# Existence Theory and Stability Analysis of Nonlinear Neutral Pantograph Equations via Hilfer-Katugampola Fractional Derivative

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**Abstract:** The aim and objectives of this paper are devoted to study some adequate results for the existence and stability of solutions of nonlinear neutral pantograph equations with Hilfer-Katugampola fractional derivative. The arguments are based upon Schauder fixed point theorem and Banach contraction principle. Further, we also study the Ulam type stability for proposed problem.

**Keywords:** Hilfer-Katugampola fractional derivative, Neutral pantograph equation, Existence, Stability, Fixed point.

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## 1. INTRODUCTION

Differential equations with fractional order are the effective tool for the modelling of many phenomenons in distinct areas of science and engineering. Hence, in the last few years, a great interest has been given to the topic of differential equations of arbitrary order. The field relevant to the existence theory of solutions of fractional differential equations (FDEs) and their problems were well analysed by many researchers, for complete study see [1-4]. In all these papers, the concerned results were derived by using some standard fixed point theorems.

Some researchers have focused their attentions to generalize the fractional derivative like Riemann-Liouville fractional derivative, Caputo fractional derivative etc. Very recently, Oliveira and his co-researcher introduced the Hilfer-Katugampola fractional derivative, see [5], which unifies some fractional derivatives (which is discussed in Section 2). The Hilfer derivative was introduced by R. Hilfer [6] that is used to interpolate the Riemann-Liouville and the Caputo fractional derivative. In the following days, many researchers had done valuable works on generalizing some fractional derivatives. In recent times, there has been a significant interest and development in Hilfer fractional derivative, which can be found in [7-12].

The stability properties of all types of equations have gained the attention of many researchers. In particular, the Ulam-Hyers stability and its four types have been considered by many authors, and analysis of this topic has grown to be one of the important studies in mathematical analysis. For detailed study on Ulam-Hyers stability, interested readers can refer to [13-17].

The pantograph equation is a special kind of delay differential equations. For more details on the recent developments of fractional pantograph equation, one can refer to [18-20]. In [18], Balachandran generalized the neutral type fractional pantograph equation. Recently, Vivek *et al*. [21]. discussed theory and analysis of nonlinear neutral pantograph equations via Hilfer fractional derivative.

Motivated by the above discussion, in this paper, we investigate the nonlinear neutral pantograph equations with Hilfer-Katugampola fractional derivative of the form

$$^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}\,(t) = \mathfrak{g}(t,\, \mathfrak{u}(t),\, \mathfrak{u}(\kappa t),\, {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(\kappa t)),\; t \in J := [a,b], \quad (1.1)$$

$$^{\rho}\mathfrak{I}^{1-\gamma}\mathfrak{u}(a) = \mathfrak{u}_a, \quad (1.2)$$

where $^{\rho}\mathfrak{D}^{\alpha,\beta}$ is the Hilfer-Katugampola fractional derivative of order $\alpha(0 < \alpha < 1)$ and type $\beta(0 \le \beta \le 1)$ and $^{\rho}\mathfrak{I}^{1-\gamma}$ is fractional integral order $1-\gamma$ ($\gamma = \alpha + \beta - \alpha\beta$). Let $R$ be a Banach space, $\mathfrak{g} : J \times R \times R \times R \to R$ is a given continuous function and $0 < \kappa < 1$.

The outline of the paper is as follows. In Section 2, we give some basic definitions and results concerning the Hilfer-Katugampola fractional derivative. In Section 3, we present our existence and uniqueness of the results. In section 4, we discuss four types of stability.

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## 2. PRELIMINARIES

In the current section, we recall some definitions and results from fractional calculus. The following observations are taken from [3,5,8,14,15,17]. Let $C[a, b]$ a space of continuous functions from $J$ into R with the norm

$$\|\mathfrak{u}\| = \sup\{|\mathfrak{u}(t)| : t \in J\}.$$

The weighted space $C_{\gamma,\rho}[a, b]$ of functions $f$ on $[a, b]$ is defined by

$$C_{\gamma,\rho}[a,b] = \left\{ f:(a,b] \to R : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \in C[a,b] \right\}, 0 \le \gamma < 1,$$

with the norm

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \right\|_C = \max_{t\in J} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \right|, C_{0,\rho}[a,b] = C[a,b].$$

