Approximation Properties of Bivariate Extension of q-Stancu-Kantorovich Operators

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Abstract: In this paper we introduce a new q-Stancu-Kantorovich operators and we study some of their approximation properties. Furthermore, a Voronovskaja type theorem is also proven.

Keywords: q-Stancu-Kantorovich operators, modulus of continuity, rate of convergence, Voronovskaja theorem.

1. INTRODUCTION

In recent years, many researchers focused their attention on the study of generalized version in q-calculus of well-known linear and positive operators [3-5, 7-9]. The goal of this paper is to introduce a Kantorovich variant of q-Stancu operators and we investigate their approximation properties and rate of convergence using modulus of continuity. We mention some basic definitions and notations from *q*-calculus. Let *q* > 0. For each nonnegative integer *k*, the *q*-integer [*k*] and *q*-factorial [*k*]! are respectively defined by

$$[k]:=\begin{cases}\dfrac{1-q^k}{1-q}, & q\neq 1,\\ k, & q=1,\end{cases}\qquad [k]!:=\begin{cases}[k][k-1]\cdots[1], & k\geq 1,\\ 1, & k=0.\end{cases}$$

For the integers *n, k* satisfying *n* ≥ *k* ≥ 0, the *q*-binomial coefficients are defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}:=\frac{[n]!}{[k]![n-k]!}.$$

We denote $(a+b)_q^k=\prod_{j=0}^{k-1}(a+bq^j)$.

The *q*-Jackson integral on the interval [0,b] is defined as

$$\int_0^b f(t)d_qt=(1-q)b\sum_{j=0}^{\infty}f\left(q^jb\right)q^j,\qquad 0<q<1,$$

provided that sums converge absolutely.

In [3], the authors introduced a *q*-type generalization of Bernstein-Kantorovich operators as follows

$$B_{n,q}^*\left(f,x\right):=\sum_{k=0}^{n}p_{n,k}\left(q;x\right)\int_0^1 f\left(\frac{[k]+q^kt}{[n+1]}\right)d_qt, \qquad (1)$$

where $f\in C[0,1]$, $0<q\leq 1$ and $p_{n,k}(q;x)=\begin{bmatrix} n\\ k\end{bmatrix}x^k(1-x)_q^{n-k}$.

In [1], inspired by Mahmudov and Sabancigil's result we introduce a q-type generalization of Stancu-Kantorovich operators as follows

$$S_{n,q}^{(\alpha,\beta)}(f,x)=\sum_{k=0}^{n}p_{n,k}(q;x)\int_0^1 f\left(\frac{[k]+q^kt+\alpha}{[n+1]+\beta}\right)d_qt, \qquad (2)$$

where 0 ≤ *α* ≤ *β*, *f* ∈ *C* [0, 1].

Lemma 1.1. [1] For all n ∈ N, x ∈ [0, 1] and 0 < q < 1, we have

$$S_{n,q}^{(\alpha,\beta)}(1,x)=1,$$

$$S_{n,q}^{(\alpha,\beta)}(t,x)=\frac{2q}{[2]}\frac{[n]}{[n+1]+\beta}x+\frac{\alpha}{[n+1]+\beta}+\frac{1}{[2]\left([n+1]+\beta\right)},$$

$$S_{n,q}^{(\alpha,\beta)}(t^2,x)=\frac{1}{\left([n+1]+\beta\right)^2}\left\{\frac{q^2(q+2)}{[3]}[n][n-1]x^2+\frac{q[n]}{[2]}\left(4\alpha+\frac{4+7q+q^2}{[3]}\right)x+\frac{2\alpha}{[2]}+\frac{1}{[3]}+\alpha^2\right\}.$$

Lemma 1.2. [1] For all n ∈ N, x ∈ [0, 1] and 0 < q < 1, we have

$$S_{n,q}^{(\alpha,\beta)}((t-x)^2,x)\leq\frac{2[n+1]^2}{\left([n+1]+\beta\right)^2}\left\{\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)+\left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^2\right\},$$

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$$S_{n,q}^{(\alpha,\beta)}((t-x)^4,x) \le \frac{8[n+1]^2}{\left([n+1]+\beta\right)^2}$$
$$\left\{\frac{C}{[n]^2}\left(x(1-x)+\frac{1}{[n]^2}\right)+\left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^4\right\},$$

where C is a positive absolute constant.

