# Computing the Hermitian Positive Definite Solutions of a Nonlinear Matrix Equation

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**Abstract:** In this paper, we consider a nonlinear matrix equation. We propose necessary and sufficient conditions for the existence of Hermitian positive definite solutions. Some necessary conditions and sufficient conditions for the existence of Hermitian positive definite solutions of this equation is also derived. Based on the Banach fixed point theorem, the existence and the uniqueness of the Hermitian positive definite solution are studied. An iterative method for obtaining the Hermitian positive definite solution of this equation is proposed. Finally, some numerical examples are presented to illustrate the performance and efficiency of the proposed algorithm.

**Keywords:** Nonlinear matrix equation, hermitian positive definite solution, iterative method.

## 1. INTRODUCTION

In this paper, we consider the nonlinear matrix equation;

$$X^s + A^* X^{-t_1} A + B^* X^{-t_2} B = I \quad (1)$$

where $I$ is an $n \times n$ identity matrix, $A, B$ are $n \times n$ nonsingular complex matrices and $s, t_1, t_2$ are positive numbers. $A^*$ stands for the conjugate transpose of the matrix $A$.

Nonlinear matrix equations with the form of (1) have many applications in engineering, control theory, dynamic programming, ladder networks statistics and so on. Several authors have studied the necessary and sufficient conditions of the existence of Hermitian positive definite (HPD) solutions of similar kinds of nonlinear matrix equations. In [7], the case $s = t_1 = t_2 = 1$ is considered and different itreative methods for computing the HPD solutions are proposed. In addition, the case $s = t_1 = 1, 0 < t_2 \le 1$ has been studied for computing the HPD solutions are proposed in [8]. In [9], the author considered the matrix equation $X + A^* X^{-t_1} A + B^* X^{-t_2} B = I \ (0 < t_1,\ t_2 \le 1)$ and proposed three different kinds of iterative methods to compute the HPD solutions. The authors [10,12] considered the matrix equation $X^s + \sum_{i=1}^{m} A_i^* X^{t_i} A_i = Q$ with $s > 0,\ 0 < t_i \le 1$ and studied the existence and the uniqueness of the HPD solution.

In this paper, we discuss the general case, namely, $X^s + A^* X^{-t_1} A + B^* X^{-t_2} B = I$ with $s, t_1, t_2 > 0$. We propose necessary and sufficient conditions for the existence of HPD solutions. Some necessary conditions and sufficient conditions for the existence of HPD solutions of this equation is also derived. Based on the Banach fixed point theorem, the existence and the uniqueness of the Hermitian positive definite solution are studied.

The paper is organized as follows: In Section 2, we give some notations and lemmas that will be needed to develop this work. Then in Section 3, we propose necessary and sufficient conditions for the existence of HPD solutions of Eq.(1). We also present some necessary conditions and sufficient conditions for the existence of HPD solutions of Eq.(1). In section 4, we propose an iterative method for the HPD solution of Eq.(1). Finally, some numerical examples are presented to illustrate the performance and the efficiency of the algorithm.

## 2. PRELIMINARIES

The following notations and the lemmas started below will be needed for developing the work:

(1) For $A, B \in C^{n\times n}$, we write $A > 0\ (\ge 0)$ if the matrix $A$ is an Hermitian positive definite (semidefinite). If $A - B > 0\ (A - B \ge 0)$, we write $A > B\ (A \ge B)$.

(2) We use $\|B\|$ and $\|B\|_F$ to denote the spectral norm and Frobenius norm of an $n \times n$ HPD matrix $B$.

(3) Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalues of an $n \times n$ HPD matrix $A$ respectively.

(4) The spectral norm is monotonic, that is if $0 < A \le B$, then $\|A\| \le \|B\|$.

**Lemma 2.1** ([1]). If $A \ge B > 0$ (or $A > B > 0$), then $A^\partial \ge B^\partial > 0$ (or $A^\partial > B^\partial > 0$) for all $\partial \in (0,1]$, and $B^\partial \ge A^\partial > 0$ (or $B^\partial > A^\partial > 0$) for all $\partial \in [-1,0)$.

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E-ISSN: 2409-5761/16 

**Lemma 2.2** ([3]). If $D$ and $E$ are Hermitian matrices of the same order with $E>0$, then;

$DED+E^{-1}\ge 2D$.

**Lemma 2.3** ([4]). If $P$ and $Q$ are Hermitian matrices of the same order with $PQ=QP$, then $P^{\partial}>Q^{\partial}>0$ for all $\partial\in(0,+\infty)$.

**Lemma 2.4** ([2]). If $\partial>0$, and $P$ and $Q$ are positive definite matrices of the same order with $P,Q\ge bI>0,$ then $\left\|P^{-\partial}-Q^{-\partial}\right\|\le \partial b^{-\partial-1}\left\|P-Q\right\|.$

**Lemma 2.5** ([5]). If $0<\partial\le 1$, $P$ and $Q$ are positive definite matrices of the same order with $P,Q\ge bI>0$, then

$$\left\|P^{\partial}-Q^{-\partial}\right\|\le \partial b^{\partial-1}\left\|P-Q\right\|,\ \left\|P^{-\partial}-Q^{-\partial}\right\|\le \partial b^{-\partial-1}\left\|P-Q\right\|.$$

**Lemma 2.6** ([5]). Let $f(x)=x^t\left(\eta-x^s\right),\ \eta>0,$ $x\ge 0$. Then

(1) $f$ is increasing on $\left[0,\left(\frac{s}{s+t}\eta\right)^{\frac{1}{s}}\right]$ and decreasing on $\left[\left(\frac{s}{s+t}\eta\right)^{\frac{1}{s}},+\infty\right)$;

(2) $f_{\max}=f\left(\left(\frac{s}{s+t}\eta\right)^{\frac{1}{s}}\right)=\frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\eta^{\frac{t}{s}+1}$.

