On Monotonic Solutions of A Nonlinear Integral Equation of Volterra Type

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Abstract: We study a nonlinear integral equation of Volterra type in the Banach space of real functions defined and continuous on a bounded and closed interval. Using a technique associated with measure of noncompactness we prove the existence of the nondecreasing solutions to a nonlinear integral equations of Volterra type in C [0, 1]. We give also one example satisfying the conditions of our main result but not satisfying the conditions of the main result in [1].

Keywords: Nonlinear volterra integral equations, measure of noncompactness, fixed point theorem.

1. INTRODUCTION

Integral equations arise naturally in applications of real world problems [2, 3, 6-9]. The theory of integral equations has been well developed with the help of various tools from functional analysis, topology and fixed-point theory.

The aim of this paper is to investigate the existence of nondecreasing solutions of an integral equation of Volterra type. Equations of such kind contain, among others, integral equations of convolution type. Our results will be established using measure of noncompactness defined in [5].

The main tool used in our investigation is the technique of measure of noncompactness which is frequently used in several branches of nonlinear analysis [4, 6, 7]. We will apply the measure of noncompactness defined in [5] in proving the existence of nondecreasing solutions to a nonlinear integral equation of Volterra type.

The results of this paper generalize and complete the results obtained earlier in the paper [1].

2. NOTATION AND AUXILARY FACTS

Assume that E is real Banach space with the norm $\|.\|$ and the zero element 0. Denote by B(x,r) the closed ball centered at x and with radius r and by B_r the ball B(0,r). If X is nonempty subset of E we denote by $\bar{X}$, Conv X the closure and the closed convex closure of X, respectively.

The symbols λX and X+Y we denote the usual algebraic operations on sets. Finally, let us denote by M_E the family of nonempty bounded subsets of E and by N_E its subfamily consisting of all relatively compact sets.

Definition 2.1 (See [4]). A function $\mu : M_E \to [0,\infty)$ is said to be *a measure of noncompactness* in the space *E* if it satisfies the following conditions:

1. The family ker $\mu=\{X \in M_E : \mu(X)=0\} \neq \varnothing$ and ker $\mu \subset N_E$.
2. $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
3. $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$
4. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, $for\ \lambda \in [0,1]$.
5. If $\{X_n\}_n$ is a sequence of closed sets from M_E such that $X_{n+1} \subset X_n\ for\ n = 1,2,...\ and\ if\ \lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family ker *μ* described above is called the *kernel of measure of noncompactness μ.* Further facts concerning measure of noncompactness and their properties may be found in [4].

Now, let us suppose that M is nonempty subset of a Banach space E and the operator T: M $\to$E is continuous and transforms bounded sets onto bounded ones. We say that T *satisfies the Darbo condition* (with constant $k \ge 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have $\mu(TX) \le k\mu(X)$.

If T satisfies the Darbo condition with $k < 1$ then it is called *a contraction with respect to* μ.

For our purposes we will only need the following fixed point theorem [4].

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Theorem 2.1. Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let F: Q → Q be a contraction with respect to *μ*. Then *F* has a fixed point in the set Q.

Remark 1. Under the assumptions of the above theorem it can be shown that, the set Fix F of fixed points of F belonging to Q is a member of ker *μ*.

Let C[0,1] denote the space of all real functions defined and continuous on the interval [0,1]. For convenience, we write I=[0,1] and C(I)=C[0,1]. The space C(I) is furnished with standard norm $\|x\| = \max\{|x(t)| : t \in I\}$. Next, we recall the definition of a measure of noncompactness in C(I) which will be used in the Section 3. This measure was introduced and studied in the paper [5].

Fix a nonempty and bounded subset X of C(I). For $\varepsilon > 0$ and $x \in X$ denote by $\omega(x,\varepsilon)$ the modulus of continuity of x defined by

$$\omega(x,\in) = \sup\{|x(t) - x(s)| : t,s \in I \text{ and } |t-s| \le \varepsilon\}.$$

Furthermore, let us put

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon).$$

Next, let us define the following quantities:

$$i(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t,s \in I \text{ and } t \le s\}$$

and

$$i(X) = \sup\{i(x) : x \in X\}.$$

Observe that $i(x)$=0 if and only if all functions belonging to X are nondecreasing on I. Finally, let us define μ(X) as

$$\mu(X) = \omega_0(X) + i(X) \cdot \tag{2.1}$$

It can be shown [5] that the function μ is a measure of noncompactness in the space C(I). Morever, the kernel ker μ consists of all sets X belonging to $M_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