Lemma 1.3. [1] Assume that $0<q_n<1$, $q_n \to 1$ and $q_n^n \to a$, a ∈ [0, 1) as n → ∞. Then we have

$$\lim_{n\to\infty}[n]S_{n,q_n}^{(\alpha,\beta)}(t-x,x) = -\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2},$$

$$\lim_{n\to\infty}[n]S_{n,q_n}^{(\alpha,\beta)}((t-x)^2,x) = -\frac{2a+1}{3}x^2+x.$$

In [1], the following Voronovskaja type theorem for *q*-Stancu-Kantorovich operators was obtained:

Theorem 1.4. [1] Let $f'' \in C[0,1]$ and q_n ∈ (0, 1), $q_n \to 1$ and $q_n^n \to a$, a ∈ [0, 1) as n → ∞. Then we have

$$\lim_{n\to\infty}[n]\left(S_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right)=$$
$$\left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)+\frac{1}{2}\left(-\frac{2a+1}{3}x^2+x\right)f''(x).$$

A convergence theorem for the q-Stancu-Kantorovich operators was established in [1]:

Theorem 1.5. [1] Let $(q_n)_n$, 0 < q_n < 1 be a sequence satisfying the following conditions

$$\lim_{n\to\infty} q_n = 1, \ \lim_{n\to\infty} q_n^n = a, \ a \in [0, 1). \tag{3}$$

Then for any f ∈ C[0, 1], the sequence $\left\{S_{n,q_n}^{(\alpha,\beta)}(f,x)\right\}$ converges to f uniformly on [0, 1].

2. CONSTRUCTION OF THE BIVARIATE OPERATORS

In this section we propose a bivariate extension of *q*-Stancu-Kantorovich operators. Let

$$S_{n_i,q_i}^{(\alpha,\beta)} : L_1[0,1] \to C[0,1], \qquad i=\overline{1,2},$$

the operators defined for any n_1, n_2 ∈ N, q_1, q_2 ∈ (0, 1) and any *g, h* ∈ [0, 1], respectively by

$$S_{n_1,q_1}^{(\alpha,\beta)}(g;x)=\sum_{k=0}^{n_1}p_{n_1,k}(q_1;x)\int_0^1 g\left(\frac{[k]+q_1^k t+\alpha}{[n_1+1]+\beta}\right)d_{q_1}t, \tag{4}$$

$$S_{n_2,q_2}^{(\alpha,\beta)}(h;x)=\sum_{k=0}^{n_2}p_{n_2,k}(q_2;x)\int_0^1 h\left(\frac{[k]+q_2^k t+\alpha}{[n_2+1]+\beta}\right)d_{q_2}t. \tag{5}$$

The parametric extensions of (4) and (5) are the operators $S_{n_1,q_1,x}^{(\alpha,\beta)}$, $S_{n_2,q_2,y}^{(\alpha,\beta)}$: L_1([0, 1]×[0, 1)) →*C*([0, 1] × [0, 1]), defined for any n_1, n_2 ∈ N and any *f* ∈ L_1([0, 1] × [0, 1]) as follows:

$$S_{n_1,q_1,x}^{(\alpha,\beta)}\left(f;x,y\right)=\sum_{k_1=0}^{n_1}p_{n_1,k_1}\left(q_1;x\right)\int_0^1 f\left(\frac{[k_1]+q_1^{k_1}t_1+\alpha}{[n_1+1]+\beta},y\right)d_{q_1}t_1,$$

$$S_{n_2,q_2,y}^{(\alpha,\beta)}\left(f;x,y\right)=\sum_{k_2=0}^{n_2}p_{n_2,k_2}\left(q_2;y\right)\int_0^1 f\left(x,\frac{[k_2]+q_2^{k_2}t_2+\alpha}{[n_2+1]+\beta}\right)d_{q_2}t_2.$$

