Infinitely Many High Energy Solutions for Kirchhoff-Schrödinger-Poisson Equation with 4-Superlinear Growth Condition

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Abstract: In this article we study the following nonlinear problem of Kirchhoff-Schrödinger-Poisson equation with pure power nonlinearity

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+Vu+\varphi u=|u|^{p-1}u, & x\in\mathbb{R}^3,\\ -\Delta\varphi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

where a,b and V are positive constants, and $3<p<5$. Using the fountain theorem, we obtain infinitely many high energy radial solutions, where some new tricks associated with the scaling technique are introduced to overcome the difficulty caused by the combination of two nonlocal terms.

Keywords: Kirchhoff-Schrödinger-Poisson equation, Fountain Theorem, High energy radial solutions, Variational methods.

1. INTRODUCTION

This article is concerned with the following nonlinear Kirchhoff-Schrödinger-Poisson type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+Vu+\varphi u=|u|^{p-1}u, & x\in\mathbb{R}^3,\\ -\Delta\varphi=u^2, & x\in\mathbb{R}^3,\end{cases} \tag{1.1}$$

where a,b and V are positive constants, and $3<p<5$, which is the special case for more general form

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+V(x)u+\mu K(x)\varphi u=f(x,u), & x\in\mathbb{R}^3,\\ -\Delta\varphi=\mu K(x)u^2, & x\in\mathbb{R}^3.\end{cases} \tag{1.2}$$

As being nonlocal terms, problem (1.2) arises in various models of physical and biological systems, and the research for related issues gives rise to more mathematical difficulties and challenges. Indeed, when $\mu=0$, problem (1.2) reduces to the following Kirchhoff type equation

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^3, \tag{1.3}$$

which corresponds to the stationary counterpart associated with more general equation than the following:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|dx\right)\frac{\partial^2 u}{\partial x^2}=0. \tag{1.4}$$

Equation (1.4) was proposed [8] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings, which took the length of the string produced by transverse vibrations into account. After the pioneer work of Lions [13], where a functional analysis approach was established, Kirchhoff type problems began to attract comprehensive attention of mathematicians. Recently, problem (1.3) has been extensively investigated by many researchers using the variational methods, see for example [3, 6, 7, 12, 19].

When $a=1$ and $b=0$, problem (1.1) reduces to the following nonlinear Schrödinger-Poisson equation

$$\begin{cases} -\Delta u+V(x)u+\mu K(x)\varphi u=f(x,u), & x\in\mathbb{R}^3\\ -\Delta\varphi=\mu K(x)u^2, & x\in\mathbb{R}^3,\end{cases} \tag{1.5}$$

which describes a charged wave interacting with its own electrostatic field [4]. Since then, there are huge literatures on the investigation of the existence and behaviors of the solutions of problem (1.5). Some of interesting results obtained by variational methods can be found in [1, 2, 5, 16, 18] and the references listed therein.

Inspired by the works mentioned above, many authors recently considered problem (1.2), see for example [11, 14, 15, 17, 21] and the references mentioned therein. Different from (1.3) and (1.5), problem (1.2) has two nonlocal terms, which implies that problem (1.2) is no longer a point-wise identity and brings some additional difficulties. Therefore, more delicate techniques are needed to deal with the effect of combination of the two nonlocal terms. Here, we

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must point out that all the existing results are concerned with the existence of positive solutions, except for [21], in which the author showed that problem (1.2) possesses infinitely many small energy solutions using the symmetric mountain pass theorem established by Kajikiya for the case that $V(x)$ satisfies some coercive condition and $f(x,u)$ is of sublinear growth at infinity, see its Theorem 1.1. For more generalized form of problem (1.2), one can find the corresponding results in recent papers [9, 10]. Motivated mainly by [11, 14, 15, 17, 21], the purpose of the present paper is to investigate the existence of infinitely many high energy solutions of problem (1.1). As far as we know, this is the first result concerned with the existence of infinitely many high energy solutions for Kirchhoff-Schrödinger-Poisson type equation.

Now, we present our main result.