Let $\delta_\rho = \left(t^\rho \frac{d}{dt}\right)$. For $n \in N$, we denote by $C^n_{\delta_{\rho,\gamma}}[a,b]$ the Banach space of functions $f$, which is continuously differentiable, with the operator $\delta_\rho$, on $[a, b]$ up to $(n - 1)$ order and the derivative $\delta^n_\rho f$ of order $n$ on $[a, b]$ such that $\delta^n_\rho f \in C_{\gamma,\rho}[a, b]$, this is

$$C^n_{\delta_{\rho,\gamma}}[a, b] = \left\{\delta^k_\rho f \in C[a, b], k = 0, 1, ..., n-1, \delta^n_\rho f \in C_{\gamma,\rho}[a, b]\right\}$$

with the norm

$$\|f\|_{C^n_{\delta_{\rho,\gamma}}} = \sum_{k=0}^{n-1} \left\|\delta^k_\rho f\right\|_C + \left\|\delta^n_\rho f\right\|_{C_{\gamma,\rho}}, \quad \|f\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{x\in R} \left|\delta^k_\rho f(x)\right|.$$

For $n = 0$, we have

$$C^0_{\delta_{\rho,\gamma}}[a, b] = C_{\gamma,\rho}[a, b]$$

**Definition 2.1.** The generalized left-sided fractional integral ${}^\rho I^\alpha f$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$\left({}^\rho\mathfrak{I}^\alpha\right) f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s)ds, t > a, \quad (2.1)$$

if the integral exists.

The generalized fractional derivative, corresponding to the generalised fractional integral (2.1), is defined for $0 \le a < t$, by

$$\left({}^\rho\mathfrak{D}^\alpha f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1} s^{\rho-1} f(s)ds, \quad (2.2)$$

if the integral exists.

**Definition 2.2.** The Hilfer-Katugampola fractional derivative with respect to $t$, with $\rho > 0$, is defined by

$$\left({}^\rho\mathfrak{D}^{\alpha,\beta} f\right)(t) = \left(\pm^\rho \mathfrak{I}^\alpha \left(t^{\rho-1}\frac{d}{dt}\right)^\rho \mathfrak{I}^{(1-\beta)(1-\alpha)}\right)(t) \quad (2.3)$$
$$= \left(\pm^\rho \mathfrak{I}^\alpha \delta_\rho {}^\rho\mathfrak{I}^{(1-\beta)(1-\alpha)}\right)(t).$$

• The operator ${}^\rho\mathfrak{D}^{\alpha,\beta}$ can be written as

$${}^\rho\mathfrak{D}^{\alpha,\beta} = {}^\rho\mathfrak{I}^{\beta(1-\alpha)}\delta_\rho {}^\rho\mathfrak{I}^{1-\gamma} = {}^\rho\mathfrak{I}^{\beta(1-\alpha)} {}^\rho\mathfrak{D}^\gamma, \gamma = \alpha + \beta - \alpha\beta.$$

• The fractional derivative ${}^\rho\mathfrak{D}^{\alpha,\beta}$ is considered as interpolator, with the convenient parameters, of the following fractional derivatives

1. Hilfer fractional derivative when $\rho \to 1$.
2. Hilfer-Hadamard fractional derivative when $\rho \to 0$.
3. Generalized fractional derivative when $\beta = 0$.
4. Caputo-type fractional derivative when $\beta = 1$.
5. Riemann-Liouville fractional derivative when $\beta = 0$, $\rho \to 1$.
6. Hadamard fractional derivative when $\beta = 0, \rho \to 0$.
7. Caputo fractional derivative when $\beta = 1, \rho \to 1$.
8. Caputo-Hadamard fractional derivative when $\beta = 1$, $\rho \to 0$.
9. Liouville fractional derivative when $\beta = 0$, $\rho \to 1$, $a = 0$.
10. Hadamard fractional derivative when $\beta = 0, \rho \to 1$, $a = -\infty$.