We construct a bivariate extension of the univariate *q*-Stancu-Kantorovich operators for *f* ∈L_1([0, 1]× [0, 1]), q_1, q_2 ∈ (0, 1) and n_1, n_2 ∈ N as follows:

$$S_{n_1,n_2,q_1,q_2}^{(\alpha,\beta)}\left(f;x,y\right)=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}p_{n_1,k_1}(q_1;x)p_{n_2,k_2}(q_2;y)$$
$$\cdot\int_0^1\int_0^1 f\left(\frac{[k_1]+q_1^{k_1}t_1+\alpha}{[n_1+1]+\beta},\frac{[k_2]+q_2^{k_2}t_2+\alpha}{[n_2+1]+\beta}\right)d_{q_1}t_1 d_{q_2}t_2.$$

Lemma 2.1. We have

i) $S_{n_1,n_2,q_1,q_2}^{(\alpha,\beta)}\left(f;x,y\right)=S_{n_1,q_1,x}^{(\alpha,\beta)}\left(S_{n_2,q_2,y}^{(\alpha,\beta)}\left(f;x,y\right)\right)$,

ii) $S_{n_1,n_2,q_1,q_2}^{(\alpha,\beta)}\left(f;x,y\right)=S_{n_2,q_2,y}^{(\alpha,\beta)}\left(S_{n_1,q_1,x}^{(\alpha,\beta)}\left(f;x,y\right)\right)$.

Proof. It follows

$$S_{n_i,q_i,x}^{(\alpha,\beta)}\left(S_{n_2,q_2,y}^{(\alpha,\beta)}\left(f;x,y\right)\right)=\sum_{k_1=0}^{n_1}p_{n_1,k_1}\left(q_1;x\right)\int_0^1\sum_{k_2=0}^{n_2}p_{n_2,k_2}\left(q_2;y\right)$$
$$\cdot\int_0^1 f\left(\frac{[k_1]+q_1^{k_1}t_1+\alpha}{[n_1+1]+\beta},\frac{[k_2]+q_2^{k_2}t_2+\alpha}{[n_2+1]+\beta}\right)d_{q_2}t_2 d_{q_1}t_1=$$
$$S_{n_1,n_2,q_1,q_2}^{(\alpha,\beta)}\left(f;x,y\right).$$

Property (ii) can be proven in a similar way.

3. APPROXIMATION PROPERTIES OF THE BIVARIATE *Q*-STANCU-KANTOROVICH OPERATORS

Using the Volkov criterion we shall prove a convergence theorem for the bivariate *q*-Stancu-Kantorovich operators. First, we recall the result due to Volkov [10].

Theorem 3.1. Let $e_{ij}(x, y) = x_i y_j$, i, j ∈ N, x, y ∈ R be the two-dimensional test functions and

I, J compact intervals of the real line. Let L_{n_1,n_2}: C(I × J) → C(I × J), (n_1, n_2) ∈ N × N be linear positive operators. If

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2} e_{ij} = e_{ij}\ ,\ (i,j)\in\left\{(0,0),(1,0),(0,1)\right\},$$

$$\lim_{n_1,n_2\to\infty} L_{n_1,n_2}\left(e_{20}+e_{02}\right)=e_{20}+e_{02},$$

uniformly on I × J, then the sequence (L_{n_1,n_2} f) converges to f uniformly on I × J for any f ∈ C(I × J).

Lemma 3.2. The bivariate q-Stancu-Kantorovich operators satisfy the equalities

a) $S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}(e_{00};x,y)=1;$

b) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(e_{10};x,y\right)=\frac{2q_1}{[2]}\frac{[n_1]}{[n_1+1]+\beta}x+\frac{\alpha}{[n_1+1]+\beta}+\frac{1}{[2]([n_1+1]+\beta)};$$

c) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(e_{01};x,y\right)=\frac{2q_2}{[2]}\frac{[n_2]}{[n_2+1]+\beta}y+\frac{\alpha}{[n_2+1]+\beta}+\frac{1}{[2]([n_2+1]+\beta)};$$

d) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(e_{11};x,y\right)=\left[\frac{2q_1}{[2]}\frac{[n_1]}{[n_1+1]+\beta}x+\frac{\alpha}{[n_1+1]+\beta}+\frac{1}{[2]([n_1+1]+\beta)}\right]\cdot\left[\frac{2q_2}{[2]}\frac{[n_2]}{[n_2+1]+\beta}y+\frac{\alpha}{[n_2+1]+\beta}+\frac{1}{[2]([n_2+1]+\beta)}\right];$$

e) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(e_{20};x,y\right)=\frac{1}{\left([n_1+1]+\beta\right)^2}\left\{\frac{q_1^2(q_1+2)}{[3]}[n_1][n_1-1]x^2+\frac{q_1[n_1]}{[2]}\left(4\alpha+\frac{4+7q_1+q_1^2}{[3]}\right)x+\frac{2\alpha}{[2]}+\frac{1}{[3]}+\alpha^2\right\};$$

f) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(e_{02};x,y\right)=\frac{1}{\left([n_2+1]+\beta\right)^2}\left\{\frac{q_2^2(q_2+2)}{[3]}[n_2][n_2-1]x^2+\frac{q_2[n_2]}{[2]}\left(4\alpha+\frac{4+7q_2+q_2^2}{[3]}\right)y+\frac{2\alpha}{[2]}+\frac{1}{[3]}+\alpha^2\right\};$$

Applying Theorem 3.1 and Lemma 3.2, the following result holds.

Theorem 3.3. If the sequences (q_{1,n_1}) and (q_{2,n_2}) satisfy conditions

(6) $\lim_{n_i\to\infty} q_{i,n_i}^{n_i}=a_i<1$ and $\lim_{n_i\to\infty} q_{i,n_i}=1$, $i=1,2$

in the interval (0, 1), then the sequence of bivariate generalized Stancu-Kantorovich $S^{(\alpha,\beta)}_{n_1,n_2,q_{1,n1},q_{2,n2}}(f;x,y)$ converges uniformly to f(x, y) for any f ∈ C ([0, 1] × [0, 1]).

Using Shisha-Mond theorem [6] for the bivariate case we study the rate of convergence of $S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}$ operators in terms of the first order modulus of smoothness .

Definition 3.1. Let f: I × J → R be a real valued bounded function, where I and J are compact intervals on the real line. The function ω_f: [0,∞) × [0,∞) → R defined by

$\omega_f(\delta_1, \delta_2) = \sup\{|f(x_1, y_1) - f(x_2, y_2)| : (x_1, y_1), (x_2, y_2) \in I \times J, |x_1 - y_1| \le \delta_1, |x_2 - y_2| \le \delta_2\}$

is called modulus of continuity of the bivariate function.

Let I and J be compact intervals on the real line and $B(I\times J)$ the set of bounded functions defined on $I\times J$.

Theorem 3.4. ([2]) Let L: C(I×J) → B(I×J) be a linear positive operator and let φ_x: I×J → R, φ_y: I×J → R be defined as $\varphi_x(s, t) = |s-x|$, respectively $\varphi_y(s, t) = |t - y|$, (s, t), (x, y) ∈ I × J. For any f ∈ C(I × J), (x,y) ∈ I × J, $\delta_1 > 0$, $\delta_2 > 0$, the following inequality

$$\left|Lf(x,y)-f(x,y)\right|\le\left|f(x,y)\right|\cdot\left|Le_{00}(x,y)-1\right|+\left\{Le_{00}(x,y)+\delta_1^{-1}\sqrt{Le_{00}(x,y)L\varphi_x^2(x,y)}+\delta_2^{-1}\sqrt{Le_{00}(x,y)L\varphi_y^2(x,y)}+\delta_1^{-1}\delta_2^{-1}\sqrt{L^2e_{00}(x,y)L\varphi_x^2(x,y)L\varphi_y^2(x,y)}\right\}\omega_f(\delta_1,\delta_2)$$

holds, where ω_f denotes the bivariate modulus of continuity.

Using Lemma 3.2 we obtain the following expressions for the first and second central moments.