Theorem 1.1 Suppose that $3<p<5$, then problem (1.1) possesses infinitely many high energy radial solutions $\{u_k\}_{k\in\mathbb{N}}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u_k|^2+Vu_k^2)dx+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u_k|^2\,dx)^2+\frac{1}{4}\int_{\mathbb{R}^3}\varphi_{u_k}u_k^2dx-\frac{1}{p+1}\int_{\mathbb{R}^3}|u_k|^{p+1}\,dx\to+\infty$$

as $k\to+\infty$.

Remark 1.2 In our Theorem 1.1, we consider problem (1.2) for the special case that $V(x)=const$ and $f(x,u)=|u|^{p-1}u$ with $3<p<5$. It is obvious that the primitive $F(u)=\int_0^u|s|^{p-1}s ds=\frac{|u|^{p+1}}{p+1}$ of $f(x,u)$ is 4-superlinear at infinity. Therefore, it is natural to ask whether problem (1.1) has the same conclusion as in our Theorem 1.1 for $1<p\le 3$ or not.

To complete the proof of Theorem 1.1, we apply the well-known fountain theorem. The main difficulty during the process of its proof is to verify the Cerami condition for the corresponding functional to problem (1.1). Due to the combination of the two nonlocal terms $(\int_{\mathbb{R}^3}|\nabla u|^2\,dx)^2$ and $\int_{\mathbb{R}^3}\varphi_u u^2dx$ in the functional, we could not reach this point as the previous routine for just with one nonlocal term. In order to demonstrate the invalidity of Cerami condition for our consideration, we adopt the scaling technique as done in some previous works. However, it is impossible for us to determine the uniform exponent in the scaling technique form. Fortunately, we can choose changing exponent α and β corresponding to the selected form $\tau^{-\alpha}u(\tau^{-\beta}x)$ for different $3<p<5$, see the details in Section 3. Based on this novelty, we can check the validity of the Cerami condition and the other hypotheses of the fountain theorem, and then finish the proof of Theorem 1.1.

The remainder of this paper is arranged as follows. In Section 2, some preliminary results and lemmas are presented. The proof of Theorem 1.1 is accomplished in Section 3.

Throughout this paper, $C>0$ denotes various positive generic constants.

2. PRELIMINARY RESULTS

In this section, we present some preliminary results, which will be used to prove Theorem 1.1. For any $1\le q<+\infty$, we denote by $\|u\|_q$ the L^q-norm of a function $u\in L^q(\mathbb{R}^3)$. In our context, for the convenience, we have introduced an equivalent norm on $H^1(\mathbb{R}^3)$ defined as

$$\|u\|=(\int_{\mathbb{R}^3}(a|\nabla u|^2+Vu^2)dx)^{\frac{1}{2}},$$

and $D^{1,2}(\mathbb{R}^3)=\{u\in L^6(\mathbb{R}^3):|\nabla u|\in L^2(\mathbb{R}^3)\}$.

We recall that by the Lax-Milgram Theorem, for every $u\in H^1(\mathbb{R}^3)$, there exists a unique $\varphi_u\in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\varphi_u=u^2$ in a weak sense. Moreover, φ_u satisfies some certain properties: $\varphi_u\ge 0$ and $\|\varphi_u\|^2_{D^{1,2}}=\int_{\mathbb{R}^3}\varphi_u u^2\le C\|u\|^4$, see [15, Lemma 2.1]. Substituting φ_u into problem (1.1), we can rewrite (1.1) into the following equivalent equation

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+Vu+\varphi_u u=|u|^{p-1}u. \tag{2.1}$$

To rule out the lack of compactness of the effect of the translations, we consider $H_r^1(\mathbb{R}^3)$ and $D_r^{1,2}(\mathbb{R}^3)$ the corresponding subspace of radial functions, respectively. Then, the embedding $H_r^1(\mathbb{R}^3)\to L^q(\mathbb{R}^3)$ for $q\in(2,6)$ is compact, see [20, Corollary 1.26]. In addition, since the embedding $H^1(\mathbb{R}^3)\to L^q(\mathbb{R}^3)$ $(2\le q\le 6)$ is continuous, see [20, Theorem 1.8], then the embedding $H_r^1(\mathbb{R}^3)\to L^q(\mathbb{R}^3)$ $(2\le q\le 6)$ is also continuous.