3. MAIN RESULT

In this section, we consider the following nonlinear integral equation of Volterra type

$$x(t) = a(\alpha(t)) + (Tx)(\beta(t)) \int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))\,ds, \quad t \in I = [0,1]. \tag{3.1}$$

The functions $\alpha(t),\beta(t),\gamma(t),g(t),a(t),f(u),\phi(t,s)$ $\varphi(u)$ and $(Tx)(t)$ are given while x=x(t) is an unknown function. This equation will be examined under the following assumptions:

(i) $\alpha,\beta,\gamma : I \to I$ continuous and nondecreasing functions on I and $g: I \to I$ continuous function.

(ii) $a \in C(I)$ and it is nondecreasing and nonnegative on the interval I.

(iii) $\phi : I \times I \to \mathbb{R}$ is continuous on $I \times I$ and the function $t \to \phi(t,s)$ is nondecreasing for each $s \in I$.

(iv) $f : \operatorname{Im}\phi. \to \mathbb{R}_+$ is a continuous and nondecreasing function on compact set $\operatorname{Im}\phi$.

(v) $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

(vi) The operator T: C(I) → C(I) is continuous and T is a positive operator, i.e. $Tx \ge 0$ *if* $x \ge 0$.

Also, there exists the nonnegative costants *c*, *d* and p>0 such that $|(Tx)(t)| \le c + d\|x\|^p$ for each $x \in C(I)$ *and* $t \in I$.

(vii) There exists r_o>0 such that the inequality $a(\|\alpha\|) + (c + dr_0^p)\|f\| M_{\varphi,r_0} \le r_o$ is satisfied where $M_{\varphi,r_0} = \max\{|\varphi(u)| : u \in [-r_o, r_o]\}$.

(viii) The operator *T* satisfies the inequality $\mu(TX) \le \theta\mu(X)$

Hypothesis (i), (ii), (iii), (iv), (v), (vi), (vii) and (viii) in main result above should be on the same vertical line as below:

on $B_{r_0}^+ = \{x \in B_{r_0} : x(t) \ge 0, t \in I\}$ for the measure of noncompactness μ defined by (2.1) with a constant θ such that $\theta\|f\| M_{\varphi,r_0} < 1$.

Then, we have the following theorem.

Theorem 3.1. Under the assumptions (i)-(viii) the equation *(3.1)* has at least one solution x=x(t) which belongs to the space C(I) and is nondecreasing on the interval I.

Proof. Let us consider two operators A, B defined on the space C(I) by

$$(Ax)(t)=a(\alpha(t))+(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))ds$$

and

$$(Bx)(t)=\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))ds.$$

Firstly, we prove that if $x\in C(I)$ then Ax ∈ C(I). To do this it is sufficient to show that if $x\in C(I)$ then $Bx\in C(I)$. $Fix \in >0$, let $x\in C(I)$ and $t_1,t_2\in I$ such that $t_1\le t_2$ and $t_2-t_1\le\varepsilon$. Then,

$$|(Bx)(t_2)-(Bx)(t_1)|$$

$$=\left|\int_0^{\gamma(t_2)} f(\phi(t_2,s))\varphi(x(g(s)))ds-\int_0^{\gamma(t_1)} f(\phi(t_1,s))\varphi(x(g(s)))ds\right|$$

$$\le\left|\int_0^{\gamma(t_2)} f(\phi(t_2,s))\varphi(x(g(s)))ds-\int_0^{\gamma(t_2)} f(\phi(t_1,s))\varphi(x(g(s)))ds\right|$$

$$+\left|\int_0^{\gamma(t_2)} f(\phi(t_1,s))\varphi(x(g(s)))ds-\int_0^{\gamma(t_1)} f(\phi(t_1,s))\varphi(x(g(s)))ds\right|$$

$$\le\int_0^{\gamma(t_2)} |f(\phi(t_2,s))-f(\phi(t_1,s))|\,|\varphi(x(g(s)))|\,ds$$

$$+\int_{\gamma(t_1)}^{\gamma(t_2)} |f(\phi(t_1,s))||\varphi(x(g(s)))|\,ds.$$

Since f is continuous on compact region, there exists $\|f\|$. Therefore, if we denote $\omega_{fo\phi}(\varepsilon,.)=sup\{|f(\phi(t_2,s))-f(\phi(t_1,s))|: t_2,t_1,s\in I \text{ and } |t_2-t_1|\le\varepsilon\}$,