**Lemma 2.7** ([7]). Let $P$ and $Q$ be positive operators on a Hilbert space such that;

$0<m_1I\le P\le M_1I, 0<m_2I\le Q\le M_2I, 0<P\le Q,$ then;

$$P^{\alpha}\ge\left(\frac{M_1}{m_1}\right)^{\alpha-1}Q^{\alpha},\ P^{\alpha}\ge\left(\frac{M_2}{m_2}\right)^{\alpha-1}Q^{\alpha}\ \text{ hold for any } \alpha\ge 1.$$

## 3. EXISTENCE CONDITIONS AND PROPERTIES OF THE HPD SOLUTIONS

**Theorem 3.1** Eq.(1) has an HPD solution if and only if there exist an HPD matrix $L$ and a column-orthonormal matrix $\begin{pmatrix} L^s \\ N_1 \\ N_2\end{pmatrix}$ such that

$$A=L^{t_1}N_1, B=L^{t_2}N_2. \tag{2}$$

In this case, Eq.(1) has an HPD solution $X=L^2$, and all the solutions can be constructed by this way.

**Proof:** Necessity. If Eq.(1) has an HPD solution $X$, then $X>0$. Let $X=L^2=L^*L$ be the Cholesky factorization, where $L$ is an HPD matrix. Then Eq.(1) can be rewritten as;

$$\begin{aligned}
&X^s+A^*X^{-t_1}A+B^*X^{-t_2}B\\
&=L^{2s}+A^*L^{-2t_1}A+B^*L^{-2t_2}B\\
&=\left(L^*\right)^sL^s+A^*\left(L^*\right)^{-t_1}L^{-t_1}A+B^*\left(L^*\right)^{-t_2}L^{-t_2}B\\
&=\left(\left(L^*\right)^s, A^*\left(L^*\right)^{-t_1}, B^*\left(L^*\right)^{-t_2}\right)\begin{pmatrix} L^s\\ L^{-t_1}A\\ L^{-t_2}B\end{pmatrix}\\
&=\begin{pmatrix} L^s\\ L^{-t_1}A\\ L^{-t_2}B\end{pmatrix}^*\begin{pmatrix} L^s\\ L^{-t_1}A\\ L^{-t_2}B\end{pmatrix}=I,
\end{aligned}$$

Let $N_1=L^{-t_1}A,\quad N_2=L^{-t_2}B,$ then we get that $A=L^{t_1}N_1, B=L^{t_2}N_2$ and $\begin{pmatrix} L^s\\ N_1\\ N_2\end{pmatrix}$ has orthonormal columns.

Sufficiency. Let $X=L^2=L^*L$. It follows from Eq.(2) that;

$$\begin{aligned}
&X^s+A^*X^{-t_1}A+B^*X^{-t_2}B\\
&=L^{2s}+N_1^*L^{t_1}L^{-2t_1}L^{t_1}N_1+N_2^*L^{t_2}L^{-2t_2}L^{t_2}N_2\\
&=L^{2s}+N_1^*N_1+N_2^*N_2\\
&=\begin{pmatrix} L^s\\ N_1\\ N_2\end{pmatrix}^*\begin{pmatrix} L^s\\ N_1\\ N_2\end{pmatrix}\\
&=I,
\end{aligned}$$

hence Eq.(1) has an HPD solution.

**Corallary 3.2** If $\begin{pmatrix} Q_1\\ Q_2\\ Q_3\end{pmatrix}$ has orthonormal columns, where $Q_1,Q_2,Q_3\in C^{n\times n}$ and $Q_1=Q_1^*\ >0$, then there exist unitary matrices $U_1\in C^{n\times n}$, $U_2\in C^{2n\times 2n}$ and diagonal matrices $I\ge W>0$, $I>T\ge 0$ with $W^2+T^2=I$ such that;

$$\begin{pmatrix} U_1^* & 0\\ 0 & U_2^*\end{pmatrix}\begin{pmatrix} Q_1\\ Q_2\\ Q_3\end{pmatrix}U_1=\begin{pmatrix} W\\ T\end{pmatrix}.$$

**Proof:** First, $\lambda(Q_1) \le 1$ and there exist unitary matrix $U_1$ and diagonal matrix $W$ such that;

$$U_1^* Q_1 U_1 = W = \begin{pmatrix} I_t & \\ & C_{n-t} \end{pmatrix}.$$

Let $\tilde{Q} = \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}$,

since $Q_1^* Q_1 + \tilde{Q}^* \tilde{Q} = I$ and

$$\left(\tilde{Q}U_1\right)^* \left(\tilde{Q}U_1\right) = U_1^* \left(I - Q_1^* Q_1\right) U_1$$
$$= I - W^* W = \begin{pmatrix} 0_t & 0 \\ 0 & I_{n-t} - C_{n-t}^2 \end{pmatrix} = T^2,$$

$\tilde{Q}U_1$ has orthonormal columns, where the first $t$ columns are zero columns. Orthormalizing the later $n-t$ columns of $\tilde{Q}U_1$ and expanding them into an orthonormal basis of $C^{n\times n}$, we use $U_2$ to denote the basis matrix and obtain that $U_2^* \tilde{Q} U_1 = T$.