Hence, we can define the corresponding energy functional $J:H_r^1(\mathbb{R}^3)\to\mathbb{R}$ to problem (2.1) as follows:

$$J(u)=\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+Vu^2)dx+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2\,dx)^2\\&+\frac{1}{4}\int_{\mathbb{R}^3}\varphi_u u^2dx-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx.\end{aligned} \tag{2.2}$$

It is standard to show that J belongs to $C^1(H_r^1(\mathbb{R}^3),\mathbb{R})$ and its Gâteaux derivative is given by

$$\langle J'(u),v\rangle = \int_{\mathbb{R}^3}(a\nabla u\nabla v+Vuv)dx + b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\int_{\mathbb{R}^3}\nabla u\nabla v dx + \int_{\mathbb{R}^3}\varphi_u uvdx - \int_{\mathbb{R}^3}|u|^{p-1}\,uvdx,\ \forall v\in H_r^1(\mathbb{R}^3). \tag{2.3}$$

It is well known that the solutions for problem (2.1) correspond to the critical points of the functional J.

In order to prove Theorem 1.1, we shall use the following fountain theorem. Let $\{e_j\}_{j=1}^n$ be a total orthonormal basis of X and define

$$X_j=\mathbb{R}e_j,\ Y_k=\bigoplus_{j=1}^{k}X_j,\ Z_k=\overline{\bigoplus_{j=k+1}^{\infty}X_j},\ k\in\mathbb{Z}. \tag{2.4}$$

Then, $X=\overline{\bigoplus_{j=1}^{\infty}X_j}$ and Y_k is finite dimensional.

Definition 2.1 Let X be a Banach space, we say that $J\in C^1(X,\mathbb{R})$ satisfies Cerami condition at the level $c\in\mathbb{R}\,((C)_c\ in\ short)$ if any sequence $\{u_n\}\subset X$ satisfying $J(u_n)\to c$ and $(1+\|u_n\|)J'(u_n)\to 0$ as $n\to+\infty$ has a convergent subsequence. J satisfies Cerami condition if J satisfies $(C)_c$ condition at any $c\in\mathbb{R}$.

Lemma 2.2 ([20, Theorem 3.6]) Let X be an infinite dimensional Banach space. Assume that $J\in C^1(X,\mathbb{R})$ satisfies $(C)_c$ condition, $J(-u)=J(u)$ for all $u\in X$. For every $k\in\mathbb{N}$, there exist $\rho_k>r_k>0$ such that

(i) $b_k:=\inf_{u\in Z_k,\|u\|=r_k}J(u)\to+\infty\ \ as\ k\to+\infty$;

(ii) $a_k:=\max_{u\in Y_k,\|u\|=\rho_k}J(u)\le 0$.

Then, J has a sequence of critical points $\{u_k\}$ such that $J(u_k)\to+\infty$.

Before going to implement the process of the proof of Theorem 1.1, as pointed out in Section 1, we need to discuss some properties for the scaling technique, which plays an essential role in testing the Cerami condition. For each given $u\in H_r^1(\mathbb{R}^3)\setminus\{0\}$, let us consider the paths defined by $\eta(\tau)=\tau^{-\alpha}u(\tau^{-\beta}\cdot),\tau>0$, where $\alpha<0$ and $\beta>0$.

Proposition 2.3 For any fixed $u\in H_r^1(\mathbb{R}^3)\setminus\{0\}$, there exists a unique $\tau_u>0$ (dependent on α and β) such that $\|\eta(\tau_u)\|^2=\|\tau_u^{-\alpha}u(\tau_u^{-\beta}\cdot)\|^2=1$. Moreover, $\tau_u\to 0$ as $\|u\|\to+\infty$.