$$M_{\varphi,\|x\|}=\max\{|\varphi(u)|: u\in[-\|x\|,\|x\|]\}$$

and

$$\omega(\gamma,\varepsilon)=\sup\{|\gamma(t_2)-\gamma(t_1)|:\ t_1,t_2\in I \text{ and } |t_2-t_1|\le\varepsilon\},$$

then we get the following inequality

$$|(Bx)(t_2)-(Bx)(t_1)|$$

$$\le\ \omega_{fo\phi}(\varepsilon,.)M_{\varphi,\|x\|}\gamma(t_2)+\|f\|M_{\varphi,\|x\|}|\gamma(t_2)-\gamma(t_1)|$$

$$\le\ \omega_{fo\phi}(\varepsilon,.)M_{\varphi,\|x\|}\gamma(t_2)+\|f\|M_{\varphi,\|x\|}\omega(\gamma,\varepsilon).$$

Now, in virtue of the uniform continuity of the function $fo\phi \text{ on } IxI$ and uniform continuity of the function γ on I we have that $\omega_{fo\phi}(\varepsilon,.)\to 0$ and $\omega(\gamma,\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus $(Bx)\in C(I)$ and consequently, $(Ax)\in C(I)$.

Furthermore, for each t∈ I we have

$$|(Ax)(t)|=\left|a(\alpha(t))+(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))ds\right|$$

$$\le a(\|\alpha\|)+(c+d\|x\|^p)\int_0^{\gamma(t)} |f(\phi(t,s))||\varphi(x(g(s)))|\,ds$$

$$\le a(\|\alpha\|)+(c+d\|x\|^p)\|f\|M_{\varphi,\|x\|}.$$

Hence,

$$\|Ax\|\le a(\|\alpha\|)+(c+d\|x\|^p)\|f\|M_{\varphi,\|x\|}$$

Thus, if $\|x\|\le r_0$ then we obtain from assumption (vii) that $\|Ax\|\le a(\|\alpha\|)+(c+dr_0^p)\|f\|M_{\varphi,r_0}\le r_0$.

Consequently, the operator A transforms the ball $B_{r_0}=B(0,r_0)$ into itself.

Now, we will prove that A is continuous on B_{r_0}. To do this let us take (x_n) sequence in $B(0,r_0)$ such that $x_n\to x$ in this case we show that $Ax_n\to Ax$. In fact, for each $t\in I$ we have

$$|(Ax_n)(t)-(Ax)(t)|$$

$$=\left|(Tx_n)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x_n(g(s)))ds-(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))ds\right|$$

$$\le\left|(Tx_n)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x_n(g(s)))ds-(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x_n(g(s)))ds\right|$$

$$+\left|(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x_n(g(s)))ds-(Tx)(\beta(t))\int_0^{\gamma(t)} f(\phi(t,s))\varphi(x(g(s)))ds\right|$$

$$\le|(Tx_n)(\beta(t))-(Tx)(\beta(t))|\int_0^{\gamma(t)} |f(\phi(t,s))||\varphi(x_n(g(s)))|\,ds$$

$$+|(Tx)(\beta(t))|\int_0^{\gamma(t)} |f(\phi(t,s))||\varphi(x_n(g(s)))-\varphi(x(g(s))|\,ds$$

In virtue of the nondecreasing of the function β we have

$$|(Ax_n)(t)-(Ax)(t)|\ \le\ \|Tx_n-Tx\|\|f\|M_{\varphi,r_0}$$

$$+(c+dr_0^p)\|f\|\int_0^{\gamma(t)}\left|\varphi\left(x_n\left(g(s)\right)\right)-\varphi\left(x\left(g(s)\right)\right)\right|ds. \qquad (3.2)$$

Now, in virtue of the uniform continuity of the function φ on $[-r_0, r_0]$, for $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_1, u_2 \in [-r_0, r_0]$ with $|u_1 - u_2| \leq \delta$, we have

$$|\varphi(u_1)-\varphi(u_2)| \leq \frac{\varepsilon}{2(c+dr_0^p)\|f\|}.$$

Also, for this $\delta > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$ we have that $\| x_n - x\| \leq \delta$ i.e. $|x_n(t) - x(t)| \leq \delta$ for all $t \in I$ and consequently

$$\left|\varphi\left(x_n(g(s))\right)-\varphi\left(x(g(s))\right)\right| \leq \frac{\varepsilon}{2(c+dr_0^p)\|f\|}$$

for all $s \in I$. Then, taking into account the previous inequalities, for $\varepsilon > 0$ and $n \geq n_0$ we have

$$\| Ax_n - Ax\| \leq \| Tx_n - Tx\|\|f\|M_{\varphi,r_0}$$
$$+(c+dr_0^p)\|f\|\frac{\varepsilon}{2(c+dr_0^p)\|f\|} \leq \| Tx_n - Tx\|\|f\|M_{\varphi,r_0}+\frac{\varepsilon}{2}. \qquad (3.3)$$

