On Supersaturated Semigroups

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Abstract: In this paper, we prove the long known open problem and converse part of the Higgin's result that any globally idempotent ideal of a supersaturated semi-group is supersaturated.

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1. INTRODUCTION AND PRELIMINARIES

Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [6], we say that U *dominates* an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta: S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It is easily seen that $Dom(U, S)$ is a subsemigroup of S containing U. A semigroup U is said to be *closed* in S if $Dom(U, S) = U$ and U is *absolutely closed* if it is closed in the class of all semigroups. If $Dom(U, S) = S$ for every properly containing semigroup S, then U is said to be *saturated*. Following Higgins [4], a semigroup U is said to be *supersaturated* if every morphic image of U is saturated. Further a semigroup U is said to be *epimorphically embedded* in S if $Dom(U, S) = S$.

A morphism $\alpha: S \to T$ is said to be an *epimorphism* (epi for short) if for all morphisms β,γ, the relation $\alpha\beta = \alpha\gamma$ implies that $\beta = \gamma$ (where β, γ are semigroup morphisms). It can be easily checked that $\alpha: S \to T$ is epi if and only if $i: S\alpha \to T$ is epi and the inclusion map $i: U \to S$ is epi if and only if $Dom(U, S) = S$. Onto morphisms are always epimorphisms, but the converse is not true in general in the category of all semigroups. Infact every epimorphism from a semigroup U is onto is just to say that U is supersaturated.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Proposition 1.1. ([6, Theorem 2.3] or [5, Theorem VII.2.13]). Let U be a subsemi-group of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0y_1 = x_1a_1y_1 = x_1a_2y_2 = x_2a_3y_2 = \cdots = x_ma_{2m-1}y_m = x_ma_{2m} \quad (1)$$

where $m \geq 1, a_i \in U (i = 0,1,...,2m), x_i, y_i \in S (i = 1,2,...,m)$, and

$a_0 = x_1a_1,\qquad a_{2m-1}y_m = a_{2m},$

$a_{2i-1}y_i = a_{2i}y_{i+1},\ x_ia_{2i} = x_{i+1}a_{2i+1}\ (1 \leq i \leq m-1).$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1,...,a_{2m}$. We refer to the equations in Proposition 1.1, in whatever follows, as *the zigzag equations*.

For any unexplained notations and conventions, one may refer to Clifford and Preston [2] and Howie [5].

2. SUPERSATURATED SEMIGROUPS

The class of supersaturated semigroups has not been explicitly considered before as there was no known example of saturated semigroup with a morphic image which was not saturated. Indeed many of the known classes of semigroups are closed under the taking of morphic images. But, In 1985, Higgins has given the example of saturated semigroups whose morphic image is not saturated, see [4].

A semigroup S is said to be right [left] reductive with respect to X if $xa = xb$ [$ax = bx$] for all x in $X \Rightarrow a = b$ ($a, b \in S$), where $\in$ is a subset of $\in$. A semigroup S is said to be globally idempotent if for all $s \in S$, there exist $x, y \in S$ such that $s = xy$ or equivalently $S^2 = S$. The following proposition is from Higgins [3].

Proposition 2.1. ([3,Theorem 8]). A semigroup U is saturated [supersaturated] if the ideal U^n is saturated [supersaturated] (for some natural number n). In particular, a finite semigroup is saturated [supersaturated] if the ideal generated by the idempotents is saturated [supersaturated].

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Before, it was unknown whether or not converse of the above proposition is true. But in this paper, author's successfully answered the question and showed that the converse part is also true. However, it is not known whether or not an ideal of a saturated [absolutely closed] semigroup is saturated [absolutely closed]. In [4], Higgins, however, has shown that the converse of the above proposition holds in some special cases and has proved that if S is supersaturated commutative semigroup, then the same is true for any globally idempotent ideal. He has, in fact, shown the following:

Proposition 2.2. ([4, Theorem 14]). Let S be a saturated semigroup and suppose that U is a commutative ideal of S such that U^n is globally idempotent for some natural number n. Then U is supersaturated.