Proof. After a direct calculation, we can obtain

$$\|\eta(\tau)\|^2=\tau^{-2\alpha+\beta}\int_{\mathbb{R}^3}a|\nabla u|^2\,dx+\tau^{-2\alpha+3\beta}\int_{\mathbb{R}^3}Vu^2dx,$$

which yields that $\lim_{\tau\to 0^+}\|\eta(\tau)\|^2=0$ and $\lim_{\tau\to+\infty}\|\eta(\tau)\|^2=+\infty$. In addition, it is obvious that $\|\eta(\tau)\|^2$ is monotone increasing. Therefore, the conclusion is evidently reached.

Remark 2.4 Based on Proposition 2.3, it allows us to define the one-to-one mapping $F_{\tau_u}:H_r^1(\mathbb{R}^3)\to S$ by $F_{\tau_u}(u)(x)=\tau_u^{-\alpha}u(\tau_u^{-\beta}x)$, where $S=\{u\in H_r^1(\mathbb{R}^3):\|u\|=1\}$ is the unite sphere. The inverse of F_{τ_u} is given by $F_{\tau_u}^{-1}(u)(x)=\tau_u^{\alpha}u(\tau_u^{\beta}x)$.

3. PROOF OF THEOREM 1.1

The purpose of this section is to finish the proof of Theorem 1.1 with the aid of the fountain theorem. For this aim, we need to demonstrate that all conditions of Proposition 2.2 are satisfied.

In what follows, with the help of Proposition 2.3 and Remark 2.4, we check the corresponding functional J satisfies the Cerami condition.

Lemma 3.1 The functional J satisfies $(C)_c$ condition for any $c\in\mathbb{R}$.

Proof. Let $u_n\subset H_r^1(\mathbb{R}^3)$ be such that

$$J(u_n)\to c\ \ and\ \ (1+\|u_n\|)J'(u_n)\to 0\quad as\quad n\to+\infty. \tag{3.1}$$

Firstly, we prove that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. To this end, we argue by contradiction. Suppose that $\|u_n\|\to+\infty$ as $n\to+\infty$. Then, by the preceding arguments in Section 2, there exists $\tau_n:=\tau_{u_n}$ such that $\|\tau_n^{-\alpha}u_n(\tau_n^{-\beta}x)\|^2=1$ and $\tau_n\to 0$ as $n\to+\infty$. Consider the sequence $\{v_n\}\subset H_r^1(\mathbb{R}^3)$ defined by $v_n(x):=\tau_n^{-\alpha}u_n(\tau_n^{-\beta}x)$, then $\|v_n\|^2=1$ and $u_n(x)=\tau_n^{\alpha}v_n(\tau_n^{\beta}x)$. Moreover, $meas\{x\in\mathbb{R}^3:v_n(x)\ne 0\}>0$ for n large enough. It follows from (2.2), (2.3) and (3.1) that there exists some constant $C>0$ such that

$$\begin{aligned} C &\geq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n \rangle = -\frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx)^2 - \\ &\frac{1}{4}\int_{\mathbb{R}^3} \varphi_{u_n} u_n^2 dx + \frac{p-1}{2(p+1)}\int_{\mathbb{R}^3} |u_n|^{p+1}\, dx \\ &= -\frac{b}{4}\tau_n^{4\alpha-2\beta}(\int_{\mathbb{R}^3} |\nabla v_n|^2\, dx)^2 - \frac{1}{4}\tau_n^{4\alpha-5\beta} \\ &\int_{\mathbb{R}^3} \varphi_{v_n} v_n^2 dx + \tau_n^{\alpha(p+1)-3\beta}\frac{p-1}{2(p+1)}\int_{\mathbb{R}^3} |v_n|^{p+1}\, dx. \end{aligned} \tag{3.2}$$

