One Dimensional Kardar-Parisi-Zhang Equation in Various Initial Condition Amplitudes

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Abstract: The Kardar-Parisi-Zhang (KPZ) equation with different initial conditions has been investigated in this paper. The numerical solutions using fixed data are performed without noise term and with two kinds of noise terms, i.e., Gaussian noise term and white noise term. The solutions to the equation have been simulated with different initial conditions of the form $A\sin\left(\frac{x}{16}\right)$. Our study introduces the obtained shape of the solutions to the KPZ equation according to noise terms with three different amplitudes A. The effect of the noise and the amplitude of the noises are presented and investigated.

Keywords: KPZ equation, Gaussian noise, white noise, amplitude, initial condition, KPZ universality class.

INTRODUCTION

The concept of surface formation was first used in the context of studying the motion of growing interfaces by Kardar, Parisi and Zhang [1, 2] in 1986. After introducing the equation, it started to take much attention from different science areas. The reason for this popularity is its applicability to surface growth phenomena in various contexts. Also, the equation in the non-equilibrium statistical mechanical models with the noise effect is nonlinear. Moreover, the equation shows an infinite number of degrees of freedom at the same time. This nonlinear generalization of the ubiquitous diffusion equation is the so-called Kardar-Parisi-Zhang model obtained from the Langevin equation

$$\frac{du}{dt} = \nu\nabla^2 u + \frac{\lambda}{2}(\nabla u)^2 + \eta(x,t),$$

where $u(x,y,t)$ stands for the height profile of the local growth. The first term on the right-hand side describes the relaxation of the interface by surface tension, which prefers a smooth surface. The second term is the lowest-order nonlinear term that can appear in the surface growth equation justified by the Eden model and originates from the fact that the surface tends to grow locally normal to itself, and it has a non-equilibrium in origin. The last term is the Langevin noise that mimics the stochastic nature of any growth process and usually has a Gaussian distribution [3-7]. The KPZ equation best describes the physical problem as the stochastic nature of the surface increases. The basics of the physics of surface growth can be found in the book of Barabasi and Stanley [8]. Hwa and Frey [9, 10] investigated the KPZ model using the self-mode-coupling method and renormalization group theory. When Green's functions are being used, it is an exhausting and sophisticated method. Many different forms of dynamic scaling have been observed in one spatial dimension

$$\frac{dh}{dt} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t), \qquad (1)$$

with $h(x,t) = x - 2\varphi\, C(bx;\, b^z t)$ as a correlation function, where φ, b and z are real constants. Later, Lässig showed that how the theoretical approach for the KPZ model can be investigated [11]. Kriecherbauer and Krug [12], in their review paper, derived the KPZ model from hydrodynamical conservation equations with a general current density relation.

Previously, in the paper [13], Quastel and Remenik used droplet initial conditions in order to obtain large deviations at a large time with half-Brownian initial conditions. This provided a remarkable relation between some solutions on the finite time of the Kardar-Parisi-Zhang (KPZ) equation and the Kadomtsev-Petviashvili (KP) equation. Also, Prolhac and Spohn [14] obtained a convolution between the Gumbel distribution and pointed out the difference between two Fredholm determinants existing for the KPZ equation. While sharp wedge initial conditions were used, the height function was of the time-dependent probability distribution function. In addition, in the papers [15-27] Brownian, flat and random initial conditions were investigated.

We are interested in our paper to obtain and investigate the numerical solution to KPZ equation (1) with Gaussian noise, white noise and without noise terms. The importance is to point out the morphology of the initial surface of the substrate and the results obtained show different surface formations. In our investigations, the parameter values are chosen as $\nu = 1$ and $\lambda = \frac{1}{2}$ when we refer to the KPZ equation.