• We consider the following parameters $\alpha$, $\beta$, $\gamma$, $\mu$ satisfying

$$\gamma = \alpha + \beta - \alpha\beta, 0 \le \gamma < 1, 0 \le \mu < 1, \alpha > 0, \beta < 1.$$

**Definition 2.3.** The equation (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{v} \in C_{1-\gamma,\rho}[a, b]$ of the inequality

$$\left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))\right| \le \varepsilon, \quad t \in J,$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\rho}[a, b]$ of equation (1.1) with

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le C_f \varepsilon, \quad t \in J,$$

**Definition 2.4.** The equation (1.1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0,\infty),[0,\infty))$ $_f(0) = 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\rho}[a,b]$ of the inequality

$$|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))| \le \varepsilon, \qquad t \in J,$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\rho}[a,b]$ of equation (1.1) with

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le \psi_f\, \varepsilon, \quad t \in J,$$

**Definition 2.5.** The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma}[a,b]$ if there exists a real number $C_f > 0$ such that for each $> 0$ and for each solution $\mathfrak{v} \in C_{1-\gamma,\rho}[a,b]$ of the inequality

$$|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))| \le \varepsilon\, \varphi(t), \; t \in J, \tag{2.4}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\rho}[a,b]$ of equation (1.1) with

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le C_f \varepsilon\, \varphi(t), \; t \in J,$$

**Definition 2.6.** The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma}[a,b]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\rho}[a,b]$ of the inequality

$$|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))| \le \varphi(t), \; t \in J, \tag{2.5}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\rho}[a,b]$ of equation (1.1) with

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le C_{f,\varphi}\, \varphi(t), \; t \in J,$$

**Remark 2.7.** It is clear that:

1. Definition 2.3 ⇒ Definition 2.4.

2. Definition 2.5 ⇒ Definition 2.6.

**Lemma 2.8.** *Let $\alpha, \beta > 0$, the semigroup property is valid. This is*,

$$({}^{\rho}\mathfrak{J}^{\alpha\rho}\mathfrak{J}^{\beta}f)(t) = ({}^{\rho}\mathfrak{J}^{\alpha+\beta})(t),$$

*and*

$$({}^{\rho}\mathfrak{D}^{\alpha\rho}\mathfrak{J}^{\alpha}f)(t) = f(t).$$

**Lemma 2.9.** *Let $t > a$, ${}^{\rho}\mathfrak{J}^{\alpha}$ and ${}^{\rho}\mathfrak{D}^{\alpha}$, according to Eqs.* (2.1) *and* (2.2), *respectively. Then*

$${}^{\rho}\mathfrak{J}^{\alpha}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta-1}.$$

$${}^{\rho}\mathfrak{D}^{\alpha}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1} = 0.$$

**Lemma 2.10.** *If $f \in C_{\gamma,\rho}[a,b]$ and ${}^{\rho}\mathfrak{J}^{1-\alpha}f \in C^{1}_{\gamma,\rho}[a,b]$, then*

$$\left({}^{\rho}\mathfrak{J}^{\alpha\,\rho}\mathfrak{D}^{\alpha}\right)(t) = f(t) - \frac{\left({}^{\rho}\mathfrak{J}^{1-\alpha}_{a+}f\right)(a)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1},$$

for all $t \in [a,b]$.

**Lemma 2.11.** *If $f \in C_{\gamma,\rho}[a,b]$, then*

$$\left({}^{\rho}\mathfrak{J}^{\alpha}f\right)(a) = \lim_{t\to a^{+}}\left({}^{\rho}\mathfrak{J}^{\alpha}\right)f(t) = 0.$$

The following Lemmas are needed in the sequel.

**Lemma 2.12.** *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $a \le t < b$ (some $b \le \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \le t < b$, such that $g(t) \le K$ for some constant $K$. Further let $u(t)$ be a nonnegative locally integrable on $a \le t < b$ function with*

$$|\mathrm{u}(t)| \le a(t) + g(t)\int_a^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathrm{u}(s)ds, \; t \in J$$

*with some $\alpha > 0$. Then*

$$|\mathrm{u}(t)| \le a(t) + \int_a^t \left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{n\alpha-1} s^{\rho-1}\right]\mathrm{u}(s)ds, \; a \le t < b.$$

**Lemma 2.13.** *[5] A function $\mathrm{u}$ is the solution of fractional initial value problem*

$$\begin{cases} {}^{\rho}\mathfrak{D}^{\alpha,\beta}_{a_+}\mathrm{u}(t) = f(t), t \in J, \\ {}^{\rho}\mathfrak{J}^{1-\gamma}_{a_+}\mathrm{u}(a) = \mathrm{u}_a, \end{cases}$$

