Further Results on Designs Arising from Some Certain Corona Graphs

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Abstract: In this paper, we determine the partially balanced incomplete block designs and association scheme which are formed by the minimum dominating sets of the graphs $C_3 \circ K_2$, and $C_4 \circ K_2$. Finally, we determine the number of minimum dominating sets of graph $G = C_n \circ K_2$ and prove that the set of all minimum dominating sets of $G = C_n \circ K_2$ forms a partially balanced incomplete block design with two association scheme.

Keywords: Minimum dominating sets, association schemes, PBIBD.

1. INTRODUCTION

In this paper by a graph, we mean a finite undirected graph without loops or multiple lines. For a graph G, let V (G) and E (G) respectively denote the point set and the line set of graph G. We say that u and v dominate each other. A set D subset of V is dominating set of G, if every vertex in V − D is adjacent to some vertex in D. The domination number γ (G) of G is the minimum cardinality of a dominating set.

Many authors have been studied PBIBD with m-association scheme which are arising from some dominating sets of some graphs. H.B.Walikar and *et al.* [5], have studied PBIBD arising from minimum dominating set of paths and cycles, Anwar and Soner [1], have studied Partial balanced incomplete block designs arising from some minimal dominating sets of SRNT graphs. Any undefined terms and notation, reader may refer to F.Harary [3]. We refer the reader to see [2, 4], for more details about PBIBD and dominating set. We concern here to study PBIBD and the association scheme which can be obtained from the minimum dominating sets in some certain ($C_n \circ K_2$) graph , then we generalize the graph ($C_n \circ K_n$) and it is open area to study the same things for the other graphs.

We can obtain different PBIBD association scheme from the ($C_n \circ K_n$) graphs by using different definitions as we will see in next sections.

2. SOME PBIBD ARISING FROM MINIMUM DOMINATING SETS OF ($C_N \circ K_2$)

2.1. Definition

Given v objects a relation satisfying the following conditions is said to be an association scheme with m classes:

i. Any two objects are either first associates, or second associates,..., or m^th associates, the relation of association being symmetric.

ii. Each object α has n_i ith associates, the number n_i being independent of α.

iii. If two objects α and β are ith associates, then the number of objects which are jth associates of α and kth associates of β is p^i_{jk} and is independent of the pair of ith associates α and β. Also $p^i_{jk} = p^i_{kj}$.

If we have association scheme for the v objects we can define a PBIBD as the following definition.

2.2. Definition

The PBIBD design is arrangement of v objects into b sets (called blocks) of size k where k < v such that

i. Every object is contained in exactly r blocks.

ii. Each block contains k distinct objects.

iii. Any two objects which are ith associates occur together in exactly λ_i blocks.

Theorem3: From ($C_3 \circ K_2$) we get PBIBD with parameters

(v= 9, k= 3, r= 9, b= 27, λ_1= 0, λ_2= 3) and association scheme of 2-classes with

$$P_1 = \begin{bmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

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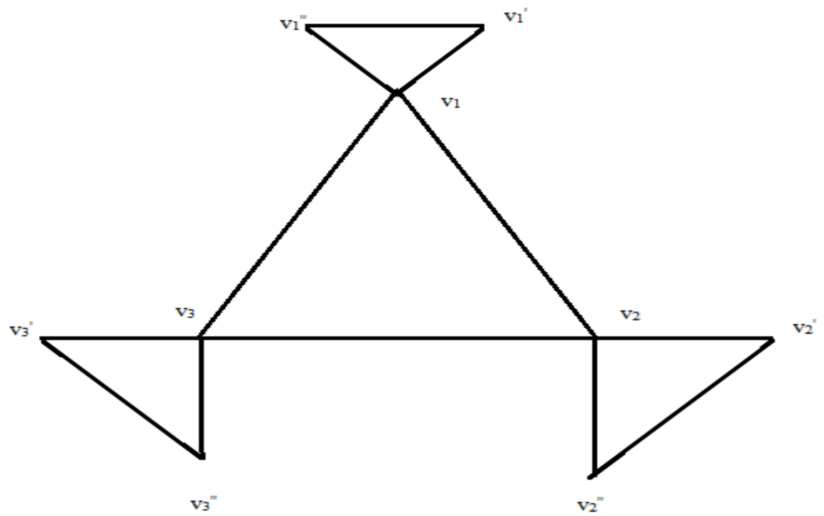

Figure 1: $C_3 \circ K_2$.