**Theorem 3.3** Eq.(1) has an HPD solution if and only if there exist a unitary matrix $P_1 \in C^{n\times n}$, a column-orthonormal matrix $P = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} \in C^{2n\times n}$,

$\left(P_{11}, P_{21} \in C^{n\times n}\right)$ and diagonal matrices $C > 0$ and $S \ge 0$ with $C^2 + S^2 = I$ such that;

$$A = \left(P_1^* C P_1\right)^{\frac{t_1}{s}} P_{11} S P_1, \tag{3}$$

$$B = \left(P_1^* C P_1\right)^{\frac{t_2}{s}} P_{21} S P_1. \tag{4}$$

**Proof**: Necessity. If Eq.(1) has an HPD solution, by Theorem 3.1, we get that;

$A = L^{t_1} N_1, B = L^{t_2} N_2$, and the matrix $\begin{pmatrix} L^s \\ N_1 \\ N_2 \end{pmatrix}$ has orthonormal columns. According to the CS decomposition Theorem and Corallary 3.2, there exists a unitary matrix

$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \in C^{3n\times 3n} (P_1 \in C^{n\times n}, P_2 \in C^{2n\times 2n})$, such that;

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} L^s \\ N_1 \\ N_2 \end{pmatrix} P_1^* = \begin{pmatrix} \Gamma \\ \Sigma \end{pmatrix} = \begin{pmatrix} I_t & 0 \\ 0 & C_{n-t} \\ 0_t & 0 \\ 0 & S_{n-t} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C \\ S \\ 0 \end{pmatrix}, \tag{5}$$

where;

$C = diag\left(1, 1, \cdots, 1, \cos\theta_{t+1}, \cdots, \cos\theta_n\right)$,
$S = diag\left(0, 0, \cdots, 0, \sin\theta_{t+1}, \cdots, \sin\theta_n\right)$,
$0 \le \theta_{t+1} \le \cdots \le \theta_n \le \dfrac{\pi}{2}$. Thus the diagonal matrices $C, S \ge 0$ and $C^2 + S^2 = I$. Notice that $L$ is an HPD matrix, by (5), we have;

$$C = P_1 L^s P_1^* > 0, \tag{6}$$

$$P_2 \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} P_1^* = \begin{pmatrix} S \\ 0 \end{pmatrix}. \tag{7}$$

Eq.(7) is equivalent to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = P_2^* \begin{pmatrix} S \\ 0 \end{pmatrix} P_1$, Let $P_2^*$ be partitioned as $P_2^* = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, in which $P_{ij} \in C^{n\times n}, i = 1,2; j = 1,2.$ Then we have

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} S \\ 0 \end{pmatrix} P_1 = \begin{pmatrix} P_{11} S P_1 \\ P_{21} S P_1 \end{pmatrix},$$

thus $N_1 = P_{11} S P_1, N_2 = P_{21} S P_1$. By (6), we obtain;

$L = \left(P_1^* C P_1\right)^{\frac{1}{s}}$. Then by $A = L^{t_1} N_1, B = L^{t_2} N_2$ in Theorem 3.1, we get that;

$$A = L^{t_1} N_1 = \left(P_1^* C P_1\right)^{\frac{t_1}{s}} P_{11} S P_1,\ B = L^{t_2} N_2 = \left(P_1^* C P_1\right)^{\frac{t_2}{s}} P_{21} S P_1.$$

Sufficiency. If $A, B$ have the decompositions as (2), then let $X = \left(P_1^* C^2 P_1\right)^{\frac{1}{s}}$, which is an HPD matrix, we obtain;

$$X^s + A^* X^{-t_1} A + B^* X^{-t_2} B$$
$$= P_1^* C^2 P_1 + \left(L^{t_1} N_1\right)^* \left(P_1^* C^2 P_1\right)^{\frac{t_1}{s}} \left(L^{t_1} N_1\right)$$
$$+ \left(L^{t_2} N_2\right)^* \left(P_1^* C^2 P_1\right)^{\frac{t_2}{s}} \left(L^{t_2} N_2\right)$$
$$= P_1^* C^2 P_1 + N_1^* L^{t_1} \left(P_1^* C^2 P_1\right)^{\frac{t_1}{s}} L^{t_1} N_1$$
$$+ N_2^* L^{t_2} \left(P_1^* C^2 P_1\right)^{\frac{t_2}{s}} L^{t_2} N_2$$

$$= P_1^* C^2 P_1 + N_1^* N_1 + N_2^* N_2$$
$$= P_1^* C^2 P_1 + \left(N_1^* \ N_2^*\right) \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$
$$= P_1^* C^2 P_1 + P_1^* \begin{pmatrix} S \\ 0 \end{pmatrix}^* P_2 P_2^* \begin{pmatrix} S \\ 0 \end{pmatrix} P_1$$
$$= P_1^* C^2 P_1 + P_1^* S^2 P_1 = I,$$

thus $X$ is an HPD solution of Eq.(1).

**Theorem 3.4** If Eq.(1) has an HPD solution, then;

$$\left(\rho(A)\right)^2 \le \frac{s}{s+t_1}\left(\frac{t_1}{s+t_1}\right)^{\frac{t_1}{s}}, \tag{8}$$

$$\left(\rho(B)\right)^2 \le \frac{s}{s+t_2}\left(\frac{t_2}{s+t_2}\right)^{\frac{t_2}{s}}. \tag{9}$$

**Proof:** If Eq.(1) has an HPD solution, by Theorem 3.3, there exist a unitary matrix $P_1 \in C^{n\times n}$, a column-orthonormal matrix $P = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} \in C^{2n\times n}(P_{11}, P_{21} \in C^{n\times n})$, and diagonal matrices $C > 0$ and $S \ge 0$ with $C^2 + S^2 = I$ such that $A = \left(P_1^* C P_1\right)^{\frac{t_1}{s}} P_{11} S P_1$, $B = \left(P_1^* C P_1\right)^{\frac{t_2}{s}} P_{21} S P_1$.