*if and only if $\mathfrak{u}$ satisfies the integral equation of the form*

$$\mathrm{u}(t) = \frac{\mathrm{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}f(s)ds.$$

**Theorem 2.14.** *(Schauder fixed point theorem [5]) Let $B$ be closed, convex and nonempty subset of a Banach space $E$. Let $N : B \to B$ be a continuous* *mapping such that $N(B)$ is a relatively compact subset of $E$. Then N has atleast one fixed point in $B$.*

## 3. EXISTENCE THEORY

Now, we give our main existence result for problem (1.1)-(1.2). Before starting and proving this result, we assume the following hypotheses:

(H1) There exist constants $K > 0$ and $L > 0$ such that

$|\mathfrak{g}(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) - \mathfrak{g}(t, \mathfrak{n}, \mathfrak{b}, \overline{\mathfrak{w}})| \leq K(|\mathfrak{u} - \mathfrak{n}| + |\mathfrak{v} - \mathfrak{b}|) + L|\mathfrak{w} - \overline{\mathfrak{w}}|$,

for any $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{n}, \mathfrak{b}, \overline{\mathfrak{w}} \in R$, and $t \in J$.

(H2) There exist $l, m, n, p \in C_{1-\gamma}[a, b]$ with $l^* = \sup_{t\in J} l(t)$ <1 such that

$$|\mathfrak{g}(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w})| \leq l(t) + m(t)|\mathfrak{u}| + n(t)|\mathfrak{v}| + p(t)|\mathfrak{w}|,$$

for $t \in J$ and $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in R$.

(H3) There exists an increasing function $\varphi \in C_{1-\gamma,\rho}[a, b]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$^{\rho}\mathfrak{J}^{\alpha}\varphi(t) \leq \lambda_\varphi\varphi(t).$$

**Theorem 3.1** *Assume that (H1)-(H2) hold. Then the problem* (1.1)-(1.2) *has at least one solution defined on* $J$.

*Proof. Consider the operator $\mathfrak{R} : C_{1-\gamma,\rho}[a, b] \to C_{1-\gamma,\rho}[a, b]$ defined by*

$$(\Re \mathsf{u})(t) = \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s, \mathsf{u}(s), \mathsf{u}(\kappa s), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathsf{u}(\kappa s))ds. \tag{3.1}$$

It can be written as

$$(\Re \mathsf{u})(t) = \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \left[{}^{\rho}\mathfrak{J}^{\alpha}g_{\mathsf{u}}(s)\right](t), \tag{3.2}$$

Where $g_u(t) := {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(t) = \mathfrak{g}(t, \mathfrak{u}(t), \mathfrak{u}(\kappa t), \mathfrak{g}_u(t))$.

Clearly, the fixed points of the operator $\mathfrak{R}$ are solutions of the problem (1.1)-(1.2).

For any $\mathfrak{u} \in C_{1-\gamma,\rho}[a, b]$ and each $t \in J$, we have

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Re \mathsf{u})(t)\right| \leq \frac{|\mathsf{u}_a|}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g_{\mathsf{u}}(s)|ds. \tag{3.3}$$

By (H2), for each $t \in J$, we have

$$\begin{aligned}|g_{\mathsf{u}}(t)| &= |\mathfrak{g}(t, \mathsf{u}(t), \mathsf{u}(\kappa t), g_{\mathsf{u}}(t))| \\ &\leq l(t) + m(t)|\mathsf{u}(t)| + n(t)|\mathsf{u}(\kappa t)| + p(t)|g_{\mathsf{u}}(t)| \\ &\leq l^* + m^*|\mathsf{u}(t)| + n^*|\mathsf{u}(\kappa t)| + p^*|g_{\mathsf{u}}(t)| \\ &\leq \frac{l^* + m^*|\mathsf{u}(t)| + n^*|\mathsf{u}(\kappa t)|}{1 - p^*}.\end{aligned} \tag{3.4}$$