Lemma 3.5. The bivariate q-Stancu-Kantorovich operators satisfies the equalities

a) $$S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(s-x;x,y\right)=\left(\frac{2q_1}{[2]}\frac{[n_1]}{[n_1+1]+\beta}-1\right)x+\frac{\alpha}{[n_1+1]+\beta}+\frac{1}{[2]([n_1+1]+\beta)},$$

b) $S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(t-y;x,y\right)=$

$$\left(\frac{2q_2}{[2]}\frac{[n_2]}{[n_2+1]+\beta}-1\right)y+\frac{\alpha}{[n_2+1]+\beta}+\frac{1}{[2]([n_2+1]+\beta)},$$

c) $S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(\varphi_x^2;x,y\right)=$

$$\left\{\frac{q_1^2(q_1+2)}{[3]}\cdot\frac{[n_1][n_1-1]}{\left([n_1+1]+\beta\right)^2}-\frac{4q_1}{[2]}\cdot\frac{[n_1]}{[n_1+1]+\beta}+1\right\}x^2+$$

$$\frac{1}{[n_1+1]+\beta}\left\{\frac{q_1[n_1]}{[2]\left([n_1+1]+\beta\right)}\left(4\alpha+\frac{4+7q_1+q_1^2}{[3]}\right)-2\left(\alpha+\frac{1}{[2]}\right)\right\}$$

$$x+\frac{1}{\left([n_1+1]+\beta\right)^2}\left(\frac{2\alpha}{[2]}+\frac{1}{[3]}+\alpha^2\right),$$

d) $S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(\varphi_y^2;x,y\right)=$

$$\left\{\frac{q_2^2(q_2+2)}{[3]}\cdot\frac{[n_2][n_2-1]}{\left([n_2+1]+\beta\right)^2}-\frac{4q_2}{[2]}\cdot\frac{[n_2]}{[n_2+1]+\beta}+1\right\}y^2$$

$$+\frac{1}{[n_2+1]+\beta}\left\{\frac{q_2[n_2]}{[2]\left([n_2+1]+\beta\right)}\left(4\alpha+\frac{4+7q_2+q_2^2}{[3]}\right)-2\left(\alpha+\frac{1}{[2]}\right)\right\}$$

$$y+\frac{1}{\left([n_2+1]+\beta\right)^2}\left(\frac{2\alpha}{[2]}+\frac{1}{[3]}+\alpha^2\right).$$

Taking Theorem 3.4 and Lemma 3.5 into account, we shall prove

Theorem 3.6. If the sequences (q_{1,n_1}) and (q_{2,n_2}) satisfy conditions (6) in the interval (0, 1), then

$$\left|S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(f;x,y\right)-f(x,y)\right|\le 4\omega_f(\delta_1,\delta_2),$$

where

$$\delta_1=\sqrt{S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(\varphi_x^2;x,y\right)} \text{ and } \delta_2=\sqrt{S^{(\alpha,\beta)}_{n_1,n_2,q_1,q_2}\left(\varphi_y^2;x,y\right)}.$$

4. A VORONOVSKAYA-TYPE THEOREM FOR BIVARIATE OPERATORS

In this section we prove a Voronovskaya type theorem for bivariate extension of *q*-Stancu-Kantorovich operators.

Theorem 4.1. Let ($q_{1,n}$) and ($q_{2,n}$) be sequences in the interval (0, 1) satisfying (6). Suppose that

f ∈ C^2 ([0, 1] × [0, 1]). Then for every (x, y) ∈ [0, 1] × [0, 1], one has

$$\lim_{n\to\infty}[n]\left\{S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(f;x,y\right)-f(x,y)\right\}$$

$$=\frac{1}{2}\left\{\left[-(1+a+2\beta)x+2\alpha+1\right]f'_x(x,y)+\left[-(1+a+2\beta)y+2\alpha+1\right]f'_y(x,y)\right.$$

$$\left.+\left(-\frac{2a+1}{3}x^2+x\right)f''_{x^2}(x,y)+\left(-\frac{2a+1}{3}y^2+y\right)f''_{y^2}(x,y)\right\}.$$