Morever, since T is a continuous operator, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $\| Tx_n - Tx\| \leq \frac{\varepsilon}{2\|f\|M_{\varphi,r_0}}$

Consequently, if we take $n \geq \max\{n_0, n_1\}$ from the inequality (3.3) we have

$$\|Ax_n - Ax\| \leq \frac{\varepsilon}{2\|f\|M_{\varphi,r_0}}\|f\|M_{\varphi,r_0}+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This fact proves that the operator A is continuous on B_{r_0}.

Consider the operator A on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by

$$B_{r_0}^+ = \{ x \in B_{r_0} : x(t) \geq 0, t \in I \}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. Now, we show that $B_{r_0}^+$ is closed. To do this firstly, let $x \in \overline{B_{r_0}^+}$ then there exists a sequence $(x_n) \subset B_{r_0}^+$ such that $\lim_{n\to\infty} x_n = x$. Then we have $x_n \to x \Rightarrow \|x_n\| \to \|x\|$. Since $\lim_{n\to\infty}\|x_n\| \leq r_0$ then $\|x\| \leq r_0$. For this reason, $x \in B_{r_0}^+$. So, $B_{r_0}^+$ is closed.

In view of the assumptions (i), (ii), (iv), (v) and (vi) if $x(t) \geq 0$ then $(Ax)(t) \geq 0$ for $t \in I$. Thus A transforms the set $B_{r_0}^+$ into itself. Furthermore, A is continuous on $B_{r_0}^+$.

Let $\emptyset \neq X \subset B_{r_0}^+$, fix $\varepsilon > 0$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \varepsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then, from the definition of the operator *B* we obtain

$$(Bx)(t_2) = \int_0^{\gamma(t_2)} f(\phi(t_2,s))\varphi(x(g(s)))ds$$

and

$$(Bx)(t_1) = \int_0^{\gamma(t_1)} f(\phi(t_1,s))\varphi(x(g(s)))ds$$

Hence, we have

$$|(Ax)(t_2) - (Ax)(t_1)|$$
$$= |\mathrm{a}(\alpha(t_2))+(Tx)(\beta(t_2))(Bx)(t_2)$$
$$-a(\alpha(t_1))-(Tx)(\beta(t_1))(Bx)(t_1)|$$
$$\leq |\mathrm{a}(\alpha(t_2))-\mathrm{a}(\alpha(t_1))| + |(Tx)(\beta(t_2))(Bx)(t_2)$$
$$-(Tx)(\beta(t_1))(Bx)(t_2)|$$
$$+|(Tx)(\beta(t_1))(Bx)(t_2)-(Tx)(\beta(t_1))(Bx)(t_1)|$$
$$\leq |\mathrm{a}(\alpha(t_2))-\mathrm{a}(\alpha(t_1))| + |(Bx)(t_2)|\,|(Tx)(\beta(t_2))-(Tx)(\beta(t_1))|$$
$$+|(Tx)(\beta(t_1))||(Bx)(t_2)-(Bx)(t_1)|$$
$$\leq \omega(\mathrm{a}, \omega(\alpha,\varepsilon)) + \omega(Tx, \omega(\beta,\varepsilon))\|f\|M_{\varphi,r_0}\gamma(t_2)$$
$$+(c+dr_0^p)(\|f\|M_{\varphi,r_0}|\gamma(t_2)-\gamma(t_1)| + \gamma(t_1)M_{\varphi,r_0}\omega_{fo\phi}(\varepsilon,.))$$
$$\leq \omega(\mathrm{a}, \omega(\alpha,\varepsilon)) + \omega(Tx, \omega(\beta,\varepsilon))\|f\|M_{\varphi,r_0}$$
$$+(c+dr_0^p)(\|f\|M_{\varphi,r_0}\omega(\gamma,\varepsilon) + \omega_{fo\phi}(\varepsilon,.)).$$