Proof. let G = (V,E) be a corona graph ($C_3 \circ K_2$). By labelling {$v_1, v_2, v_3, v_1', v_2', v_3', v_1'', v_2'', v_3''$} as in Figure **1**. we can define PBIBD as follows:

The point set is the vertices and the block set is the minimum dominating sets {v_1, v_2, v_3},

{v_1', v_2', v_3'}, {v_1'', v_2'', v_3''}, {v_1, v_2, v_3'}, {v_1, v_2, v_3''}, {v_1', v_2, v_3}, {v_1'', v_2, v_3}, {v_1, v_2', v_3},

{v_1, v_2'', v_3}, {v_1', v_2', v_3}, {v_1', v_2', v_3''}, {v_1, v_2', v_3'}, {v_1'', v_2', v_3'}, {v_1', v_2, v_3'}, {v_1', v_2'', v_3'},

{v_1'', v_2'', v_3}, {v_1'', v_2'', v_3'}, {v_1, v_2'', v_3''}, {v_1', v_2'', v_3''}, {v_1'', v_2, v_3''}, {v_1'', v_2', v_3''}, {v_1'', v_2', v_3}, {v_1'', v_2, v_3'}, {v_1, v_2'', v_3'}, {v_1', v_2'', v_3}, {v_1', v_2, v_3''} and {v_1, v_2', v_3''}.

We define association scheme as follows, for any α, $\beta \in V(G)$, α is first associate of β if α and β appear in zero or three minimum dominating sets and α is second associate of β otherwise.

Table 1:

Elements	First Associates	Second Associates
v_1	v_1', v_1''	$v_2, v_3, v_2', v_3', v_2'', v_3''$
v_1'	v_1, v_1''	$v_2, v_3, v_2', v_3', v_2'', v_3''$
v_1''	v_1, v_1'	$v_2, v_3, v_2', v_3', v_2'', v_3''$
v_2	v_2', v_2''	$v_1, v_3, v_1', v_3', v_1'', v_3''$
v_2'	v_2, v_2''	$v_1, v_3, v_1', v_3', v_1'', v_3''$
v_2''	v_2, v_2'	$v_1, v_3, v_1', v_3', v_1'', v_3''$
v_3	v_3', v_3''	$v_1, v_2, v_1', v_2', v_1'', v_2''$
v_3'	v_3, v_3''	$v_1, v_2, v_1', v_2', v_1'', v_2''$
v_3''	v_3, v_3'	$v_1, v_2, v_1', v_2', v_1'', v_2''$

Theorem4. From ($C_4 \circ K_2$) we get PBIBD with parameters

($v = 12$, $k = 4$, $r = 27$, $b = 81$, $\lambda_1 = 0$, $\lambda_2 = 4$) and association scheme of 2-classes with

$$P_1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Proof. let G = (V,E) be a corona graph ($C_4 \circ K_2$). By labelling {$v_1, v_2, v_3, v_4, v_1', v_2', v_3', v_4', v_1'', v_2'', v_3'', v_4''$} as in Figure **1**.we can define PBIBD as follows:

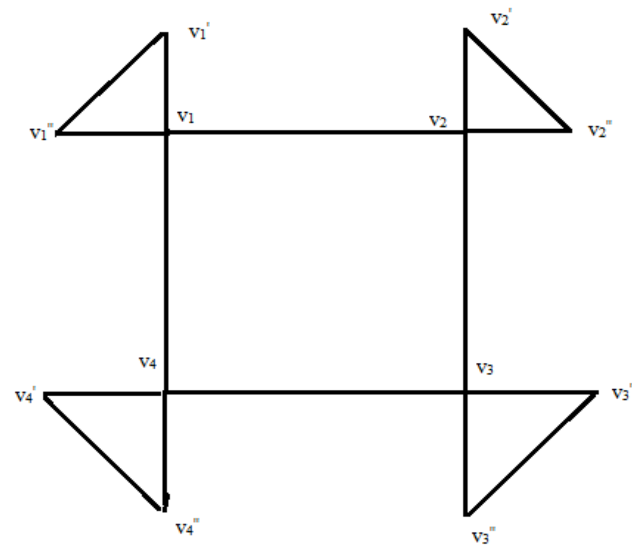

Figure 2: $C_4 \circ K_2$.

The point set is the vertices and the block set is the minimum dominating sets {v_1, v_2, v_3, v_4},

{v_1', v_2', v_3', v_4'}, {$v_1'', v_2'', v_3'', v_4''$}, {$v_1, v_2, v_3', v_4'$}, {$v_1, v_2, v_3', v_4''$}, {$v_1, v_2, v_3'', v_4'$},

$\{v_1, v_2, v_3', v_4\}, \{v_1, v_2, v_3'', v_4\}, \{v_1, v_2, v_3'', v_4''\}, \{v_1, v_2, v_3, v_4'\}, \{v_1, v_2, v_3, v_4''\}, \{v_1', v_2, v_3, v_4\}, \{v_1'', v_2, v_3, v_4\}, \{v_1', v_2, v_3, v_4'\}, \{v_1'', v_2, v_3, v_4'\}, \{v_1', v_2, v_3, v_4''\}, \{v_1'', v_2, v_3, v_4''\},$

$\{v_1, v_2', v_3, v_4\}, \{v_1, v_2'', v_3, v_4\}, \{v_1', v_2', v_3, v_4\}, \{v_1', v_2'', v_3, v_4\}, \{v_1'', v_2', v_3, v_4\},$

$\{v_1'', v_2'', v_3, v_4\}, \{v_1, v_2, v_3'', v_4\}, \{v_1, v_2', v_3', v_4\}, \{v_1, v_2', v_3'', v_4\}, \{v_1, v_2'', v_3', v_4\},$

$\{v_1, v_2'', v_3'', v_4\}, \{v_1', v_2', v_3', v_4''\}, \{v_1', v_2', v_3, v_4'\}, \{v_1', v_2', v_3, v_4''\}, \{v_1', v_2', v_3'', v_4\},$

$\{v_1', v_2', v_3'', v_4'\}, \{v_1', v_2', v_3'', v_4''\}, \{v_1, v_2', v_3', v_4'\}, \{v_1'', v_2', v_3', v_4'\}, \{v_1'', v_2', v_3', v_4\},$

$\{v_1, v_2', v_3', v_4''\}, \{v_1'', v_2', v_3', v_4''\}, \{v_1', v_2, v_3', v_4'\}, \{v_1', v_2'', v_3', v_4'\}, \{v_1, v_2'', v_3', v_4'\},$

$\{v_1'', v_2, v_3', v_4'\}, \{v_1'', v_2'', v_3', v_4'\}, \{v_1', v_2, v_3'', v_4'\}, \{v_1', v_2'', v_3, v_4'\}, \{v_1', v_2'', v_3'', v_4'\},$

$\{v_1'', v_2'', v_3', v_4\}, \{v_1'', v_2'', v_3', v_4''\}, \{v_1'', v_2'', v_3, v_4'\}, \{v_1'', v_2'', v_3, v_4''\}, \{v_1, v_2'', v_3'', v_4'\},$

$\{v_1'', v_2'', v_3'', v_4\}, \{v_1', v_2, v_3'', v_4''\}, \{v_1', v_2'', v_3'', v_4''\}, \{v_1, v_2', v_3'', v_4''\}, \{v_1, v_2'', v_3'', v_4''\},$

