbol{r}-\boldsymbol{v}\| \int_\eta^\theta |E_{p+k_1}(\mathcal{T},\theta)| \left(\int_\eta^{\mathcal{T}} 1\Delta_{\alpha,\mathcal{T}}\mathfrak{z}\right)\Delta_{\alpha,\theta}\mathcal{T}.$$

Using the definition of M_α, we obtain

$$|(\mathcal{T}\boldsymbol{r})(\theta)-(\mathcal{T}\boldsymbol{v})(\theta)| \leq \quad \widehat{\aleph}M_\alpha(\alpha_1+\alpha_2)\|\boldsymbol{r}-\boldsymbol{v}\| \int_\eta^\theta |E_{p+k_1}(\mathcal{T},\theta)|\Delta_{\alpha,\theta}\mathcal{T}.$$

Hence,

$$|(\mathcal{T}\boldsymbol{r})(\theta)-(\mathcal{T}\boldsymbol{v})(\theta)| \leq \widehat{\aleph}M_\alpha(\alpha_1+\alpha_2)E_2\|\boldsymbol{r}-\boldsymbol{v}\|.$$

Taking supremum over $\theta \in \mathfrak{T}$, we get

$$\|\mathcal{T}\boldsymbol{r}-\mathcal{T}\boldsymbol{v}\| \leq \widehat{\aleph}M_\alpha(\alpha_1+\alpha_2)E_2\|\boldsymbol{r}-\boldsymbol{v}\|.$$

The condition

$$\widehat{\aleph}M_\alpha(\alpha_1+\alpha_2)E_2 < 1$$

implies that the mapping $\mathcal{T}$ satisfies the contraction property on the complete metric space $\mathbb{F}_N[\varepsilon]$. Consequently, an application of Banach's contraction principle ensures the existence of a single element

$$\boldsymbol{r}^* \in \mathbb{F}_N[\varepsilon]$$

such that $\mathcal{T}\boldsymbol{r}^*=\boldsymbol{r}^*$. Therefore, $\boldsymbol{r}^*$ constitutes the unique solution to problem (1)–(2).

Corollary 3.2 Suppose that assumptions (A) and (D) are fulfilled. In addition, assume that inequality (8) holds together with

$$\widehat{\aleph} M_{\alpha}(\alpha_1+\alpha_2)E_2<1.$$

Then the problem (1)–(2) possesses exactly one solution belonging to $\mathbb{F}_N[\varepsilon]$.

4. Ulam Stability Analysis

In this section, we investigate the Hyers–Ulam stability of problem (1)–(2).

Definition 4.1 Equation (1) is said to possess Hyers–Ulam stability if there exists a constant $L>0$ such that for every $\varepsilon>0$ and every function

$$\boldsymbol{r}\in C_{rd}^{1}(\mathfrak{T},\mathbb{R})$$

satisfying

$$\left|D^{\alpha}\boldsymbol{r}(\theta)+p(\theta)\boldsymbol{r}^{\sigma}(\theta)-\int_{\eta}^{\theta}\aleph(\theta,\mathfrak{z})\,Q(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))\Delta_{\alpha,\theta}\mathfrak{z}\right|\leq\varepsilon, \tag{9}$$

for all $\theta\in\mathbb{T}^{\kappa}$, there exists a solution

$$x\in C_{rd}(\mathfrak{T},\mathbb{R})$$

of (1)–(2) such that

$$|\boldsymbol{r}(\theta)-x(\theta)|\leq\varepsilon L,\ \text{ for all }\ \theta\in\mathfrak{T}.$$

Definition 4.2 Equation (1) possesses generalized Hyers–Ulam stability if there exists a continuous function

$$\theta:\mathbb{R}^{+}\to\mathbb{R}^{+}$$

with $\theta(0)=0$ such that for every function

$$\boldsymbol{r}\in C_{rd}^{1}(\mathfrak{T},\mathbb{R})$$

satisfying (9), there exists a solution

$$x\in C_{rd}(\mathfrak{T},\mathbb{R})$$

of (1)–(2) satisfying

$$|\boldsymbol{r}(\theta)-x(\theta)|\leq\theta(\varepsilon),\qquad\forall\theta\in\mathfrak{T}.$$

Remark 4.1 *A function*

$$\boldsymbol{r}\in C_{rd}^{1}(\mathfrak{T},\mathbb{R})$$

satisfies (9) if and only if there exists a function

$$G\in C_{rd}(\mathfrak{T},\mathbb{R})$$

such that

$$|G(\theta)|\leq\varepsilon,\qquad\forall\theta\in\mathfrak{T},$$

and

$$D^{\alpha}\boldsymbol{r}(\theta)+p(\theta)\boldsymbol{r}^{\sigma}(\theta)=\int_{\eta}^{\theta}\aleph(\theta,\mathfrak{z})Q\,(\mathfrak{z},\boldsymbol{r}(\mathfrak{z}),\boldsymbol{r}(\lambda\boldsymbol{r}(\mathfrak{z})))\Delta_{\alpha,\theta}\mathfrak{z}+G(\theta).$$