Proof. Let $f\in C^2$ ([0, 1] × [0, 1]) and (x_0, y_0) ∈ [0, 1]×[0, 1] be fixed point. By the Taylor formula, it follows

$$f(t,s)=f(x_0,y_0)+f'_x(x_0,y_0)(t-x_0)+f'_y(x_0,y_0)(s-y_0)$$

$$+\frac{1}{2}\left\{f''_{x^2}(x_0,y_0)(t-x_0)^2+2f''_{xy}(x_0,y_0)(t-x_0)(s-y_0)+f''_{y^2}(x_0,y_0)(s-y_0)^2\right\}$$

$$+\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right),$$

where (*t*, *s*) ∈ [0, 1]×[0, 1] and $\lim_{(t,s)\to(x_0,y_0)}\varphi(t,s)=0$. From the linearity of $S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}$, we have

$$S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(f(t,s);x_0,y_0\right)=f(x_0,y_0)+f'_x(x_0,y_0)S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(t-x_0;x_0,y_0\right)$$

$$+f'_y(x_0,y_0)S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(s-y_0;x_0,y_0\right)+\frac{1}{2}\left\{f''_{x^2}\,S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left((t-x_0)^2;x_0,y_0\right)\right.$$

$$+2f''_{xy}(x_0,y_0)S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left((t-x_0)(s-y_0);x_0,y_0\right)$$

$$\left.+f''_{y^2}(x_0,y_0)S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left((s-y_0)^2;x_0,y_0\right)\right\}$$

$$+S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);x_0,y_0\right)$$

$$=f(x_0,y_0)+f'_x(x_0,y_0)S^{(\alpha,\beta)}_{n,q_{1,n}}\left(t-x_0;x_0\right)+f'_y(x_0,y_0)S^{(\alpha,\beta)}_{n,q_{2,n}}\left(s-y_0;y_0\right)$$

$$+\frac{1}{2}\left\{f''_{x^2}(x_0,y_0)S^{(\alpha,\beta)}_{n,q_{1,n}}\left(\left(t-x_0\right)^2;x_0\right)+2f''_{xy}(x_0,y_0)S^{(\alpha,\beta)}_{n,q_{1,n}}\left(t-x_0;x_0\right)S^{(\alpha,\beta)}_{n,q_{2,n}}\left(s-y_0;y_0\right)\right.$$

$$\left.+f''_{y^2}(x_0,y_0)S^{(\alpha,\beta)}_{n,q_{2,n}}\left(\left(s-y_0\right)^2;y_0\right)\right\}$$

$$+S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);x_0,y_0\right)$$

By the Hölder inequality, we have

$$\left|S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);x_0,y_0\right)\right|$$

$$\le\left\{S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi^2(t,s);x_0,y_0\right)\right\}^{1/2}$$
$$\left\{S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\left((t-x_0)^2+(s-y_0)^2\right)^2;x_0,y_0\right)\right\}^{1/2}$$

$$\le\sqrt{2}\left\{S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi^2(t,s);x_0,y_0\right)\right\}^{1/2}$$

$$\cdot\left\{S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left((t-x_0)^4;x_0,y_0\right)+S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left((s-y_0)^4;x_0,y_0\right)\right\}^{1/2}.$$

Using Theorem 3.3, we obtain

$$\lim_{n\to\infty}S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi^2(t,s);x_0,y_0\right)=\varphi^2(x_0,y_0)=0,$$

but from Lemma 1.2 we have

$$[n]S^{(\alpha,\beta)}_{n,q_{1,n}}\left((t-x_0)^4;x_0\right)=O\left(\frac{1}{[n]}\right),$$
$$[n]S^{(\alpha,\beta)}_{n,q_{2,n}}\left((s-y_0)^4;y_0\right)=O\left(\frac{1}{[n]}\right),$$

therefore

$$\lim_{n\to\infty}S^{(\alpha,\beta)}_{n,n,q_{1,n},q_{2,n}}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);x_0,y_0\right)=0.$$

Then using Lemma 1.3 theorem is proved.