By replacing (3.4) in the inequality (3.3), we get

$$\begin{aligned}&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Re \mathsf{u})(t)\right| \\ &\leq \frac{|\mathsf{u}_a|}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left[\frac{l^* + m^*|\mathsf{u}(s)| + n^*|\mathsf{u}(\kappa s)|}{1 - p^*}\right]ds \\ &\leq \frac{|\mathsf{u}_a|}{\Gamma(\gamma)} + \frac{1}{(1 - p^*)\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(l^*\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}ds\right. \\ &\left.+ m^*\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\mathsf{u}(s)|ds + n^*\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\mathsf{u}(\kappa s)|ds\right) \\ &\leq \frac{|\mathsf{u}_a|}{\Gamma(\gamma)} + \frac{l^*}{(1 - p^*)\Gamma(\alpha + 1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma+1} + \frac{B(\gamma, \alpha)}{(1 - p^*)\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}(m^* + n^*)\|\mathsf{u}\|_{C_{1-\gamma,\rho}}.\end{aligned}$$

Hence,

$$\|\Re \mathsf{u}\|_{C_{1-\gamma,\rho}} \leq \frac{|\mathsf{u}_a|}{\Gamma(\gamma)} + \frac{l^*}{(1 - p^*)\Gamma(\alpha + 1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma+1} + \frac{B(\gamma, \alpha)}{(1 - p^*)\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}(m^* + n^*)\|\mathsf{u}\|_{C_{1-\gamma,\rho}} := r'.$$

This proves that $\mathfrak{R}$ transforms the ball $\mathfrak{B}_{r'} := \{\mathfrak{u} \in C_{1-\gamma,\rho}[a, b] : \|\mathfrak{u}\| \leq r'\}$ into itself. We shall show that the operator $\mathfrak{R} : \mathfrak{B}_{r'} \to \mathfrak{B}_{r'}$ satisfies all the conditions of Schauder fixed point theorem. The proof will be given in several steps.

**Step 1:** $\mathfrak{R} : \mathfrak{B}_{r'} \to \mathfrak{B}_{r'}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to u$ in $C_{1-\gamma,\rho}[a, b]$. Then for each $t \in J$,

$$\begin{aligned}&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Re \mathsf{u}_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Re \mathsf{u})(t)\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g_{\mathsf{u}_n}(\cdot) - g_{\mathsf{u}}(\cdot)|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}B(\gamma, \alpha)\|g_{\mathsf{u}_n}(\cdot) - g_{\mathsf{u}}(\cdot)\|_{C_{1-\gamma,\rho}}.\end{aligned}$$

Since $\mathfrak{u}_n \to \mathfrak{u}$ as $n \to \infty$ and $g_u$ is continuous (i.e., $f$ is continuous), then by the Lebesgue dominated convergence theorem, we have

$$\|\mathfrak{R}\mathfrak{u}_n - \mathfrak{R}\mathfrak{u}\|_{C_{1-\gamma,\rho}} \to 0 \quad \text{as} \quad n \to \infty.$$

**Step 2:** $\mathfrak{R}(\mathfrak{B}_{r'})$ is uniformly bounded.

This is clear since $\mathfrak{R}(\mathfrak{B}_{r'}) \subset \mathfrak{B}_{r'}$ and $\mathfrak{B}_{r'}$ is bounded.

**Step 3:** $\mathfrak{R}(\mathfrak{B}_{r'})$ is equicontinuous.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and let $\mathfrak{u} \in \mathfrak{B}_{r'}$. Thus, we have

$$\left|\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\mathfrak{R}\mathrm{u})(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\mathfrak{R}\mathrm{u})(t_1)\right|$$

$$\leq \left|\frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathrm{u}}(s)ds\right.$$

$$\left. - \frac{1}{\Gamma(\alpha)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^{t_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathrm{u}}(s)ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[\frac{l^* + m^*|\mathrm{u}(s)| + n^*|\mathrm{u}(\kappa s)|}{1-p^*}\right]ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.$$

$$\left. - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1}\left[\frac{l^* + m^*|\mathrm{u}(s)| + n^*|\mathrm{u}(\kappa s)|}{1-p^*}\right]ds.$$