THE NOISE TERMS AND INITIAL CONDITIONS

In some of our previous studies [28-35], the role of the additional noise term makes the KPZ solutions interesting. However, when we conducted experiments

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without noise term in literature [36, 37, 38], it introduced the phenomenal effect of different initial conditions to the surface. Now, our aim is to look for the solutions of (1) with initial condition $A\sin\left(\frac{x}{16}\right)$ applying various amplitudes. Furthermore, we will see numerical results with Gaussian noise and white noise terms with all physical parameters (ν, λ, A). Here, A is the initial condition parameter.

In the simulations, the solutions to equation (1) with various noise terms have been presented. Also, the case without noise term effect is studied. Furthermore, three different amplitudes are applied for the initial condition of the form

$$h(x,0) = \mathrm{A}\sin\left(\frac{x}{16}\right) \tag{2}$$

with amplitude *A.* The numerical analyses have been conducted on the MATLAB R2019b with parameters given below (3). In the calculations, the initial time t has been chosen as 0.01 until 100. The following values are fixed throughout the study

$$x \in [-100, 100], t \in [0.01, 100], N = 100, \Delta t = 0.01/100. \tag{3}$$

In the process of the simulation obtaining the solution of the KPZ equation (1), the initial conditions which mimic the initial surface structure are chosen as in equations (4)-(6). The amplitude parameters are taken in a maximally simple form to determine relational changes of the initial conditions and the noise term differences. To get a better understanding first, we determine the solution (1) without noise term with different amplitude values

$$h(x,0) = 10\sin(x/16)\ , \tag{4}$$

$$h(x,0) = \sin(x/16), \tag{5}$$

$$h(x,0) = 0.1\sin\left(\frac{x}{16}\right). \tag{6}$$

The Case: Without Noise Term $\eta(x,t) = 0$

The research result using the traveling wave Ansatz with closed forms to obtain analytic solutions to KPZ equation (1) without noise term was presented in paper [3]. The KPZ partial differential equation (1) with $\eta(x,t) = 0$ leads to the ordinary differential equation (ODE) of

$$\nu f''(\omega) + f'(\omega)\left[c\ \frac{\lambda}{2} f'(\omega)\right] = 0. \tag{7}$$

The solution to equation (7) can be given as

$$f(\omega) = \frac{2}{\lambda}\ln\left(\frac{\lambda\left[c_1 \nu e^{\frac{c\eta}{\nu}}\right]}{2\nu c}\right)\nu, \tag{8}$$

where c_1 and c_2 are the constants of the integration. It should be noted that this is a linear function equation $f(\omega) = a\omega + b$ which is presented in a complicated form. The parameters for c_1 is set, and it equals 0, which gives a constant solution. Physically it means that there is a continuous surface growing till infinity which is quite unphysical. Therefore, some additional noise is needed to have physically realistic surface growing phenomena. However, in our situation, when there is no noise term $(\eta(x,t) = 0)$, the surface is flat without any wavy effect.

When there is an initial condition effect, and its amplitude difference is investigated, then the wavy surface of the graph appears as Figure **1** without any curve or wedge surface effect.

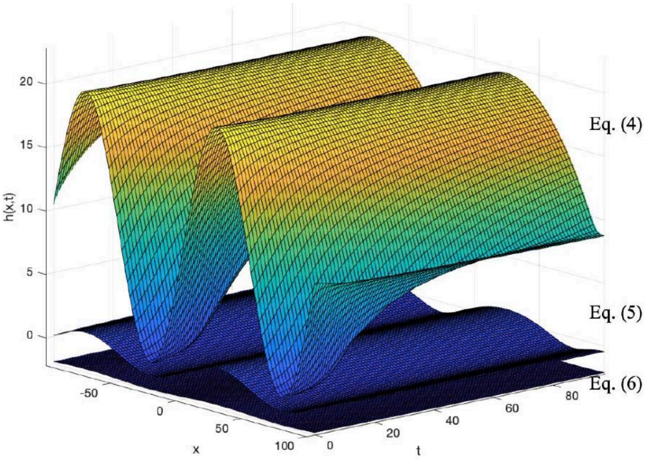

Figure 1: The solution graph to (1) without noise term and with given values (3) and various initial conditions (4)-(6).