Khan and Shah [7] generalized this proposition by taking U as a permutative globally idempotent ideal satisfying a permutation identity $x_1x_2 \cdots x_n = x_{i1}x_{i2} \cdots x_{in}$ for which $i_1 = 1$ and $i_n \neq n$ and thus, relaxed the commutativity of U. Further, Alam and Khan [1] extend this proposition by taking U as a permutative globally idempotent ideal satisfying a seminormal permutation identity and, thus, relaxed the right semicommutativity of U. In fact, they have proved the following:

Theorem 2.3. Let S be a supersaturated semigroup and let U be any ideal of S satisfying a seminormal permutation identity. If U^n is globally idempotent for some natural number n, then U is supersaturated.

In this paper, finally, we are able to prove the converse of Higgin's result that any ideal of a supersaturated semigroup is supersaturated.

The following proposition of Alam and khan [1] is very important in proving our main theorem.

Proposition 2.4. ([1, Lemma 2.7]) Suppose that a globally idempotent semigroup U is not super-saturated. Then there exists a non-surjective epimorphism ϕ: $U \to V$ such that V is right and left reductive with respect to $U\phi$.

Theorem 2.5. Let S be a supersaturated semigroup and let U be any ideal of S. If U^n is globally idempotent for some natural number n, then U is supersaturated.

Proof. If we prove that U^n is supersaturated, then the theorem follows by Proposition 2.1. So, we assume that U is globally idempotent ideal satisfying a semi-normal permutation identity. Suppose to the contrary that U were not super-saturated. Then, by Proposition 2.4, there exists a non-surjective epimorphism ϕ: $U \to \bar{V}$ such that $\bar{V}$ is right and left reductive with respect to $U\phi$ (which we shall denote by $\bar{U}$ up to isomorphism).

Let $\rho = \phi \circ \phi^{-1} \cup /S$. Then, clearly ρ is an equivalence relation on S. Next, we show that ρ is a congruence on S. For this we are required to show that if $u,v \in U$ and $w \in S\backslash U$, then $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$ and $(wu)\phi = (wv)\phi$.

To prove the first equality, suppose that $u, v \in U$ and $w \in S\backslash U$ and $(uw)\phi \neq (vw)\phi$. Since $\bar{V}$ is left reductive with respect to $\bar{U}$, there exists $x \in U$ such that

$$(uw)\phi x\phi = (vw)\phi x\phi. Now((uw)x)\phi = ((vw)x)\phi$$
$$\Rightarrow (u(wx))\phi = (v(wx))\phi \Rightarrow (wx)\phi u\phi = (wx)\phi v\phi.$$
$$Hence u\phi = v\phi.$$

Therefore, the statement

$u\phi = v\phi \Rightarrow (uw)\phi = (vw)\phi \Rightarrow \rho$ is a right congruence.

Next we show that ρ is a left congruence. Suppose that $u,v \in U$ and $w \in S\backslash U$ and $(wu)\phi \neq (wv)\phi$. Since V is right reductive with respect to $\bar{U}$, there exists $x \in U$ such that $x\phi(wu)\phi \neq x\phi(wv)\phi$. Now $(x(wu))\phi \neq (x(wv))\phi \Rightarrow ((xw)u)\phi \neq ((xw)v))\phi \Rightarrow (xw)\phi u\phi \neq (xw)\phi v\phi$. Hence $u\phi \neq v\phi$.

Again we conclude that the statement $u\phi = v\phi$ implies that $(wu)\phi = (wv)\phi$. Therefore ρ is a left congruence and, hence, a congruence. Denote S/ρ by S. Then $U^\natural = \bar{U}$ (up to isomorphism).