Where $\omega(\alpha,\varepsilon)$, $\omega(\beta,\varepsilon)$, $\omega(\gamma,\varepsilon)$ and $\omega_{fo\phi}(\varepsilon,.)$ are defined as

$\omega(\alpha, \varepsilon) = \sup\{|\alpha(t) - \alpha(s)| : t, s \in I \text{ and } |t-s| \le \varepsilon\}$,

$\omega(\beta, \varepsilon) = \sup\{|\beta(t) - \beta(s)| : t, s \in I \text{ and } |t-s| \le \varepsilon\}$,

$\omega(\gamma, \varepsilon) = \sup\{|\gamma(t) - \gamma(s)| : t, s \in I \text{ and } |t-s| \le \varepsilon\}$ and

$\omega_{fo\phi}(\varepsilon, .) = \sup\{|f(\phi(t,s)) - f(\phi(t',s))| : t, t' s \in I \text{ and } |t-t'| \le \varepsilon\}$

Thus, we get

$\omega(Ax, \varepsilon) \le \omega(\mathrm{a}, \omega(\alpha, \varepsilon)) + \omega(Tx, \omega(\beta, \varepsilon))\|f\|M_{\varphi, r_0}$ $+ (c + d r_0^p)(\|f\|M_{\varphi, r_0}\, \omega(\gamma, \varepsilon) + \omega_{fo\phi}(\varepsilon, .))$. If we take the supremum at this inequality over all x, we have the inequality

$$\omega(AX, \varepsilon) \le \omega(\mathrm{a}, \omega(\alpha, \varepsilon)) + \omega(TX, \omega(\beta, \varepsilon))\|f\|M_{\varphi, r_0} + (c + d r_0^p)(\|f\|M_{\varphi, r_0}\omega(\gamma, \varepsilon) + \omega_{fo\phi}(\varepsilon, .)).$$

In virtue of the continuity of the functions $\mathrm{a}, \alpha, \beta$ and γ on I uniform continuity of the function $fo\phi$ on $I \times I$ we have that $\omega(\alpha, \varepsilon), \omega(\beta, \varepsilon)$ and $\omega(\gamma, \varepsilon) \to 0$ furthermore $\omega_{fo\phi}(\varepsilon, .)$ and $\omega(\mathrm{a}, \omega(\alpha, \varepsilon)) \to 0$ as $\varepsilon \to 0$.

Consequently, we have

$$\omega_0(AX) \le \|f\|M_{\varphi, r_0}\omega_0(TX). \quad (3.4)$$

Let $\emptyset \ne X \subset B_{r_0}^+$, $x \in X, t_1, t_2 \in I$ and $t_1 \le t_2$. Then,

$$|(Ax)(t_2) - (Ax)(t_1)| - [(Ax)(t_2) - (Ax)(t_1)]$$
$$= |a(\alpha(t_2)) + (Tx)(\beta(t_2))(Bx)(t_2)$$
$$- \mathrm{a}(\alpha(t_1)) - (Tx)(\beta(t_1))(Bx)(t_1)|$$
$$- [\mathrm{a}(\alpha(t_2)) + (Tx)(\beta(t_2))(Bx)(t_2)$$
$$- \mathrm{a}(\alpha(t_1)) - (Tx)(\beta(t_1))(Bx)(t_1)]$$
$$\le |\mathrm{a}(\alpha(t_2)) - \mathrm{a}(\alpha(t_1))| - [(\mathrm{a}(\alpha(t_2)) - \mathrm{a}(\alpha(t_1))]$$
$$+ |(Tx)(\beta(t_2))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_1)|$$
$$- [(Tx)(\beta(t_2))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_1)]$$
$$\le |(Tx)(\beta(t_2))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_2)|$$
$$+ |(Tx)(\beta(t_1))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_1)|$$
$$- [(Tx)(\beta(t_2))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_2)]$$
$$- [(Tx)(\beta(t_1))(Bx)(t_2) - (Tx)(\beta(t_1))(Bx)(t_1)]$$
$$\le [|(Tx)(\beta(t_2)) - (Tx)(\beta(t_1))| -$$
$$((Tx)(\beta(t_2)) - (Tx)(\beta(t_1)))]\,(Bx)(t_2)$$
$$+ (Tx)(\beta(t_1))\,[\,|(Bx)(t_2) - (Bx)(t_1)|$$
$$- ((Bx)(t_2) - (Bx)(t_1))].$$