$$\lambda(A) = \lambda\left(\left(P_1^* C P_1\right)^{\frac{t_1}{s}} P_{11} S P_1\right)$$
$$= \lambda\left(P_1^* C^{\frac{t_1}{s}} P_1 P_{11} S P_1\right)$$
$$= \lambda\left(C^{\frac{t_1}{s}} P_1 P_{11} S\right).$$

Then $\left(\rho(A)\right)^2 \le \left\| C^{\frac{t_1}{s}} P_1 P_{11} S \right\|_2^2 = \left\| P_1 P_{11} C^{\frac{t_1}{s}} S \right\|_2^2 \le \left\| C^{\frac{t_1}{s}} S \right\|_2^2,$

Let $C = diag\left(x_1, x_2, \cdots, x_n\right), S = diag\left(y_1, y_2, \cdots, y_n\right),$ where $x_i \in (0,1], x_i^2 + y_i^2 = 1.$

Then;

$$\rho(A) = \max_i \left\{ x_i^{\frac{t_1}{s}} \left(1 - x_i^2\right)^{\frac{1}{2}} \right\}$$
$$\le \max_{x\in(0,1]} \left\{ x^{\frac{t_1}{s}} \left(1 - x^2\right)^{\frac{1}{2}} \right\} = \frac{s}{s+t_1}\left(\frac{t_1}{s+t_1}\right)^{\frac{t_1}{s}}.$$

The proof of (10) is similar to that of (9), thus omitted here.

**Theorem 3.5** If Eq.(1) has an HPD solution $X$, then $\lambda_{\min}\left(A^*A + B^*B\right) \le \left(\frac{t}{s+t}\right)^{\frac{t}{s}} \frac{s}{s+t}$, and $X \le \hat{\alpha} I$, where $t = \min\left\{t_1, t_2\right\}$, and $\hat{\alpha}$ is a solution of equation;

$$y^t\left(1 - y^s\right) = \lambda_{\min}\left(A^*A + B^*B\right) \text{ in } \left[\left(\frac{t}{s+t}\right)^{\frac{1}{s}}, 1\right].$$

**Proof:** Consider the sequence defined as follows:

$\alpha_0 = 1,\ \alpha_{k+1} = \left(1 - \frac{\lambda_{\min}\left(A^*A + B^*B\right)}{\alpha_k^t}\right), k = 0,1,2,\cdots$ Let $X$ be an HPD solution of Eq.(1), then $X = \left(I - A^* X^{-t_1} A - B^* X^{-t_2} B\right)^{\frac{1}{s}} < I = \alpha_0 I.$ Suppose that $X < \alpha_k I$, then by Lemma 2.3, we have

$$X^s = I - A^* X^{-t_1} A - B^* X^{-t_2} B$$
$$< I - A^*\left(\alpha_k I\right)^{-t_1} A - B^*\left(\alpha_k I\right)^{-t_2} B$$
$$< I - \frac{A^*A + B^*B}{\alpha_k^t}$$
$$< \left(1 - \frac{\lambda_{\min}\left(A^*A + B^*B\right)}{\alpha_k^t}\right) I$$
$$= \alpha_{k+1}^s I.$$

Therefore $X < \alpha_{k+1} I$. Then by induction, we obtain;

$$X < \alpha_k I, k = 0,1,2,\cdots.$$

Notice that the sequence $\alpha_k$ is monotonically decreasing and positive, hence $\alpha_k$ is convergent. Let $\lim_{k\to\infty} \alpha_k = \hat{\alpha}$, then;

$$\hat{\alpha} = \left(1 - \frac{\lambda_{\min}\left(A^*A + B^*B\right)}{\alpha_k^t}\right)^{\frac{1}{s}},$$

where $\hat{\alpha}$ is a solution of equation;

$$y^t\left(1 - y^s\right) = \lambda_{\min}\left(A^*A + B^*B\right).$$

Consider the function $f(y) = y^t\left(1 - y^s\right)$, since;

$$\max_{y\in[0,1]} = f\left(\left(\frac{t}{s+t}\right)^{\frac{1}{s}}\right) = \left(\frac{t}{s+t}\right)^{\frac{t}{s}} \frac{s}{s+t},$$

from which it follows that;

$$\lambda_{\min}\left(A^*A+B^*B\right)\le\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\frac{s}{s+t}.$$

Next we shall prove that $\hat{\alpha}=\left[\left(\frac{t}{s+t}\right)^{\frac{1}{s}},1\right]$.

Obviously, $\hat{\alpha}\le 1$. On the other hand, for the sequence $\alpha_k$, since $\alpha_0=1>\left(\frac{t}{s+t}\right)^{\frac{1}{s}}$, we may assume that $\alpha_k>\left(\frac{t}{s+t}\right)^{\frac{1}{s}}$, then

$$\alpha_{k+1}=\left(1-\frac{\lambda_{\min}\left(A^*A+B^*B\right)}{\alpha_k^t}\right)^{\frac{1}{s}}$$

$$\ge\left(1-\frac{1}{\alpha_k^t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\frac{s}{s+t}\right)^{\frac{1}{s}}$$

$$>\left(1-\frac{1}{\left(\frac{t}{s+t}\right)^{\frac{t}{s}}}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\frac{s}{s+t}\right)^{\frac{1}{s}}$$

$$=\left(\frac{t}{s+t}\right)^{\frac{1}{s}}.$$

Hence $\alpha_k>\left(\frac{t}{s+t}\right)^{\frac{1}{s}},k=0,1,2,\cdots.$ So $\hat{\alpha}=\lim_{k\to\infty}\alpha_k\ge\left(\frac{t}{s+t}\right)^{\frac{1}{s}}$. Consequently, we have $\hat{\alpha}=\left[\left(\frac{t}{s+t}\right)^{\frac{1}{s}},1\right]$.