Theorem 4.1 Assume that condition (A) holds and

$$\widehat{\aleph} M_\alpha (\alpha_1 + \alpha_2) E_2 < 1.$$

Then equation (1)–(2) possesses Hyers–Ulam stability with HUS constant

$$L := \frac{E_2}{1 - \widehat{\aleph} M_\alpha (\alpha_1 + \alpha_2) E_2}. \tag{10}$$

Proof. Let

$$\boldsymbol{r} \in C^1_{rd}(\mathfrak{T}, \mathbb{R})$$

satisfy (9). Then, by Remark 4.1,

$$D^\alpha \boldsymbol{r}^\Delta(\theta) + p(\theta) \boldsymbol{r}^\sigma(\theta) = \int_\eta^\theta \aleph(\theta, \mathfrak{z}) Q\,(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\lambda \boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\theta} \mathfrak{z} + G(\theta),$$

where

$$|G(\theta)| \le \varepsilon.$$

Using the variation of constants formula, we obtain

$$\begin{aligned} \boldsymbol{r}(\theta) = \quad & \boldsymbol{r}_\eta E_g(\theta, \eta) \\ & + \int_\eta^\theta E_{p+k_1}(\mathcal{T}, \theta) \left(\int_\eta^\tau \aleph(\mathcal{T}, \mathfrak{z}) Q\,(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\lambda \boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\mathcal{T}} \mathfrak{z} \right) \Delta_{\alpha,\theta} \mathcal{T} \\ & + \int_\eta^\theta E_{p+k_1}(\mathcal{T}, \theta) G(\mathcal{T}) \Delta_{\alpha,\theta} \mathcal{T}. \end{aligned}$$

Hence,

$$\begin{aligned} |\boldsymbol{r}(\theta) \quad & - \boldsymbol{r}_\eta E_g(\theta, \eta) - \int_\eta^\theta E_{p+k_1}(\tau, \theta) \left(\int_\eta^{\mathcal{T}} \aleph(\mathcal{T}, \mathfrak{z}) Q\,(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\lambda \boldsymbol{r}(\mathfrak{z}))) \Delta_{\alpha,\mathcal{T}} \mathfrak{z} \right) \Delta_{\alpha,\theta} \mathcal{T}| \\ \le \quad & \varepsilon \int_\eta^\theta |E_{p+k_1}(\mathcal{T}, \theta)| \Delta_{\alpha,\theta} \mathcal{T} \\ \le \quad & \varepsilon E_2. \end{aligned}$$

Let x be the exact solution of (1)–(2). Then

$$\begin{aligned} |\boldsymbol{r}(\theta) - x(\theta)| \le \quad & \varepsilon E_2 + \int_\eta^\theta |E_{p+k_1}(\mathcal{T}, \theta)| \\ & \times \left(\int_\eta^{\mathcal{T}} |\aleph(\mathcal{T}, \mathfrak{z})|\, |Q(\mathfrak{z}, \boldsymbol{r}(\mathfrak{z}), \boldsymbol{r}(\lambda \boldsymbol{r}(\mathfrak{z}))) - Q(\mathfrak{z}, x(\mathfrak{z}), x(\lambda x(\mathfrak{z})))| \Delta_{\alpha,\mathcal{T}} \mathfrak{z} \right) \Delta_{\alpha,\theta} \mathcal{T}. \end{aligned}$$

Using assumption (A) and condition (D), we get

$$|\boldsymbol{r}(\theta) - x(\theta)| \le \varepsilon E_2 + \widehat{\aleph}(\alpha_1 + \alpha_2) \|\boldsymbol{r} - x\| \int_\eta^\theta |E_{p+k_1}(\mathcal{T}, \theta)| \left(\int_\eta^{\mathcal{T}} 1 \Delta_{\alpha,\mathcal{T}} z \right) \Delta_{\alpha,\theta} \mathcal{T}.$$

By the definition of M_α,

$$|\boldsymbol{r}(\theta) - x(\theta)| \le \varepsilon E_2 + \widehat{\aleph} M_\alpha (\alpha_1 + \alpha_2) E_2 \|\boldsymbol{r} - x\|.$$

Taking supremum over $\theta \in \mathfrak{T}$, we obtain

$$\|\boldsymbol{r} - x\| \le \varepsilon E_2 + \widehat{\aleph} M_\alpha (\alpha_1 + \alpha_2) E_2 \|\boldsymbol{r} - x\|.$$

Therefore,

$$\|\boldsymbol{r} - x\| \le \frac{\varepsilon E_2}{1 - \widehat{\aleph} M_\alpha (\alpha_1 + \alpha_2) E_2}.$$

Hence equation (1)–(2) possesses Hyers–Ulam stability with HUS constant given by (10).