Thus, we get

$$\left|\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\mathfrak{R}\mathrm{u})(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\mathfrak{R}\mathrm{u})(t_1)\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left[\begin{array}{l}\dfrac{l^*}{\alpha(1-p^*)}\left(\dfrac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha} + \\ \dfrac{(m^* + n^*)B(\gamma,\alpha)}{1-p^*}\left(\dfrac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha+\gamma-1}\|\mathrm{u}\|_{C_{1-\gamma,\psi}}\end{array}\right]$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\begin{array}{l}\left(\dfrac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\dfrac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \\ \left(\dfrac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\dfrac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1}\end{array}\right] s^{\rho-1}$$

$$\left[\frac{l^* + m^*|\mathrm{u}(s)| + n^*|\mathrm{u}(\kappa s)|}{1-p^*}\right]ds.$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Step 1-3 together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{R}$ is continuous and compact. From Schauder's theorem, we conclude that $\mathfrak{R}$ has a fixed point $\mathfrak{u}$, which is a solution of the problem (1.1)-(1.2).

**Lemma 3.2.** *Using the hypotheses (H1),(H2) and if*

$$\left(\frac{2K}{(1-L)\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha} B(\gamma,\alpha)\right) < 1, \tag{3.5}$$

*then the proposed problem* (1.1)-(1.2) *has a unique solution*.

## 4. STABILITY ANALYSIS

In this section, we study the Ulam-Hyers stability for the proposed problem (1.1)-(1.2).

**Remark 4.1.** A function $\mathfrak{v} \in C_{1-\gamma,\rho}[a, b]$ is a solution of the inequality

$$\left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))\right| \leq \varepsilon, \qquad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma,\rho}[a, b]$ (which depends on solution $\mathfrak{u}$) such that

1. $|g(t)| \leq \varepsilon, \quad \forall\, t \in J.$
2. ${}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t)) + g(t), \quad t \in J.$

One can have similar remarks for the inequalities (2.4) and (2.5).

**Remark 4.2** A solution of the Hilfer-Katugampola type nonlinear neutral pantograph inequality

$$\left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))\right| \leq \varepsilon, \qquad t \in J,$$

is called a fractional $\varepsilon$-solution of the problem (1.1).

**Theorem 4.3** Assume that (H1),(H2) and (3.5) hold, then the problem (1.1)-(1.2) is Ulam-Hyers stable.

*Proof.* Let $\varepsilon > 0$ and let $\mathfrak{v} \in C_{1-\gamma,\rho}[a, b]$ be a function which satisfies the inequality:

$$\left|{}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^\rho\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))\right| \leq \varepsilon, \text{ for any } t \in J, \tag{4.1}$$

and let $\mathfrak{u} \in C_{1-\gamma,\rho}[a, b]$ be the unique solution of the following nonlinear neutral pantograph equation

$${}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t)), \quad t \in J.$$
$${}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{u}(a) = {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(a) = \mathfrak{u}_a,$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

Using Lemma 2.13, we obtain

$$\mathsf{u}(t) = \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} + \left[{}^{\rho}\mathfrak{J}^{\alpha} g_{\mathsf{u}}\right](t).$$

By integration of the inequality (4.1) and using Remark 4.1, we obtain

$$\left|\mathfrak{v}(t) - \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathsf{v}}(s)ds\right|$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(\frac{b^{\rho} - a^{\rho}}{\rho}\right)^{\alpha}.$$

We have

$$|\mathfrak{v}(t) - \mathsf{u}(t)|$$
$$\leq \left|\mathfrak{v}(t) - \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathsf{v}}(s)ds\right|$$
$$+ \left|\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(g_{\mathsf{v}}(s) - g_{\mathsf{u}}(s)\right)ds\right|$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|g_{\mathsf{v}}(s) - g_{\mathsf{u}}(s)\right|ds$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha+1)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha} + \frac{2K}{(1-L)\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}|\mathfrak{v}(s) - \mathsf{u}(s)|ds,$$

and to apply Lemma 2.12, we obtain

$$|\mathfrak{v}(t) - \mathsf{u}(t)| \leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{b^{\rho} - a^{\rho}}{\rho}\right)^{\alpha}\left(1 + \frac{\nu 2K}{(1-L)\Gamma(\alpha+1)}\left(\frac{b^{\rho} - a^{\rho}}{\rho}\right)^{\alpha}\right)\varepsilon$$
$$:= C_f\varepsilon.$$

Where $\nu = \nu\ (\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\varepsilon) = C_f\varepsilon$; $\psi(0) = 0$, then the problem (1.1)-(1.2) is generalized Ulam-Hyers stable.