$\{v_1'', v_2', v_3, v_4''\}, \{v_1'', v_2', v_3'', v_4''\}, \{v_1'', v_2, v_3', v_4''\}, \{v_1'', v_2, v_3'', v_4''\}, \{v_1, v_2', v_3, v_4'\},$

$\{v_1, v_2'', v_3, v_4''\}, \{v_1, v_2', v_3, v_4''\}, \{v_1, v_2'', v_3, v_4'\}, \{v_1', v_2, v_3', v_4\}, \{v_1'', v_2, v_3', v_4\},$

$\{v_1', v_2, v_3'', v_4\}, \{v_1'', v_2, v_3'', v_4\}, \{v_1'', v_2', v_3'', v_4'\}, \{v_1'', v_2, v_3'', v_4\}, \{v_1', v_2'', v_3', v_4''\},$

$\{v_1, v_2, v_3', v_4''\}, \{v_1, v_2'', v_3', v_4\}, \{v_1', v_2, v_3', v_4''\}, \{v_1', v_2'', v_3, v_4\}, \{v_1, v_2, v_3'', v_4'\},$

$\{v_1'', v_2, v_3, v_4'\}, \{v_1, v_2', v_3'', v_4\}$ and $\{v_1'', v_2', v_3, v_4\}$

We define association scheme as follows, for any $\alpha, \beta \in V(G)$, α is first associate of β if α and β appear in zero or three minimum dominating sets and α is second associate of β otherwise.

Theorem 5: Let $G \cong (C_n \circ K_2)$. Then the minimum dominating sets of G is 3^n.

Proof: Let $G \cong (C_n \circ K_2)$. Then $\gamma(G) = n$. We need to find out all the sets of size n. For this, we have many possibilities:

Case1: All the vertices of the minimum dominating set is from inside that is from C_n. Then there is only one minimum dominating set.

Case 2: The vertices of minimum dominating set i.e; not from the vertices of C_n. The number of ways to select minimum dominating sets of size n from outside is 2^n.

Case 3: We select some vertices of minimum dominating sets from inside and some from outside. So we start by selecting one vertex from inside and (n-1) vertices from outside. There are $\binom{n}{1}2^{(n-1)}$ ways. Similarly 2 vertex from inside and (n-2) vertices from outside. There are $\binom{n}{2}2^{(n-2)}$ ways. By continuing in same way till (n-1) vertices from inside and one from outside, there are $\binom{n}{n-1}2$ ways. Hence the total number of minimum dominating sets is

$$2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} + \ldots\ldots + \binom{n}{n-1}2 + 1$$

Table 2:

Elements	First Associates	Second Associates
v_1	v_1', v_1''	$v_2, v_3, v_4, v_2', v_3', v_4', v_2'', v_3'', v_4''$
v_1'	v_1, v_1''	$v_2, v_3, v_4, v_2', v_3', v_4', v_2'', v_3'', v_4''$
v_1''	v_1, v_1'	$v_2, v_3, v_4, v_2', v_3', v_4', v_2'', v_3'', v_4''$
v_2	v_2', v_2''	$v_1, v_3, v_4, v_1', v_3', v_4', v_1'', v_3'', v_4''$
v_2'	v_2, v_2''	$v_1, v_3, v_4, v_1', v_3', v_4', v_1'', v_3'', v_4''$
v_2''	v_2, v_2'	$v_1, v_3, v_4, v_1', v_3', v_4', v_1'', v_3'', v_4''$
v_3	v_3', v_3''	$v_1, v_2, v_4, v_1', v_2', v_4', v_1'', v_2'', v_4''$
v_3'	v_3, v_3''	$v_1, v_2, v_4, v_1', v_2', v_4', v_1'', v_2'', v_4''$
v_3''	v_3, v_3'	$v_1, v_2, v_4, v_1', v_2', v_4', v_1'', v_2'', v_4''$
v_4	v_4', v_4''	$v_1, v_2, v_3, v_1', v_2', v_3', v_1'', v_2'', v_3''$
v_4'	v_4, v_4''	$v_1, v_2, v_3, v_1', v_2', v_3', v_1'', v_2'', v_3''$
v_4''	v_4, v_4'	$v_1, v_2, v_3, v_1', v_2', v_3', v_1'', v_2'', v_3''$

$$= \sum_{i=0}^{n} \binom{n}{i} 2^{n-1}$$

$= 3^n$.