Corollary 4.1 Assume that condition (A) holds and

$$\widehat{\aleph} M_\alpha(\alpha_1+\alpha_2)E_2 < 1.$$

Then equation (1)–(2) possesses generalized Hyers–Ulam stability.

Proof. Define $\theta(\varepsilon)=\varepsilon L$, where L is given by (10). Clearly, $\theta \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $\theta(0)=0$. From the previous theorem,

$$\|\boldsymbol{r}-x\| \leq \theta(\varepsilon).$$

Therefore, equation (1)–(2) possesses generalized Hyers–Ulam stability.

Example 4.1 Consider the finite time scale $T=\{0,1,3,4,7\}$, and the conformable integro-dynamic equation

$$D^\alpha r(\theta)+\frac{1}{2}r^\sigma(\theta)=\int_0^\theta \frac{1}{1+\theta^2}\left(\frac{1}{20}r(\mathcal{T})+\frac{1}{20}r\left(\frac{1}{2}r(\mathcal{T})\right)\right)\Delta\mathcal{T}, \qquad \theta \in \mathbb{T}^\kappa,$$

subject to the initial condition $r(0)=1$. Here,

$$p(\theta)=\frac{1}{2}, \qquad \aleph(\theta,\mathcal{T})=\frac{1}{1+\theta^2},$$

and

$$\mathcal{Q}(\mathcal{T},u,v)=\frac{1}{20}(u+v).$$

Assume that $r(\mathbb{T}) \subseteq \{0,2,6\}$. Then

$$\frac{1}{2}r(\mathcal{T}) \in \{0,1,3\} \subseteq \mathbb{T},$$

which guarantees that the deviating term $r \rightleftarrows \left(\frac{1}{2}r(\mathcal{T})\right)$ is well defined on the time scale $\mathbb{T}$. We now verify assumptions (A)–(D). For any $u_1,u_2,v_1,v_2 \in \mathbb{R}$,

$$\begin{aligned}|\mathcal{Q}(\mathcal{T},u_1,v_1)-\mathcal{Q}(\mathcal{T},u_2,v_2)| &= \left|\frac{1}{20}(u_1+v_1)-\frac{1}{20}(u_2+v_2)\right| \\ &\leq \frac{1}{20}|u_1-u_2|+\frac{1}{20}|v_1-v_2|.\end{aligned}$$

Hence assumption (A) is satisfied with

$$\alpha_1=\alpha_2=\frac{1}{20}.$$

Furthermore,

$$|\mathcal{Q}(\mathcal{T},u,v)|=\frac{1}{20}|u+v| \leq \frac{1}{20}(|u|+|v|).$$

For $(u,v) \in [-1,1]\times[-1,1]$, we obtain

$$|\mathcal{Q}(\mathcal{T},u,v)| \leq \frac{1}{20}(1+1)=\frac{1}{10}.$$

Therefore assumption (B) holds with

$$M=\frac{1}{10}.$$

Next,

$$|\aleph(\theta,\mathcal{T})|=\frac{1}{1+\theta^2} \leq 1, \qquad (\theta,\mathcal{T}) \in \mathbb{T}\times\mathbb{T},$$

and thus $\widehat{\aleph} = 1$. Also,

$$\sup_{\theta\in\mathbb{T}} \int_0^\theta |\aleph(\theta,\mathcal{T})| \ \Delta\tau \leq \sup_{\theta\in\mathbb{T}} \int_0^\theta 1\Delta\mathcal{T} \ = \ 7$$

Hence

$$L \ = \sup_{\theta\in\mathbb{T}} \int_0^\theta |\aleph(\theta,\mathcal{T})| \ \Delta\mathcal{T} \leq \ 7$$

Therefore assumption (C) is satisfied. Finally, taking $\varepsilon = 1, \lambda = \frac{1}{2}$, we obtain

$$\begin{aligned} \widehat{\aleph}L(\alpha_1 + (\lambda\varepsilon + 1)\alpha_2) \quad &= 1 \cdot 7\left(\frac{1}{20} + \left(\frac{1}{2} + 1\right)\frac{1}{20}\right) \\ &= 7\left(\frac{1}{20} + \frac{3}{40}\right) \\ &= 7 \cdot \frac{5}{40} = \frac{7}{8} < 1. \end{aligned}$$

Hence assumption (D) is fulfilled. Consequently, all the hypotheses of Theorems 3.1 and 3.2 are satisfied. Therefore, the above conformable integro-dynamic problem possesses a unique solution on the time scale $\mathbb{T} = \{0,1,3,4,7\}$.

Conflict of Interest

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