Matrix Transforms of Summability Domains of Zweier Method

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Abstract. Let $Z_{1/2}$ be the Zweier method, and B and M matrices with real or complex entries. In the present paper, we find necessary and sufficient conditions for M to be transform from the summability domain of $Z_{1/2}$ into the summability domain of B if B is lower triangular. For an infinite matrix B, we consider only sufficient conditions.

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1. INTRODUCTION

Let ω be the set of all sequences with real or complex entries. Let $A=(a_{nk})$ be a matrix with real or complex entries and

$$A_n x := \sum_k a_{nk} x_x, \quad Ax := \left(A_n x\right)$$

for every $x := (x_k) \in \omega$. Throughout this paper, we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. A sequence $x \in \omega$ is said to be *A*-summable (or summable by the summability method A) if the sequence Ax is convergent. The set of all *A*-summable sequences is said to be the summability domain of *A* and denoted by c_A. A method A is said to be lower triangular if $a_{nk}=0$ for $k>n$, and normal if A is lower triangular with $a_{nn} \neq 0$ for every *n*.

The Zweier method $Z_{1/2}$ is defined by the lower triangular matrix $Z_{1/2}=(a_{nk})$ where (see [4], p. 14) $a_{00}=1/2$ and

$$a_{nk} = \begin{cases} 1/2, \text{if } k=n-1 \text{ and } k=n; \\ 0, \quad \text{if } k=n-1 \end{cases}$$

for $n \geq 1$. Let $B=(b_{nk})$ and $M=(m_{nk})$ be matrices with real or complex entries. In this paper, we find necessary and sufficient conditions for M to be transform from $c_{Z_{1/2}}$ into c_B for lower triangular B. For an infinite matrix B we obtain only sufficient conditions. If M transforms c_A into c_B, we write $M \in \left(c_A, c_B\right)$.

The paper has been organized as follows. In Section 2, some auxiliary results have been introduced. In Section 3, necessary and sufficient conditions for M to be transform from $c_{Z_{1/2}}$ into c_B for lower triangular B has been studied. In Section 4, sufficient conditions for $M \in \left(c_{Z_{1/2}}, c_B\right)$ in the case of an infinite matrix *B* have been presented.

2. AUXILIARY RESULTS

Let throughout this paper $A=(a_{nk})$ be a normal method with its inverse $A^{-1}=(\eta_{nk})$, and

$$\eta_k := \sum_{l=0}^{k} \eta_{kl}.$$

For a lower triangular method $B=(b_{nk})$ and an arbitrary matrix $M=(m_{nk})$ throughout this paper, we use the following notations:

$$h_{jl}^n := \sum_{k=l}^{l+j} m_{nk}\eta_{kl},$$

$$G=(g_{nk})=BM, \qquad \text{that is,}$$

$$g_{nk} := \sum_{l=0}^{n} b_{nl} m_{lk},$$

and

$$\chi_{nl}^r := \sum_{k=l}^{l+r} g_{nk}\eta_{kl}.$$

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Lemma 2.1. ([1], p. 93). The matrix transformation Mx exists for each $x \in c_A$ if and only if

there exist finite limits $\lim_j h_{jl}^n := h_{nl}$, (2.1)

series $\sum_k m_{nk}\eta_k$ are convergent for every n (2.2)

$$\sum_l \left|h_{jl}^n\right| = O_n(1). \quad (2.3)$$

Lemma 2.2. ([1], p. 95-96; see also [3], p.40-41). Let $B=(b_{nk})$ be a lower triangular method and $M=(m_{nk})$ an arbitrary matrix. Then, $M \in \left(c_A, c_B\right)$ if and only if conditions (2.1) – (2.3) are fulfilled and

there exists the finite limit $\lim_n \sum_k g_{nk}\eta_k$, (2.4)

there exist finite limits $\lim_n \chi_{nl}, w$ (2.5)

$$\sum_k \left|\chi_{nl}\right| = O(1), \quad (2.6)$$

where

$$\chi_{nl} := \lim_r \chi_{nl}^r.$$

Remark 2.3. The existence of limits χ_{nl} follows from conditions (2.1) – (2.3). If M is lower triangular, then conditions (2.1) – (2.3) are redundant in Lemma 2.2.