This completes the proof.

**Theorem 3.6** Suppose Eq.(1) has an HPD solution, then if $\begin{pmatrix}A\\B\end{pmatrix}$ is column full rank, then $\lambda_{\max}(X)<1$; if $\begin{pmatrix}A\\B\end{pmatrix}$ is not column full rank, then $\lambda_{\max}(X)=1$.

**Proof:** Eq.(1) is equivalent to;

$$X^s+\begin{pmatrix}A\\B\end{pmatrix}^*\begin{pmatrix}X^{-t_1}&\\&X^{-t_2}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=I. \qquad (10)$$

(1) If $\begin{pmatrix}A\\B\end{pmatrix}$ is column full rank, then $\begin{pmatrix}A\\B\end{pmatrix}^*\begin{pmatrix}X^{-t_1}&\\&X^{-t_2}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$ is positive definite matrix, hence;

$X^s=I-\begin{pmatrix}A\\B\end{pmatrix}^*\begin{pmatrix}X^{-t_1}&\\&X^{-t_2}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}<I,$ we can obviously obtain that $\lambda_{\max}(X)<1$. If $\begin{pmatrix}A\\B\end{pmatrix}$ is not column full rank, let $rank\left(\frac{A}{B}\right)=r<n,$ by Schur theorem, there exists a unitary matrix $T$ such that $\begin{pmatrix}A\\B\end{pmatrix}=T^*\begin{pmatrix}A_1&0\\B_1&0\end{pmatrix}T$.

Let $Y=TXT^*$, then Eq.(11) has an HPD solution $X$ if and only if;

$$Y^s+\begin{pmatrix}A_1&0\\B_1&0\end{pmatrix}^*Y^{-t_1}\begin{pmatrix}A_1&0\\B_1&0\end{pmatrix}=I \qquad (11)$$

has an HPD solution $Y$. Substituting;

$$Y^s=\begin{pmatrix}Y_{11}&Y_{21}\\Y_{21}^*&Y_{22}\end{pmatrix},Y^{-t_1}=\begin{pmatrix}\tilde{Y}_{11}&\tilde{Y}_{21}\\\tilde{Y}_{21}^*&\tilde{Y}_{22}\end{pmatrix},$$

Then (11) turns into the following form

$\begin{pmatrix}*&Y_{21}\\Y_{21}^*&Y_{22}\end{pmatrix}=I,$ and therefore

$$Y_{21}=0,Y_{22}=I,\ X=T^*\begin{pmatrix}Y_{11}^*&0\\0&I\end{pmatrix}T,$$

$$\lambda_{\max}(X)=\max\left\{\lambda_{\max}\left(Y_{11}^{\frac{1}{s}}\right),1\right\}\ge 1.$$

On the other hand, $\lambda_{\max}(X)\le 1$, so we can get that $\lambda_{\max}(X)=1$.

**Theorem 3.7** Suppose Eq.(1) has an HPD solution $X$, then $X\in(M,N)$, where

$$M=\frac{1}{2}\begin{pmatrix}\left(\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(AA^*)}\right)^{\frac{1-t_1}{t_1}}\left(AA^*\right)^{\frac{1}{t_1}}+\\\left(\frac{\lambda_{\min}(BB^*)}{\lambda_{\max}(BB^*)}\right)^{\frac{1-t_2}{t_2}}\left(BB^*\right)^{\frac{1}{t_2}}\end{pmatrix},$$

$$N = \left(I - A^* A - B^* B\right)^{\frac{1}{s}}. \tag{12}$$

**Proof:** Let $X$ be an HPD solution of Eq.(1), then it follows that $0 < X^s < I$ and Lemma 2.3 that $X^{-t_1} > I$ , $X^{-t_2} > I$. Hence;

$X^s = I - A^* X^{-t_1} A - B^* X^{-t_2} B < I - A^* A - B^* B.$ Thus we have;

$X < (I - A^* A - B^* B)^{\frac{1}{s}} = N.$ On the other hand, from $A^* X^{-t_1} A < I,$ it follows that;

$$A^* X^{-\frac{t_1}{2}} X^{-\frac{t_1}{2}} A < I,$$
$$X^{-\frac{t_1}{2}} A A^* X^{-\frac{t_1}{2}} < I,$$
$$A A^* < I.$$

When $0 < t_1 < 1,$ since $\frac{1}{t_1} > 1,$. and

$\lambda_{\min}(AA^*) I \le AA^* \le \lambda_{\max}(AA^*) I,$ by Lemma 2.7, we get that;

$X > \left(\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(AA^*)}\right)^{\frac{1-t_1}{t_1}} \left(AA^*\right)^{\frac{1}{t_1}}.$ When $t_1 \ge 1,$ since $0 < \frac{1}{t_1} \le 1,$ by Lemma 2.1, we have $X > \left(AA^*\right)^{\frac{1}{t_1}}.$ Hence, for $\forall t_1 \in R^+$ , we have;

$X > \left(\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(AA^*)}\right)^{\frac{1-t_1}{t_1}} \left(AA^*\right)^{\frac{1}{t_1}}.$ When $0 < t_2 < 1,$ we have;

$X > \left(\frac{\lambda_{\min}(BB^*)}{\lambda_{\max}(BB^*)}\right)^{\frac{1-t_2}{t_2}} \left(BB^*\right)^{\frac{1}{t_2}}.$ When $t_1 \ge 1,$ we have $X > \left(BB^*\right)^{\frac{1}{t_2}}.$ Thus for $\forall t_2 \in R^+$ , we have;

$X > \left(\frac{\lambda_{\min}(BB^*)}{\lambda_{\max}(BB^*)}\right)^{\frac{1-t_2}{t_2}} \left(BB^*\right)^{\frac{1}{t_2}}.$ Hence we have;

$$X > \frac{1}{2}\left(\left(\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(AA^*)}\right)^{\frac{1-t_1}{t_1}} \left(AA^*\right)^{\frac{1}{t_1}} + \left(\frac{\lambda_{\min}(BB^*)}{\lambda_{\max}(BB^*)}\right)^{\frac{1-t_2}{t_2}} \left(BB^*\right)^{\frac{1}{t_2}}\right) = M.$$

This completes the proof.