Now we form the amalgam A of $\bar{S}$ and $\bar{V}$ with core $\bar{U}$. We extend the partial operation on A to an associative multiplication. For this take any $a \in \bar{S}\backslash\bar{U}$ ($= S\backslash U$), $v \in \bar{V}\backslash\bar{U}$ and factorize v as $v = u_1y_1 = y'_1 u'_1$ (where $u_1, u'_1 \in \bar{U}$; $y_1, y'_1 \in \bar{V}\backslash\bar{U}$). Now define $av = (au_1)y_1$ and $va = y'_1(u'_1 a)$. We first show that this is a well defined binary operation. For this suppose that $v = u_2y_2 = y'_2 u'_2$ (where $u_2, u'_2 \in U$ and $y_2, y'_2 \in \bar{V}\backslash\bar{U}$). Then for any $x \in \bar{U}$, as $u_1y_1 = u_2y_2$, we have

$$(xa)u_1y_1 = (xa)u_2y_2$$

$\Rightarrow ((xa)u_1)y_1 = ((xa)u_2)y_2$ (by associativity of $\bar{V}$)

$\Rightarrow (x(au_1))y_1 = (x(au_2))y_2$ (by associativity of $\bar{S}$)

$\Rightarrow x((au_1)y_1) = x((au_2)y_2)$ (by associativity of $\bar{V}$).

As $\bar{V}$ is right reductive with respect to $\bar{U}$, we have that $(au_1)y_1=(au_2)y_2$ and, therefore, the operation is well defined.

Now, we verify the associativity of the above operation. For any $a' \in S \setminus U$, we have $a'(av) = a'(au_1y_1) = (a'au_1)y_1 = (a'a)u_1y_1 = (a'a)v$ (by associativity of $\bar{V}$ and $\bar{S}$ respectively).

Similarly, for any $v' \in V$, it can be shown that $(av)v' = a(vv')$. The only case that requires some attention is to show that $(av)a' = a(va')$, where $a' \in S, v \in \bar{V}$. For this, factorize v as $v = a_1ya_2$ (where $a_1, a_2 \in \bar{V} \setminus U$). Now

$(av)a' = (a(a_1ya_2))a'$ (as $v = a_1ya_2$)

$= ((aa_1)ya_2)a'$ (as $aa_1 \in \bar{U} \subseteq \bar{S}$)

$= (aa_1)(ya_2)a'$ (by associativity of $\bar{V}$)

$= (aa_1)y(a_2a')$ (as $a_2a \in \bar{U} \subseteq \bar{S}$)

$= (aa_1)(ya_2a')$ (by associativity of $\bar{V}$)

$= a(a_1ya_2a')$ (by associativity of $\bar{V}$)

$= a(va'),$

as required.

We, now, have $\bar{S} \neq A = Dom(\bar{S}, A)$.

This contradicts the fact that S is supersaturated.

Hence the theorem is proved. 2

Open Question. Is an ideal of a saturated [supersaturated] semigroup a saturated [super-saturated] semigroup?

REFERENCES

[1] Alam N and Khan NM. Epimorphisms, closed and supersaturated semi-groups, Commun Algebra 2014; 42: 3137-3146. http://dx.doi.org/10.1080/00927872.2013.781609

[2] Clifford AH and Preston GB. The Algebraic Theory of Semigroups. Math Surveys No.7, Amer Math Soc Providence RI 1961; I. 1967; II.

[3] Higgins PM. Saturated and closed varieties of semigroups. J Aust Math Soc 1984; 36: 153-175. http://dx.doi.org/10.1017/S1446788700024629

[4] Higgins PM. Epimorphisms, dominions and semigroups. Algebra Univers 1985; 21: 225-233. http://dx.doi.org/10.1007/BF01188058

[5] Howie JM. An Introduction to semigroup. Theory London Math Soc Mono-graph. Academic Press San Diego 1976; Vol.7.

[6] Isbell JR. Epimorphisms and dominions. Proc of the conference on Cate-gorical Algebra, LaJolla, Lange and Springer, Berlin 1966; 232-246. http://dx.doi.org/10.1007/978-3-642-99902-4_9

[7] Khan NM and Shah AH. Epimorphisms, dominions and permutative semi-groups, Semigroup Forum 2010; 80: 181-190. http://dx.doi.org/10.1007/s00233-009-9198-1

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