**Theorem 4.4** *Assume that (H1), (H2), (H3) and (3.5) hold. Then, the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.*

*Proof.* Let $\mathfrak{v} \in C_{1-\gamma,\rho}[a, b]$ be solution of the inequality

$$|{}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t))| \leq \varepsilon\varphi(t), t \in J, \varepsilon > 0, \tag{4.2}$$

and let $\mathfrak{u} \in C_{1-\gamma,\rho}[a, b]$ the unique solution of the following nonlinear neutral pantograph equation

$${}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathfrak{v}(\kappa t)), \quad t \in J,$$
$${}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{u}(a) = {}^{\rho}\mathfrak{J}^{1-\gamma}\mathfrak{v}(a) = \mathfrak{u}_a,$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

Using Lemma 2.13, we get

$$\mathsf{u}(t) = \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} + \left[{}^{\rho}\mathfrak{J}^{\alpha} g_{\mathsf{u}}\right](t).$$

By integration of the inequality (4.2), we get

$$\left|\mathfrak{v}(t) - \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathsf{v}}(s)ds\right| \leq \varepsilon\lambda_{\varphi}\varphi(t). \tag{4.3}$$

On the other hand, we have

$$|\mathfrak{v}(t) - \mathsf{u}(t)|$$
$$\leq \left|\mathfrak{v}(t) - \frac{\mathsf{u}_a}{\Gamma(\gamma)}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} g_{\mathsf{v}}(s)ds\right|$$
$$+ \frac{2K}{(1-L)\Gamma(\alpha)}\int_0^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}|\mathfrak{v}(s) - \mathsf{u}(s)|ds$$
$$\leq \varepsilon\lambda_{\varphi}\varphi(t) + \frac{2K}{(1-L)\Gamma(\alpha)}\int_0^t\left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}|\mathfrak{v}(s) - \mathsf{u}(s)|ds.$$

By applying Lemma 2.12, we get

$$|\mathfrak{v}(t) - \mathsf{u}(t)| \leq \varepsilon\lambda_{\varphi}\varphi(t) + \frac{2K\nu_1}{1-L}\varepsilon\lambda_{\varphi}^2\varphi(t).$$

Then for any $t \in J$, and by (H3), we have

$$|\mathfrak{v}(t) - \mathsf{u}(t)| \leq \left[\left(1 + \frac{2K\nu_1\lambda_{\varphi}}{1-L}\right)\lambda_{\varphi}\right]\varepsilon\varphi(t)\ |$$
$$= C_f\varepsilon\varphi(t),$$

where $\nu_1$ $-\nu_1$ ($\alpha$) is a constant, which completes the proof of Theorem 4.4.

**Example 4.5** Consider the following Hilfer-Katugampola type nonlinear neutral pantograph problem

$$^{\rho}\mathfrak{D}^{\alpha,\beta}\mathrm{u}(t) = 1 + \frac{e^{-t}}{9+e^{t}}\left(\mathrm{u}(t) + \mathrm{u}\left(\frac{t}{2}\right) + {}^{\rho}\mathfrak{D}^{\alpha,\beta}\mathrm{u}\left(\frac{t}{2}\right)\right), \quad t \in J := [0,1], \tag{4.4}$$

$$^{\rho}\mathfrak{I}_{0^+}^{1-\gamma}\mathrm{u}(0) = 0, \quad \gamma = \alpha + \beta - \alpha\beta. \tag{4.5}$$

Where $0 < \kappa < 1$, $a = 0$, $b = 1$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\kappa = \frac{1}{2}$, and choose $\gamma = \frac{2}{3}$.

Set

$$f(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = 1 + \frac{e^{-t}}{9+e^{t}}(\mathfrak{u} + \mathfrak{v} + \mathfrak{w}), \quad for\ any\ \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in R.\ t \in J.$$

Clearly, the function $f$ satisfies the hypotheses of Theorem 3.1.