Theorem 6: Let G ≅ ≅ ($C_n \circ K_2$). Any two vertices in G either belong to zero minimum dominating set or 3^{n-2} minimum dominating sets.

Proof: By labeling the vertices of the graph $C_n \circ K_2$ = {$v_1, v_2, \ldots, v_n, v_1', v_1'', v_2', v_2'', \ldots, v_n', v_n''$} where

{$v_1, v_2, \ldots, v_n$} are the vertices of C_n and { $v_1', v_1'', v_2', v_2'', \ldots, v_n', v_n''$} are the vertices of K_2.

Suppose A = {$v_1, v_2, \ldots, v_n$} and B = { $v_1', v_1'', v_2', v_2'', \ldots, v_n', v_n''$}. Let u, v be any two vertices, we have the following cases:

Case1: u and v belong to A then there are 3^{n-2} minimum dominating sets containing u and v.

Case2: u and v belong to B then there are 3^{n-2} ways to select minimum dominating sets containing u and v.

Case3: Let u ϵ A and vϵ B we have two sub cases:

Case(i). Let u and v in the same triangle then there does not exists any minimum dominating sets containing u and v.

Case(ii). If u and v are from the different triangle then there are 3^{n-2} ways to select minimum dominating sets.

Theorem 7: Let G ≅ ($C_n \circ K_2$). Then every vertex v ϵ V(G) contained in 3^{n-1} minimum dominating sets.

Proof: Let G ≅ ($C_n \circ K_2$). The vertices of G can be partitioned into n sets, each set containing 3 vertex as the triangles $\Delta 1, \Delta 2, \ldots \Delta$□. Let v ϵ V(G) be any vertex such that v ϵ Δ□ for some

1 ≤ i ≤ n. Any minimum dominating set containing v will contain (n−1) vertices from the other triangle Δ□ where i ≠ j. But it is not allowed to take two vertex from the same triangle so we need to take one vertex from each triangle. Hence the ways to select (n− 1) vertices from the Δ□ where i ≠ j is

$\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1} \ldots\ldots \binom{3}{1} = 3^{n-1}$.

Theorem 8: For any graph G ≅ ($C_n \circ K_2$), there is PBIBD and association scheme associate with G as the following parameters, (v = 3n, k = n, r = 3^{n-1}, b = 3^n, λ_1= 0, λ_2= 3^{n-2}) and association scheme of 2-classes with

$$P_1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3(n-1) \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 3(n-2) \end{bmatrix}$$

Proof: Above theorem follows from the previous theorems.

$P^1_{11} = 1$

If we select α, β any two vertices such that α, β does not belong to any dominating set. Then the number of vertices which appear with α and β together is only one.

$P^1_{12} = 1$

There are no minimum dominating sets which contains with α in zero minimum dominating set and β in 3^{n-2} dominating set. Similarly for $P^1_{21} = 0$.

$P^1_{22} = 3(n-1)$

The number of vertices which appear with α in 3^{n-1}dominating sets are 3(n−1) and β also in 3^{n-1}dominating sets are 3(n − 1).

$P^2_{11} = 0$

α, β does not belong to the same triangle , so the value of $P^2_{11} = 0$

$P^2_{12} = 2$

The number of vertices which belongs with α to zero vertices and with β to 3^{n-2} minimum dominating set is only two. Similarly for P^2_{21}= 2.

$P^2_{22} = 3(n-2)$

α, β belong to different triangles. The number of vertices which appear with α in 3^{n-2} minimum dominating sets and with β also in 3^{n-2} minimum dominating sets.

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