Lemma 2.4. ([1], p. 100-101; see also [2], p. 7-8). Let $B=(b_{nk})$ and $M=(m_{nk})$ be arbitrary matrices with real or complex entries satisfying, correspondingly, conditions

$$\sum_k \left|b_{nk}\right| = O_n(1) \quad (2.7)$$

and

$$m_{nk} = O_k(1) . \quad (2.8)$$

If conditions (2.1) and (2.2) are fulfilled and

$$\sum_{k=0}^{r} m_{nk}\eta_k = O(1), \quad (2.9)$$

$$\sum_l \left|h_{jl}^n\right| = O(1), \quad (2.10)$$

then

finite limits $\lim_r \chi_{nl}^r := \chi_{nl}$ exist. (2.11)

If, in addition to it, conditions (2.4) – (2.6) are fulfilled, then $M \in \left(c_A, c_B\right)$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR

$$M \in \left(c_{Z_{1/2}}, c_B\right)$$

Let throughout this Section, unless otherwise stated, $B=(b_{nk})$ be a lower triangular method and $M=(m_{nk})$ an arbitrary matrix.

Proposition 3.1. The matrix transformation Mx exists for each $x \in c_{Z_{1/2}}$ if and only if

series $\sum_k (-1)^k m_{nk}$ are convergent for every *n* (3.1)

series $\sum_k m_{n,2k}$ are convergent for every n (3.2)

$$\sum_l \left|\sum_{k=l}^{l+j} (-1)^{k-l} m_{nk}\right| = O_n(1). \quad (3.3)$$

Proof. It is sufficient to show that all conditions of Lemma 2.1 are satisfied. First we see that for every $x=(x_k) \in c_{Z_{1/2}}$ the system

$$y_n = A_n x;\ n = 0,1,...$$

for $A = Z_{1/2}$ can be presented in the form

$$y_n = \frac{x_{n-1}+x_n}{2};\ n = 0,1,....$$

It is easy to show that the solution of this system is

$$x_k = 2\sum_{l=0}^{k} (-1)^{k-l} y_l;\ k = 0,1,....$$

Hence the inverse matrix of the Zweier method $Z_{1/2}$ is the lower triangular matrix $A^{-1} = (\eta_{kl})$, where

$$\eta_{kl} = 2(-1)^{k-l}. \quad (3.4)$$

This implies that

$$h_{jl}^{n}=2\sum_{k=l}^{l+j}(-1)^{k-l}m_{nk}\ . \tag{3.5}$$

Therefore, condition (3.1) is equivalent to condition (2.1), and condition (3.3) to condition (2.3). As

$$\eta_k=\sum_{l=0}^{k}\eta_{kl}=2\sum_{l=0}^{k}(-1)^{k-l},$$

then

$$\eta_{2k}=2 \text{ and } \eta_{2k+1}=0;\ k=0,1,.... \tag{3.6}$$

Consequently, condition (3.2) is equivalent to (2.2). Thus, by Lemma 2.1, the transform Mx exists for each $x\in c_{Z_{1/2}}$.

Now we prove the main result of this section.

Theorem 3.2. A matrix $M\in\left(c_{Z_{1/2}},c_B\right)$ if and only if conditions (3.1) – (3.3) are fulfilled and

there exists the finite limit $\lim\limits_n\sum\limits_k g_{n,2k}$, (3.7)

there exist finite limits $\lim\limits_n g_{nk}$, (3.8)

there exists the finite limit $\lim\limits_n\sum\limits_k(-1)^k g_{nk}$, (3.9)

$$\sum_l\left|\sum_{k=l}^{\infty}(-1)^{k-l}g_{nk}\right|=O(1). \tag{3.10}$$

Proof. It is sufficient to show that all conditions of Lemma 2.2 are satisfied. First we see that conditions (3.1) – (3.3) are equivalent to (2.1) – (2.3) by Proposition 3.1. Due to (3.6), condition (3.7) is equivalent to (2.4). By (3.4) we obtain

$$\chi_{nl}^{r}=2\sum_{k=l}^{l+r}(-1)^{k-l}g_{nk}. \tag{3.11}$$

It follows from (3.1) – (3.3) by Remark 2.3, that condition (2.11) holds. Hence

$$\chi_{nl}=2\sum_{k=l}^{\infty}(-1)^{k-l}g_{nk}. \tag{3.12}$$

This implies that condition (3.10) is equivalent to (2.6), and condition (2.5) is equivalent to the condition

there exist finite limits $\lim\limits_n\sum\limits_{k=l}^{\infty}(-1)^{k-l}g_{nk}$. (3.13)

From (3.13) follows the validity of (3.9). As

$$g_{nl}=\chi_{nl}+\chi_{n,l+1},$$

then, due to (3.12), condition (3.8) also follows from (3.13). Conversely, from (3.8) and (3.9) follows the validity of (3.13). Therefore, (3.8) and (3.9) are equivalent to condition (2.5). Thus, $M\in\left(c_{Z_{1/2}},c_B\right)$ by Lemma 2.2.