**Theorem 3.8** If $A^* X^{-t_1} A + B^* X^{-t_2} B \le I - M^s$ for all $X \in \left[M, I\right]$, and

$p = \frac{1}{s}\left(t_1 \lambda_{\min}^{-t_1 - s}\left(M\right)\left\|A\right\|_F^2 + t_2 \lambda_{\min}^{-t_2 - s}\left(M\right)\left\|B\right\|_F^2\right) < 1,$ where $M$ is defined by(15), then Eq.(1) has a unique HPD solution.

**Proof:** By the definition of $M$, we have $M > 0.$ Hence $\lambda_{\min}(M) > 0.$ We consider the map;

$F(X) = \left(I - A^* X^{-t_1} A + B^* X^{-t_2} B\right)^{\frac{1}{s}}$ and let $X \in \Phi = \left\langle X \middle\| M \le X \le I\right\rangle.$ Obviously, $\Phi$ is a convex, closed and bounded set and $F(X)$ is continuous on $\Phi$ .

By the hypothesis of the theorem, we obtain;

$I \ge \left(I - A^* X^{-t_1} A + B^* X^{-t_2} B\right)^{\frac{1}{s}} \ge \left(I - I + M^s\right)^{\frac{1}{s}} = M$, i.e., $M \le F\left(X\right) \le I.$ Hence $F\left(\Phi\right) \subseteq \Phi.$ For arbitrary $X, Y \in \Phi,$ we get that;

$$A^* X^{-t_1} A + B^* X^{-t_2} B \le I - M^s,$$
$A^* Y^{-t_1} A + B^* Y^{-t_2} B \le I - M^s.$ Hence

$$\begin{aligned} F\left(X\right) &= \left(I - A^* X^{-t_1} A + B^* X^{-t_2} B\right)^{\frac{1}{s}} \\ &\ge \left(I - I + M^s\right)^{\frac{1}{s}} \ge \lambda_{\min}\left(M\right) I, \end{aligned} \tag{13}$$

$$\begin{aligned} F\left(Y\right) &= \left(I - A^* Y^{-t_1} A + B^* Y^{-t_2} B\right)^{\frac{1}{s}} \\ &\ge \left(I - I + M^s\right)^{\frac{1}{s}} \ge \lambda_{\min}\left(M\right) I. \end{aligned} \tag{14}$$

From (13) and (14), we have;

$$\begin{aligned} &\left\|F\left(X\right)^s - F\left(Y\right)^s\right\|_F \\ &= \left\|\sum_{i=0}^{s-1} F\left(X\right)^i \left(F\left(X\right) - F\left(Y\right)\right) F\left(Y\right)^{s-i-1}\right\|_F \\ &= \left\|vec\left[\sum_{i=0}^{s-1} F\left(X\right)^i \left(F\left(X\right) - F\left(Y\right)\right) F\left(Y\right)^{s-i-1}\right]\right\| \\ &= \left\|\sum_{i=0}^{s-1} vec\left[F\left(X\right)^i \left(F\left(X\right) - F\left(Y\right)\right) F\left(Y\right)^{s-i-1}\right]\right\| \\ &= \left\|\sum_{i=0}^{s-1} \left(F\left(Y\right)^{s-i-1} \otimes F\left(X\right)^i\right) vec\left(F\left(X\right) - F\left(Y\right)\right)\right\| \\ &\ge \sum_{i=0}^{s-1} \lambda_{\min}^{s-1}\left(M\right) \left\|vec\left(F\left(X\right) - F\left(Y\right)\right)\right\| \\ &= s \lambda_{\min}^{s-1}\left(M\right) \left\|F\left(X\right) - F\left(Y\right)\right\|_F. \end{aligned} \tag{15}$$

According to the definition of the map of $F$, we obtain;

$$\begin{aligned} &F(X)^s - F(Y)^s \\ &= \left(I - A^*X^{-t_1}A + B^*X^{-t_2}B\right) - \left(I - A^*Y^{-t_1}A + B^*Y^{-t_2}B\right) \quad (16) \\ &= A^*\left(Y^{-t_1} - X^{-t_1}\right)A + B^*\left(Y^{-t_2} - X^{-t_2}\right)X^{-t_2}B. \end{aligned}$$

Combining (15) and (16), by Lemma 2.4 and Lemma 2.5 , we have;

$$\begin{aligned} &\left\|F(X) - F(Y)\right\|_F \\ &\le \frac{1}{s\lambda_{\min}^{s-1}(M)}\left\|F(X)^s - F(Y)^s\right\|_F \\ &= \frac{1}{s\lambda_{\min}^{s-1}(M)}\left\|A^*\left(Y^{-t_1} - X^{-t_1}\right)A + B^*\left(Y^{-t_2} - X^{-t_2}\right)X^{-t_2}B\right\|_F \\ &\le \frac{1}{s\lambda_{\min}^{s-1}(M)}\left(\|A\|_F^2\left\|Y^{-t_1} - X^{-t_1}\right\|_F + \|B\|_F^2\left\|Y^{-t_2} - X^{-t_2}\right\|_F\right) \\ &\le \frac{1}{s\lambda_{\min}^{s-1}(M)}\left(t_1\lambda_{\min}^{-t_1-1}(M)\|A\|_F^2 + t_2\lambda_{\min}^{-t_2-1}(M)\|B\|_F^2\right)\|X - Y\|_F \\ &= \frac{1}{s}\left(t_1\lambda_{\min}^{-t_1-s}(M)\|A\|_F^2 + t_2\lambda_{\min}^{-t_2-s}(M)\|B\|_F^2\right)\|X - Y\|_F \end{aligned}$$

Let $p = \frac{1}{s}\left(t_1\lambda_{\min}^{-t_1-s}(M)\|A\|_F^2 + t_2\lambda_{\min}^{-t_2-s}(M)\|B\|_F^2\right)$, since $p < 1$, by Banach fixed point theorem, the map $F(X)$ has a unique fixed point in $\Phi$ and Eq.(1) has a unique HPD solution in $[M, I]$.