Now we will prove $((Bx)(t_2) - (Bx)(t_1)) \ge 0$. In fact, notice that

$$((Bx)(t_2) - (Bx)(t_1)) =$$
$$\int_0^{\gamma(t_2)} f(\phi(t_2, s))\varphi(x(g(s)))ds$$
$$- \int_0^{\gamma(t_1)} f(\phi(t_1, s))\varphi(x(g(s)))ds$$
$$= \int_0^{\gamma(t_2)} f(\phi(t_2, s))\varphi(x(g(s)))ds$$
$$- \int_0^{\gamma(t_2)} f(\phi(t_1, s))\varphi(x(g(s)))ds$$
$$+ \int_0^{\gamma(t_2)} f(\phi(t_1, s))\varphi(x(g(s)))ds$$
$$- \int_0^{\gamma(t_1)} f(\phi(t_1, s))\varphi(x(g(s)))ds$$
$$= \int_0^{\gamma(t_2)} (f(\phi(t_2, s)) - f(\phi(t_1, s)))\varphi\left(x(g(s))\right)ds$$
$$+ \int_{\gamma(t_1)}^{\gamma(t_2)} f(\phi(t_1, s))\varphi(x(g(s)))ds.$$

Since $t \to \phi(t, s)$ is nondecreasing we have $\phi(t_2, s) \ge \phi(t_1, s)$. On the other hand, in view of the assumption *(iv)*, $f(\phi(t_2, s)) - f(\phi(t_1, s)) \ge 0$. Furthermore, since $x(g(s)) \ge 0$ we have $\varphi(x(g(s)))ds \ge 0$. Then,

$$\int_0^{\gamma(t_2)} (f(\phi(t_2, s)) - f(\phi(t_1, s)))\varphi(x(g(s)))ds \ge 0. \quad (3.5)$$

On the other hand, since $f \geq 0$, $\varphi(x(g(s))) \geq 0$ and γ is nondecreasing function we obtain that

$$\int_{\gamma(t_1)}^{\gamma(t_2)} f(\phi(t_1,s))\varphi(x(g(s)))ds \geq 0. \tag{3.6}$$

Consequently, from (3.5) and (3.6) we observe that $((Bx)(t_2) - (Bx)(t_1)) \geq 0$. Hence,

$$|(Ax)(t_2) - (Ax)(t_1)| \; -((Ax)(t_2) - (Ax)(t_1))$$
$$\leq [|(Tx)(\beta(t_2)) - (Tx)(\beta(t_1))| \; - ((Tx)(\beta(t_2)) - (Tx)(\beta(t_1)))] \times$$
$$\int_0^{\gamma(t_2)} f(\phi(t_2,s))\varphi(x(g(s)))ds$$
$$\leq \|f\| M_{\varphi,r_0} i(Tx).$$

Thus, if we take the supremum at this inequality over all $t_2, t_1 \in I$, for $t_1 \leq t_2$, we have the inequality

$$i(Ax) \leq \|f\| M_{\varphi,r_0} i(Tx).$$

Consequently, if we take the supremum over all $x \in B_{r_0}^+$ we get

$$i(AX) \leq \|f\| M_{\varphi,r_0} i(TX). \tag{3.7}$$

Finally, combining (3.4) and (3.7), we obtain

$$\begin{aligned}\mu(AX) &= \omega_0(AX) + i(AX)\\ &\leq \|f\| M_{\varphi,r_0}\omega_0(TX) + \|f\| M_{\varphi,r_0} i(TX)\\ &= \|f\| M_{\varphi,r_0}\, \mu(TX)\\ &\leq \|f\| M_{\varphi,r_0}\, \theta\mu(X).\end{aligned}$$

Thus, Theorem 2.1 (recall $\|f\| M_{\varphi,r_0}\, \theta < 1$) guarantees that there exists $x \in B_{r_0}^+$ a solution of (3.1). Furthermore, such a solution is nondecreasing in view of Remark 1.