5. NUMERICAL EXAMPLES

In this section we give some numerical results using Matlab and we show that when $0 < q_1 < 1$, $0 < q_2 < 1$ and n_1, n_2 are increasing, the maximum error is smaller, as follows from the convergence properties of *q*-Stancu-Kantorovich operators.

Let us consider function $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2y^2 + x^2y - y^2$.

Example 5.1. For $n_1 = 100$, $n_2 = 100$, a = 2, b = 3, $q_1 = 0.6$, $q_2 = 0.6$, it follows the maximum error is 0.687918104177673.

Example 5.2. For $n_1 = 400$, $n_2 = 400$, a = 2, b = 3, $q_1 = 0.8$, $q_2 = 0.8$, it follows the maximum error is 0.445396492537402.

Example 5.3. For $n_1 = 800$, $n_2 = 800$, a = 2, b = 3, $q_1 = 0.9$, $q_2 = 0.9$, it follows the maximum error is 0.275784239758968.

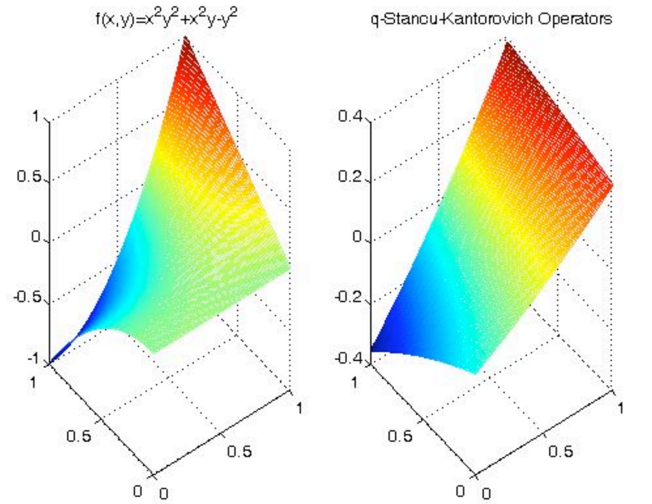

Figure 1:

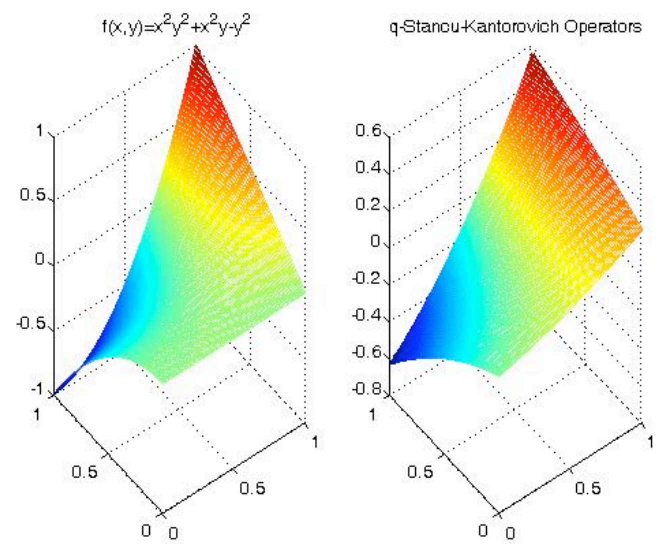

Figure 2:

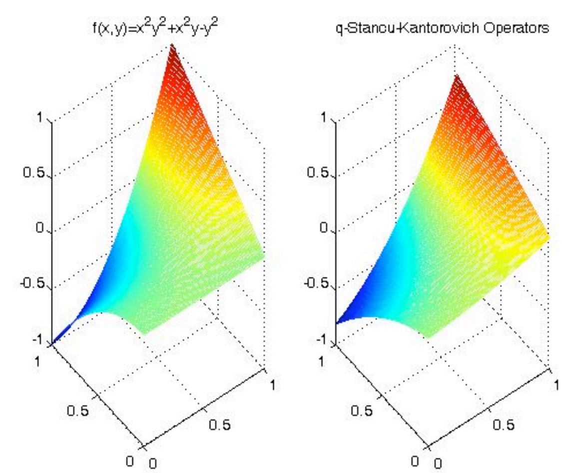

Figure 3:

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