For any fixed $p \in (3,5)$, we can choose reasonable $\alpha < 0$ and $\beta > 0$ such that

$$2\beta < \alpha(3-p), \tag{3.3}$$

which leads to one contradiction as $n \to +\infty$ in the above inequality (3.2). Thus, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. So, up to a subsequence, we may assume that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ and $u_n \to u$ in $L^q(\mathbb{R}^3)$, $2 < q < 6$. Meanwhile, from (2.3) and (3.1), one deduces that

$$\begin{aligned} o_n(1) &= \langle J'(u_n) - J'(u), u_n - u \rangle = (a + b\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx)\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 \\ &dx + \int_{\mathbb{R}^3} V|u_n - u|^2\, dx + (\int_{\mathbb{R}^3} \varphi_{u_n} u_n(u_n - u)dx - \int_{\mathbb{R}^3} \varphi_u u(u_n - u)dx) \\ &-b\int_{\mathbb{R}^3}(|\nabla u|^2 - |\nabla u_n|^2)dx \int_{\mathbb{R}^3} \nabla u \nabla(u_n - u)dx \\ &-(\int_{\mathbb{R}^3} |u_n|^{p-1} u_n(u_n - u)dx - \int_{\mathbb{R}^3} |u|^{p-1} u(u_n - u)dx). \end{aligned} \tag{3.4}$$

In view of the Hölder's inequality, we can easily obtain that

$$\begin{aligned} &\int_{\mathbb{R}^3} |u_n|^{p-1} u_n(u_n - u)dx \to 0 \\ &and \quad \int_{\mathbb{R}^3} |u|^{p-1} u(u_n - u)dx \to 0, \end{aligned} \tag{3.5}$$

since $u_n \to u$ in $L^q(\mathbb{R}^3)$ for $2 < q < 6$. In addition, on account of the continuity of the embedding $H_r^1(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ and the properties of φ_{u_n}, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \varphi_{u_n} u_n(u_n - u)dx \leq \|\varphi_{u_n}\|_6 \|u_n\|_2 \|u_n - u\|_3 \\ &\leq C\|\varphi_{u_n}\|_{D^{1,2}} \|u_n\|_2 \|u_n - u\|_3 \leq C\|u_n\|^3 \|u_n - u\|_3 \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.6}$$

Similarly, it holds that

$$\int_{\mathbb{R}^3} \varphi_u u(u_n - u)dx \to 0 \quad as \quad n \to \infty. \tag{3.7}$$

On the other hand, due to the fact that the embedding $H_r^1(\mathbb{R}^3) \to D_r^{1,2}(\mathbb{R}^3)$ is continuous, we also arrive to the conclusion that

$$b\int_{\mathbb{R}^3}(|\nabla u_n|^2 - |\nabla u|^2)dx \int_{\mathbb{R}^3} \nabla u \nabla(u_n - u)dx = o_n(1). \tag{3.8}$$

Thus, (3.4)-(3.8) lead to

$$\begin{aligned} o_n(1) &= \langle J'(u_n) - J'(u), u_n - u \rangle \geq \|u_n - u\|^2 + \\ &(\int_{\mathbb{R}^3} \varphi_{u_n} u_n(u_n - u)dx - \int_{\mathbb{R}^3} \varphi_u u(u_n - u)dx) \\ &-b\int_{\mathbb{R}^3}(|\nabla u|^2 - |\nabla u_n|^2)dx \int_{\mathbb{R}^3} \nabla u \nabla(u_n - u)dx \\ &-(\int_{\mathbb{R}^3} |u_n|^{p-1} u_n(u_n - u)dx - \int_{\mathbb{R}^3} |u|^{p-1} u(u_n - u)dx), \end{aligned}$$

which yields $\|u_n - u\| \to 0$ as $n \to +\infty$.

Remark 3.2 Due to the fact that $3 < p < 5$, we could not choose uniform α and β to guarantee that (3.3) holds true. However, for any fixed $p \in (3,5)$, one can always determine suitable $\alpha < 0$ and $\beta > 0$ to ensure that (3.3) makes sense.

Now, we are in the position to establish the proof of Theorem 1.1. In view of Lemma 3.1, we divide the process into three steps. Let $\{e_j\}$ be an orthonornal basis of $H_r^1(\mathbb{R}^3)$ and define X_j, Y_k and Z_k the same as in (2.4).

Proof. Claim 1: Assume that $3 < p < 5$, then there exists $r_k > 0$ such that $\inf_{u \in Z_k, \|u\| = r_k} J(u) \to +\infty$ as $k \to +\infty$.