## 4. AN ITERATIVE METHOD FOR SOLVING THE EQUATION (1)

In this section, we consider the iterative method for solving the Eq.(1). We proposed the following algorithm:

**Algorithm 4.1.**

$$\begin{cases} X_0 = \alpha I, \\ X_{k+1} = \left(I - A^*X_k^{-t_1}A - B^*X_k^{-t_2}B\right)^{\frac{1}{s}}, k = 0,1,2,3\cdots \end{cases}$$

**Theorem 4.1** Suppose $A, B$, $0 < \gamma < \eta < 1$ and $0 < t_1 < t_2$ , satisfying the following conditions:

(1) $\eta^{t_1}\left(1 - \eta^s\right)I \le A^*A + B^*B \le \gamma^{t_2}\left(1 - \gamma^s\right)I,$

(2) $\frac{1}{s}\gamma^{1-s}\left(t_1\gamma^{-t_1-1}\|A\|^2 + t_2\gamma^{-t_2-1}\|B\|^2\right) < 1.$

Then the sequence $\{X_k\}$ generated by Algorithm 4.1 converges to the HPD solution $X$ of the Eq. (1), where $\alpha \in [\gamma, \eta]$, $\gamma I \le X \le \eta I$.

**Proof:** Since $X_0 = \alpha I$ , we have $\gamma I \le X_0 \le \eta I$.

Suppose that $\gamma I \le X_k \le \eta I$ , by Lemma 2.3, we have;

$$\frac{1}{\eta^{t_1}}I \le X_k^{-t_1} \le \frac{1}{\gamma^{t_1}}I \Rightarrow \frac{1}{\eta^{t_1}}A^*A \le A^*X_k^{-t_1}A \le \frac{1}{\gamma^{t_1}}A^*A,$$

$$\frac{1}{\eta^{t_2}}I \le X_k^{-t_2} \le \frac{1}{\gamma^{t_2}}I \Rightarrow \frac{1}{\eta^{t_2}}B^*B \le B^*X_k^{-t_2}B \le \frac{1}{\gamma^{t_2}}B^*B.$$

Thus;

$$\begin{aligned} &\left(I - \frac{1}{\gamma^{t_1}}A^*A - \frac{1}{\gamma^{t_2}}B^*B\right)^{\frac{1}{s}} \le \left(I - A^*X_k^{-t_1}A - B^*X_k^{-t_2}B\right)^{\frac{1}{s}} \\ &\le \left(I - \frac{1}{\eta^{t_1}}A^*A - \frac{1}{\eta^{t_2}}B^*B\right)^{\frac{1}{s}}. \end{aligned}$$

By (1) in Theorem 4.1, we zoom the inequality separately as follows:

$$\begin{aligned} X_{k+1} &\ge \left(I - \frac{1}{\gamma^{t_1}}A^*A - \frac{1}{\gamma^{t_2}}B^*B\right)^{\frac{1}{s}} \\ &\ge \left(I - \frac{1}{\gamma^{t_2}}\left(A^*A + B^*B\right)\right)^{\frac{1}{s}} \\ &\ge \left(I - \frac{1}{\gamma^{t_2}}\gamma^{t_2}\left(1 - \gamma^s\right)I\right)^{\frac{1}{s}} = \gamma I, \\ X_{k+1} &\le \left(I - \frac{1}{\eta^{t_1}}A^*A - \frac{1}{\eta^{t_2}}B^*B\right)^{\frac{1}{s}} \\ &\le \left(I - \frac{1}{\eta^{t_1}}\left(A^*A + B^*B\right)\right)^{\frac{1}{s}} \\ &\le \left(I - \frac{1}{\eta^{t_1}}\eta^{t_1}\left(1 - \eta^s\right)I\right)^{\frac{1}{s}} = \eta I. \end{aligned}$$

i.e., $\gamma I \le X_{k+1} \le \eta I$.

By induction we have $\gamma I \le X_p \le \eta I, p = 1,2,3,\cdots$ According to Lemma 2.4 and Lemma 2.5, we obtain;

$$\begin{aligned} &\left\|X_{k+1} - X_k\right\| \\ &= \left\|\left(I - A^*X_k^{-t_1}A - B^*X_k^{-t_2}B\right)^{\frac{1}{s}} - \left(I - A^*X_{k-1}^{-t_1}A - B^*X_{k-1}^{-t_2}B\right)^{\frac{1}{s}}\right\| \\ &\le \frac{1}{s}\gamma^{s\left(\frac{1}{s}-1\right)}\left\|A^*\left(X_{k-1}^{-t_1} - X_k^{-t_1}\right)A + B^*\left(X_{k-1}^{-t_2} - X_k^{-t_2}\right)B\right\| \\ &\le \frac{1}{s}\gamma^{1-s}\left\|X_{k-1}^{-t_1} - X_k^{-t_1}\right\|\|A\|^2 + \frac{1}{s}\gamma^{1-s}\left\|X_{k-1}^{-t_2} - X_k^{-t_2}\right\|\|B\|^2 \\ &\le \frac{1}{s}\gamma^{1-s}t_1\gamma^{-t_1-1}\left\|X_{k-1} - X_k\right\|\|A\|^2 + \frac{1}{s}\gamma^{1-s}t_2\gamma^{-t_2-1}\left\|X_{k-1} - X_k\right\|\|B\|^2 \\ &= \frac{1}{s}\gamma^{1-s}\left(t_1\gamma^{-t_1-1}\|A\|^2 + t_2\gamma^{-t_2-1}\|B\|^2\right)\left\|X_{k-1} - X_k\right\| \end{aligned}$$