Example 3.1. Let us consider the equation

$$x(t) = t^2 + \frac{x^2(t)}{5}\int_0^t \ln(1+\sqrt{t+s})\,\frac{1}{x^2(s)+1}ds, \qquad t \in I = [0,1]. \tag{3.8}$$

Let $a(t)=t$, $\alpha(t) = t^2$, $\beta(t)=t$, $\gamma(t) = t$, $g(s)=s$. These functions satisfies the assumptions (i) and (ii). The function $\phi(t,s)=\sqrt{t+s}$ satisfies the assumption (iii). Let $f: [0, \sqrt{2}] \to \mathbb{R}_+$ be given by $f(u)=\ln(1+u)$ and it satisfies assumption (iv). Let $\varphi(x) = \frac{1}{x^2+1}$ and this function satisfies assumption (v). Also, this function has an absolute maximum at x=0. We have that $M_{\varphi,r} = \varphi(0)=1$. Let $(Tx)(t) = \frac{x^2(t)}{5}$ and c=0, $d=\frac{1}{5}$, $p=2$. From $\alpha(t) = t^2$ and $a(t)=t$, it is seen that $a(\|\alpha\|)=1$ and besides $\|f\| = \ln(1+\sqrt{2})$.

The operator T from C(I) to C(I) is a positive operator. Let us show that the operator T is continuous. To do this let x_0 be arbitrarily element chosen from C(I). For $\|x - x_0\| < \delta$, we have the following estimate:

$$\begin{aligned}\|Tx - Tx_0\| &= \max_{t\in I}\left|\frac{x^2(t)}{5} - \frac{x_0{}^2(t)}{5}\right|\\ &= \frac{1}{5}\max_{t\in I}|\,x^2(t) - x_0{}^2(t)\,|\\ &= \frac{1}{5}\max_{t\in I}[|\,x(t) - x_0(t)\,|\,|\,x(t) + x_0(t)\,|]\end{aligned}$$

and

$$|\,x(t)\,| = |x(t) - x_0(t) + x_0(t)| \leq |x(t) - x_0(t)| + |x_0(t)|$$
$$\leq \|x - x_0\| + \|x_0\| < \delta + \|x_0\|. \tag{3.9}$$

From the inequality (3.9) we obtain

$$|\,x(t) + x_0(t)\,| \leq |x(t)| + \|x_0\| \leq \delta + 2\|x_0\|. \tag{3.10}$$

From the inequality (3.10) we obtain

$$\begin{aligned}\|Tx - Tx_0\| &= \frac{1}{5}\max_{t\in I}[|x(t) - x_0(t)|\,|x(t) + x_0(t)|]\\ &\leq \frac{1}{5}(\delta + 2\|x_0\|)\,\max_{t\in I}|\,x(t) - x_0(t)\,|\\ &= \frac{1}{5}(\delta + 2\|x_0\|)\|x - x_0\| \;\leq \frac{1}{5}(\delta + 2\|x_0\|)\delta.\end{aligned}$$

Taking $\frac{1}{5}(\delta + 2\|x_0\|)\delta = \varepsilon$ we get

$$\begin{aligned}\delta^2 + 2\|x_0\|\,\delta - 5\,\varepsilon = 0 \;&\Rightarrow (\delta + \|x_0\|)^2 - \|x_0\|^2 - 5\varepsilon = 0\\ &\Rightarrow (\delta + \|x_0\|)^2 = \|x_0\|^2 + 5\varepsilon\\ &\Rightarrow \;\delta + \|x_0\| = \sqrt{\|x_0\|^2 + 5\varepsilon}.\end{aligned}$$

If δ is chosen as

$$\delta = \sqrt{\|x_0\|^2 + 5\varepsilon} - \|x_0\| > 0\,.$$

it is seen that the operator T is continuous at the point x_0. Since x_0 is an arbitrarily element chosen from C(I), T is continuous on C(I).

On the other hand, for each x∈ C(I) and each t∈I the inequality

$$|(Tx)(t)| \le c + d\|x\|^p, \qquad (p > 0)$$

is provided. In fact,

$$\left|\frac{x^2(t)}{5}\right| \le \frac{1}{5}|x^2(t)| = \frac{1}{5}|x(t)|^2 \le \frac{1}{5}\|x\|^2$$

where $c = 0, d = \frac{1}{5}, p = 2$.

Thus, the assumption (vi) is satisfied. There exists r_0 positive solution that provides the inequality

$$a(\|\alpha\|) + (c + dr^p)\|f\|M_{\varphi,r} \le r.$$

Actually, since $\|\alpha\| = 1$, $a(\|\alpha\|) = 1$, $M_{\varphi,r} = 1$ and for any number r_0 constant which provides the inequality $1.29614 \le r \le 4.37683$,

this constant r_0 is a solution of the following inequality

$$1 + \frac{1}{5}r^2 \ln(1+\sqrt{2}) \le r.$$

For example $r_0=2$ is a solution of this inequality. Thus the assumption (vii) is satisfied.