For any $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}}, \overline{\mathfrak{w}} \in R$, and $t \in J$.

$$|f(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) - f(t, \overline{\mathfrak{u}}, \overline{\mathfrak{v}}, \overline{\mathfrak{w}})| \le \frac{1}{10}(|\mathfrak{u} - \overline{\mathfrak{u}}| + |\mathfrak{v} - \overline{\mathfrak{v}}|) + \frac{1}{10}(|\mathfrak{w} - \overline{\mathfrak{w}}|.$$

Hence, the hypothesis (H1) is satisfied with $K = L\frac{1}{10}$

Thus, the condition from (3.5)

$$\left(\frac{2K}{(1-L)\Gamma(\alpha)}\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\alpha} B(\gamma,\alpha)\right) = 0.4592 < 1,$$

It follows from Lemma 3.2 that the problem (4.4)-(4.5) has a unique solution. Moreover, Theorem 4.3 implies that the problem (4.4)-(4.5) is Ulam-Hyers stable.

## REFERENCES


[1] Ahmad B, Ntouyas SK. Initial value problems of fractional order Hadamard-type functional differential equations. Electron J Diff Equ. 2015; 77: 1-9.

[2] Furati KM, Tatar NE. An existence results for a nonlocal fractional differetial problem. J Fractional Calculus, 2004; 26: 43-51.

[3] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations, in: Mathematics Studies, 2006; 204: Elsevier.

[4] Podlubny I. Fractional Differential Equations, in: Mathematics in Science and Engineering, 1999; 198: Acad. Press.

[5] Oliveira DS. E. Capelas de oliveira, Hilfer-Katugampola fractional derivative. Comp Appl Math. 2018; 37: 3672-3690.

[6] Hilfer R. Application of fractional Calculus in Physics, World Scientific, Singapore, 1999.

[7] Furati KM, Kassim MD, Tatar NE. Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput Math Appl. 2012; 64: 1616-1626.

[8] Gu H, Trujillo JJ. Existence of mild solution for evolution equation with Hilfer fractional derivative. Appl Math Comput. 2014; 257: 344-354.

[9] Katugampola UN. Existence and uniqueness results for a class of generalized fractional differential equations Bull Math Anal App. 2014; 6(4): 1-15.

[10] Kamocki R, Obcznnski C. On fractional Cauchy-type problems containing Hilfer derivative. Electron J Qual Theory Differ Equ. 2016; 50: 1-12.

[11] Vivek D, Kanagarajan K, Harikrishnan S. Existence and uniqueness results for pantograph equations with generalized fractional derivative. J Nonlinear Anal Appl. 2017; 2017(1): 1-8.

[12] Vivek D, Kanagarajan K, Harikrishnan S. Existence results for implicit differential equations with generalized fractional derivative. J Nonlinear Anal Appl. 2017; (Accepted article-ID 00371).

[13] Abbas S, Benchohra M, Sivasundaram S. Dynamics and Ulam stability for Hilfer type fractional differential equations. Nonlinear Stud. 2016; 4: 627-637.

[14] Ibrahim RW, Jalab HA. Existence of Ulam stability for iterative fractional differential equations based on fractional entropy, Entropy 2015; 17(5): 3172.

[15] Ibrahim RW. Generalized Ulam-Hyers stability for fractional differential equations. Int J Math. 2012; 23: 1-9.

[16] Muniyappan P, Rajan S. Hyers-Ulam-Rassias stability of fractional differential equation. Int J Pure Appl Math. 2015; 102: 631-642.

[17] Wang J, Lv L, Zhou Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electron J Qual Theory Differ Equ. 2011; 63: 1-10.

[18] Balachandran K, Kiruthika S, Trujillo JJ. Existence of solutions of Nonlinear fractional pantograph equations. Acta Math Sci. 2013; 33B: 712-720.

[19] Iserles A. On the generalized pantograph functional differential equation. Eur J Appl Math. 1993; 4(1): 1-38.

[20] Vivek D, Kanagarajan K, Sivasundaram S. Dynamics and stability of pantograph equations via Hilfer fractional derivative. Nonlinear Stud. 2016; 23(4): 685-698.

[21] Vivek D, Kanagarajan K, Sivasundaram S. Theory and analysis of nonlinear neutral pantograph equation via Hilfer fractional derivative. Nonlinear Stud. 2017; 24(3): 699-712.


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