Remark 3.3. If M is lower triangular, then conditions (3.1) – (3.3) are redundant in Theorem 3.2.

Now we consider the case if $M=(m_{nk})$ is a multiplicative matrix; *i.e.*,

$$m_{nk}=t_n v_k;\ (t_n)\in\omega,(v_k)\in\omega. \tag{3.14}$$

Proposition 3.4. Let M be defined by (3.14), where (v_k) is a positive monotonically decreasing sequence and the series $\sum_k v_k$ is convergent. Then, $M\in\left(c_{Z_{1/2}},c_B\right)$ if and only if $(t_n)\in c_B$.

Proof. It is sufficient to show that all conditions of Theorem 3.2 are satisfied. As

$$\sum_k(-1)^k m_{nk}=t_n\sum_k(-1)^k v_k$$

and

$$\sum_k m_{n,2k}=t_n\sum_k v_{2k},$$

then conditions (3.1) and (3.2) are fulfilled, since $v_k\geq 0$ and $\sum_k v_k$ is convergent. Also condition (3.3) holds. Indeed,

$$\sum_l\left|\sum_{k=l}^{l+j}(-1)^{k-l}m_{nk}\right|=|t_n|\sum_l\left|\sum_{k=l}^{l+j}(-1)^{k-l}v_k\right|$$
$$<|t_n|\sum_l v_l=O_n(1),$$

since (v_k) is monotonically decreasing and positive. Thus condition (3.3) holds.

We can write that

$$g_{nk}=v_k\sum_{l=0}^{n}b_{nl}t_l,$$

$$\sum_k g_{n,2k}=\sum_k v_{2k}\sum_{l=0}^{n}b_{nl}t_l,$$

$$\sum_k(-1)^k g_{nk}=\sum_k(-1)^k v_k\sum_{l=0}^{n}b_{nl}t_l$$

This implies that conditions (3.7) – (3.9) are fulfilled if and only if $(t_n)\in c_B$. As

$$\sum_l\left|\sum_{k=l}^{\infty}(-1)^{k-l}g_{nk}\right|=\sum_l\left|\sum_{k=l}^{\infty}(-1)^{k-l}v_k\sum_{j=0}^{n}b_{nj}t_j\right|=$$
$$\left|\sum_{j=0}^{n}b_{nj}t_j\right|\sum_l\left|\sum_{k=l}^{\infty}(-1)^{k-l}v_k\right|\le\left|\sum_{j=0}^{n}b_{nj}t_j\right|\sum_l v_k=O(1)$$

due to $(t_n)\in c_B$, then condition (3.10) also holds. Consequently, $M\in\left(c_{Z_{1/2}},c_B\right)$ by Theorem 3.2.

Remark 3.3. The assumptions for (v_k) presented in Proposition 3.4 holds if, for example, v_k is defined by $v_k=1/(k+1)^{\alpha}$ with $\alpha>1$.

4. SUFFICIENT CONDITIONS FOR $M\in\left(c_{Z_{1/2}},c_B\right)$ IN CASE OF INFINITE MATRIX *B*

In this Section we obtain sufficient conditions for $M\in\left(c_{Z_{1/2}},c_B\right)$ if B is an infinite matrix.

Theorem 4.1. Let $B=(b_{nk})$ and $M=(m_{nk})$ be infinite matrices satisfying conditions (2.7), (2.8), (3.1), (3.2) and

$$\sum_{k=0}^{r}m_{n,2k}=O(1), \qquad (4.1)$$

$$\sum_l\left|\sum_{k=l}^{l+j}(-1)^{k-l}m_{nk}\right|=O(1). \qquad (4.2)$$

Then

the series $\sum_k(-1)^k g_{nk}$ is convergent for each n. (4.3)

If, in addition to it, conditions (3.7) - (3.10) are fulfilled, then $M\in\left(c_{Z_{1/2}},c_B\right)$.

Proof. As equalities (3.5), (3.11), (3.12) hold and condition (3.13) is equivalent conditions (3.8) and (3.9) (see the proof of Theorem 3.2), then the proof of Theorem 4.1 immediately follows from Lemma 2.4.

From Theorem 4.1 and from the proof of Proposition 3.4 we immediately obtain the following result.

Corollary 4.2. Let $B=(b_{nk})$ be an infinite matrix satisfying condition (2.7) and M is a matrix defined by (3.14), where (v_k) is a positive monotonically decreasing sequence, the series $\sum_k v_k$ is convergent and (t_n) is a bounded sequence. If $(t_n)\in c_B$, then $M\in\left(c_{Z_{1/2}},c_B\right)$.

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