Let $q=\frac{1}{s}\gamma^{1-s}\left(t_1\gamma^{-t_1-1}\|A\|^2+t_2\gamma^{-t_2-1}\|B\|^2\right)$, By (2) in Theorem4.1, we obtain;

$\|X_{k+1}-X_k\|\le q\|X_k-X_{k-1}\|$, And then we have $\|X_{k+1}-X_k\|\le q^k\|X_1-X_0\|$, By Banach fixed point theorem, $\{X_k\}$ have the limit $X$.

**Note:**

When $0.9\le\gamma<\eta<1$, the condition (1) and (2) hold. 0.9 is estimated by MATLAB 7.0, there is no strict proof here.

## 5. NUMERICAL EXAMPLES

In this section, we give some numerical examples to illustrate the efficiency of the Algorithm 4.1. All computations are performed on a Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz computer. All the tests are performed by MATLAB, version 7.0. We stop the practical iteration when the residual;

$$\left\|X^s+A^*X^{-t_1}A+B^*X^{-t_2}B-I\right\|_F\le 1.0\times10^{-10}.$$

Example 5.1 Let $t_1=2, t_2=4, s=5, \alpha=0.91$ and

$$A=\begin{bmatrix}0.015 & -0.006 & -0.041\\ -0.106 & 0.038 & -0.043\\ -0.023 & -0.054 & -0.112\end{bmatrix},\ B=\begin{bmatrix}0.013 & 0.029 & 0.028\\ 0.025 & 0.133 & 0.047\\ 0.028 & 0.047 & 0.255\end{bmatrix}.$$

By using Algorithm 4.1 and iterating ten steps, we obtain the HPD solution of Eq. (1) as follows:

$$X\approx X_{10}=\begin{bmatrix}0.9972 & -0.0005 & -0.0034\\ -0.0005 & 0.9947 & -0.0055\\ -0.0034 & -0.0055 & 0.9811\end{bmatrix},$$

with the residual;

$$\left\|X_{10}^s+A^*X_{10}^{-t_1}A+B^*X_{10}^{-t_2}B-I\right\|_F\le 6.4410\times10^{-11}.$$

It is not difficult to varify that $\gamma I\le X_{10}\le\eta I$. The residual;

$$R_{10}(X)=\left\|X_{10}^s+A^*X_{10}^{-t_1}A+B^*X_{10}^{-t_2}B-I\right\|_F=6.4410\times10^{-11}$$

shows that the algorithm is effective.

Example 5.2 Let $t_1=0.5, t_2=1.5, s=2, \alpha=0.9$ and;

$$A=\begin{bmatrix}0.025 & -0.045 & 0.039 & 0.026\\ -0.019 & 0.057 & -0.038 & 0.019\\ 0.044 & -0.073 & 0.093 & -0.034\\ 0.021 & 0.016 & -0.013 & 0.023\end{bmatrix},$$

$$B=\begin{bmatrix}0.215 & -0.045 & 0.039 & 0.024\\ -0.015 & 0.127 & -0.038 & 0.017\\ 0.024 & -0.027 & 0.293 & -0.034\\ 0.012 & 0.016 & -0.012 & 0.021\end{bmatrix}.$$

By using Algorithm 4.1 and iterating ten steps, we obtain the HPD solution of Eq. (1) as follows:

$$X\approx X_{10}=\begin{bmatrix}0.9730 & 0.0095 & -0.0127 & -0.0016\\ 0.0095 & 0.9840 & 0.0148 & -0.0027\\ -0.0127 & 0.0148 & 0.9429 & 0.0075\\ -0.0016 & -0.0027 & 0.0075 & 0.9968\end{bmatrix},$$

with the residual;

$$\left\|X_{10}^s+A^*X_{10}^{-t_1}A+B^*X_{10}^{-t_2}B-I\right\|_F\le 5.8837\times10^{-11}.$$

It is not difficult to varify that $\gamma I\le X_{10}\le\eta I$. The residual;

$$R_{10}(X)=\left\|X_{10}^s+A^*X_{10}^{-t_1}A+B^*X_{10}^{-t_2}B-I\right\|_F=5.8837\times10^{-11}$$

shows that the algorithm is effective.

## 6. CONCLUSIONS

Compared to previous results in [12], our results are more general. In this paper, we discuss the HPD solutions of Eq.(1). We derive necessary and sufficient conditions for the existence of the HPD solutions of Eq.(1). We propose an iterative method for obtaining the HPD solution of Eq. (1). Numerical results show that the proposed algorithm is quite efficient. However, we have not found a suitable iterative algorithm for finding the optimal solution of the nonlinear matrix equation. We only add two constraint conditions to make the nonlinear matrix equation converge to an HPD solution by Algorithm 4.1.

## ACKNOWLEDGEMENTS

This work was supported by the Science and technology project of Shandong Province (J11LA04).

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Received on 20-03-2016 Accepted on 08-06-2016 Published on 14-07-2016

DOI: http://dx.doi.org/10.15377/2409-5761.2016.03.01.4