By [20, Lemma 3.8], for any $3 < p < 5$, we have $\beta_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_{p+1} \to 0$ as $k \to +\infty$, where β_k is also dependent on p. For each $k \geq 1$, choosing $r_k := (\frac{p+1}{4\beta_k^{p+1}})^{\frac{1}{p-1}}$. Then, for $3 < p < 5$, we have $r_k \to +\infty$ as $k \to +\infty$. Subsequently, for $u \in Z_k$ with $\|u\| = r_k$, in view of (2.2), we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u|^2\, dx)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3} \varphi_u u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\|u\|^2 \\ &- \frac{\beta_k^{p+1}}{p+1}\|u\|^{p+1} \geq \frac{1}{4}\|u\|^2 = \frac{r_k^2}{4}, \end{aligned}$$

which implies that Claim 1 holds true.

Claim 2: For the subspace $Y_k \subset E$, there exists $\rho_k > r_k$ such that $\max_{u \in Y_k, \|u\| = \rho_k} J(u) \leq 0$.

By Proposition 2.3 and Remark 2.4, we note that for any $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $\tau_u > 0$ such that $\|\tau_u^{-\alpha}u(\tau_u^{-\beta})x\|=1$ and $\tau_u\to 0$ as $\|u\|\to+\infty$. Let $v(x):=\tau_u^{-\alpha}u(\tau_u^{-\beta}x)$, then $\|v\|=1$ and $u(x)=\tau_u^{\alpha}v(\tau_u^{\beta}x)$. Therefore, one has

$$\begin{aligned} J(u)&=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)^2+\frac{1}{4}\int_{\mathbb{R}^3}\varphi_u u^2dx\\ &-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx\\ &\le\frac{1}{2}\tau_u^{2\alpha-\beta}\|v\|^2+\frac{1}{2}\tau_u^{2\alpha-3\beta}\|v\|^2+\frac{b}{4a^2}\tau_u^{4\alpha-2\beta}\|v\|^4\\ &+\frac{C}{4}\tau_u^{4\alpha-5\beta}\|v\|^4-\frac{\tau_u^{\alpha(p+1)-3\beta}}{p+1}\|v\|_{p+1}^{p+1}. \end{aligned} \tag{3.9}$$

Since, on the finite dimensional subspace Y_k, all the norms are equivalent, there exists $C_k>0$ such that

$$\|u\|_{p+1}\ge C_k\|u\|,\ \forall u\in Y_k. \tag{3.10}$$

Hence, combining (3.9) with (3.10), we can conclude that

$$\begin{aligned} J(u)\le\ &\frac{1}{2}\tau_u^{2\alpha-\beta}\|v\|^2+\frac{1}{2}\tau_u^{2\alpha-3\beta}\|v\|^2+\frac{b}{4a^2}\tau_u^{4\alpha-2\beta}\|v\|^4+\frac{C}{4}\tau_u^{4\alpha-5\beta}\|v\|^4\\ &-\frac{\tau_u^{\alpha(p+1)-3\beta}}{p+1}C_k^{p+1}\|v\|^{p+1}:=\xi(\tau_u). \end{aligned}$$

Choosing the same α and β as in (3.3) for any fixed $p\in(3,5)$, then $\lim_{\|u\|\to+\infty}J(u)\le\lim_{\tau_u\to 0}\xi(\tau_u)=-\infty$. Therefore, we infer that there exists $R_k=R(Y_k)>0$ such that $J(u)<0$ for all $u\in Y_k$ with $\|u\|\ge R_k$. Hence, choose $\rho_k>\max\{R_k,r_k\}$, and so the claim is proved.

Claim 3: Obviously, $J(0)=0, J\in C^1(H_r^1(\mathbb{R}^3),\mathbb{R})$ and J is even. Moreover, we notice from Lemma 3.1, Claim 1 and Claim 2 that all the conditions of Lemma 2.2 are verified. Thus, problem (1.1) possesses a sequence of radial nontrivial solutions $\{u_k\}_{k\in\mathbb{N}}\subset H_r^1(\mathbb{R}^3)$ such that $J(u_k)\to+\infty$.

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