Let $\emptyset \ne X \subset B_2^+$, $x \in X$, $t_1 \le t_2$, $t_2 - t_1 \le \varepsilon$ and $t_1, t_2 \in I$. In this case, we have the following estimate:

$$\begin{aligned}|(Tx)(t_2) - (Tx)(t_1)| &= \left|\frac{x^2(t_2)}{5} - \frac{x^2(t_1)}{5}\right| \\ &\le \frac{1}{5}|x(t_2) + x(t_1)||x(t_2) - x(t_1)| \\ &\le \frac{1}{5}(|x(t_2)| + |x(t_1)|)|x(t_2) - x(t_1)| \\ &\le \frac{1}{5}(\|x\| + \|x\|)|x(t_2) - x(t_1)| \\ &\le \frac{4}{5}|x(t_2) - x(t_1)|\end{aligned}$$

so,

$$\sup_{t_1,t_2 \in I} |(Tx)(t_2) - (Tx)(t_1)| \le \frac{4}{5} \sup_{t_1,t_2 \in I} |x(t_2) - x(t_1)|$$

thus we get

$$\omega(Tx, \varepsilon) \le \frac{4}{5}\omega(x, \varepsilon)$$

which implies

$$\sup_{x \in X} \omega(Tx, \varepsilon) \le \frac{4}{5} \sup_{x \in X} \omega(x, \varepsilon).$$

Then, we have

$$\omega(TX, \varepsilon) \le \frac{4}{5}\omega(X, \varepsilon)$$

and

$$\lim_{\varepsilon \to 0} \omega(TX, \varepsilon) \le \frac{4}{5} \lim_{\varepsilon \to 0} \omega(X, \varepsilon)$$

i.e.

$$\omega_0(TX) \le \frac{4}{5}\omega_0(X). \tag{3.11}$$

Let $X \ne \emptyset$, $X \subset B_2^+$, $x \in X$, $t_1 \le t_2$ and $t_1, t_2 \in I$. In this case, we have the following estimate:

$$\begin{aligned}&|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)] \\ &= \left|\frac{x^2(t_2) - x^2(t_1)}{5}\right| - \left[\frac{x^2(t_2) - x^2(t_1)}{5}\right] \\ &\le \frac{1}{5}|x(t_2) - x(t_1)||x(t_2) + x(t_1)| \\ &- \frac{1}{5}[(x(t_2) - x(t_1))(x(t_2) + x(t_1))] \\ &= \frac{1}{5}(|x(t_2)| + |x(t_1)|)[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\ &\le \frac{1}{5}(\|x\| + \|x\|)[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \\ &\le \frac{4}{5}[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))]\end{aligned}$$

so,

$$|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)]$$

$$\le \frac{4}{5}[|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]].$$

In this inequality, if we take the supremum over all $t_1, t_2 \in I$, we get

$$\sup_{t_1,t_2 \in I} \{|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)]\}$$

$$\le \frac{4}{5} \sup_{t_1,t_2 \in I} \{|(x(t_2) - x(t_1))| - [(x(t_2) - x(t_1))]\}$$

which yields

$$i(Tx) \le \frac{4}{5} i(x).$$

Again, if we take the supremum over all x, we get the inequality

$$\sup_{x\in X} i(Tx) \le \frac{4}{5} \sup_{x\in X} i(x)$$

and we have

$$i(TX) \le \frac{4}{5} i(X). \tag{3.12}$$

From the inequalities (3.11) and (3.12) we get

$$\mu(TX) \le \frac{4}{5} \mu(X).$$

Then, θ *can be taken as* $\theta = \frac{4}{5}$. *Also, for* $M_{\varphi,2} = 1$ *and* $\|f\| = \ln(1+\sqrt{2})$ *the inequality* $\theta \|f\| M_{\varphi,2} < 1$ *holds. Thus, the assumption (viii) is satisfied and in view of the Theorem* 3.1*, the equation* (3.8) *has at least one solution which is nondecreasing*.

Remark 2. In the Example 3.1, since

$$|(Tx)(t)| \le 0 + \frac{1}{5}\|x\|^2$$

for all $x \in C(I)$ and $t \in I$, the condition (vi)

$$|(Tx)(t)| \le c + d\|x\|$$

in [1] doesn't hold. Hence, the main result given in [1] is not applicable to the our integral equation (3.8).

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