As it is presented in Figure **1**, the amplitude of the initial condition is an important parameter for the formation of the surface even without noise term. When the amplitude parameter of the initial condition equals 10, the starting part of the wavy surface fluctuates between given values. After a certain time, this surface becomes flat; see details on the results and numerical data in paper [10].

The Case: With Noise Term $\eta(x,t) \neq 0$

Gaussian Noise Term

Kinetic roughening is an important part of universality classes. For this reason, we pay attention to the time-dependent noise affecting the surface growth. Generally, these surface fluctuations occur by driving flux or atoms acting on the system. However, this does not mean that its amplitude is directly the square root of the average external flux. For example, research on beam molecular epitaxy growth for electrochemical or chemical vapor deposition showed that the noise term from the Langevin equation for the interface is more affected. However, it cannot be overemphasized. Universal behavior refers to asymptotic properties, far exceeding all existing transients (induced by, e.g., physical instability acting on the system) and crossovers (due to competition among various physical mechanisms, each of which is dominant for a different range in time and space). Considering the type of system and the asymptotic properties, the equations containing the additive noise term adequately describe the phenomenon. The Gaussian-type noise is uncorrelated in time and space [12]. Applying similarity transformation $h(x,t) = f(w)$ and $w = \frac{x}{\sqrt{t}}$ to (1) with Gaussian noise, one gets the ODE of

$$\nu f''(w) + 0.5f'(w)[w + \lambda f'(w)] + ae^{\frac{-w^2}{n}} = 0, \qquad (9)$$

where value a is chosen as 1. There is no general formula available for the solution to (9) for arbitrary parameters $\lambda,\ \mu,\ a$. Fortunately, if two parameters are fixed, e.g., $\nu = 1$, $\lambda = \frac{1}{2}$ and $a = 1$, then there is a closed expression (an analytical solution) available

$$f(w) = -\frac{1}{2\lambda}\ln\left[1 + \tan\left\{\sqrt{\lambda a\pi}\cdot \mathrm{erf}\left(\sqrt{\frac{w}{2}}\right) + c_1\right\}^2\right] + c_2 \quad , \qquad (10)$$

where erf means the error function and c_1 and c_2 are integration constants, see [39, 40].

We will consider the case of sharp wedge initial condition for positive, small ε as

$$h_\varepsilon(x,0) = -\frac{|x|}{\varepsilon} \text{ with } \varepsilon > 0, \varepsilon \to 0. \qquad (11)$$

The solution to equation (1) with (11) is presented as follows

$$h(x,t) = -\frac{t}{24} - \frac{x^2}{2t} + \left(\frac{t}{2}\right)^{\frac{1}{3}}\eta(x,t). \qquad (12)$$

The flattening parabola should be viewed as the top part of the droplet in the experiment of [28, 29] and $\eta(x,t)$ represents the superimposed fluctuations. Eventually, the KPZ equation holds in greater generality. In particular, it is also applied for interface motion and growth models in higher dimensions. For surveys on the earlier developments, we refer to [8, 42, 43, 44]. Recently, the KPZ equation has been used as a challenging test ground for non-equilibrium renormalization group techniques [46].

The subtraction $\frac{x^2}{2t}$ from equation (12) is uniquely fixed by the requirement that $\eta(x,t)$ is independent of x for any given $t > 0$. In fact, by the scaling invariance of the KPZ equation, for fixed t, higher-order correlations also depend only on the relative distance in x. Noise term $\eta(x,t)$ depends on t. However, to find out its value is less obvious. Because a diverging uniform shift in *h*-direction is the construction of the solution. This is most easily explained for the initial condition $h_\varepsilon(x,0)$. The construction of the solution requires

$$\lim_{\varepsilon\to 0} e^{h_\varepsilon(x,0)} = \delta(x), \qquad (13)$$

which presents that $h_\varepsilon(x,0) = -\varepsilon^{-1}|x| - \log(2\varepsilon)$ with $\log(2\varepsilon)$ diverging as $\varepsilon \to 0$ (see [18]).

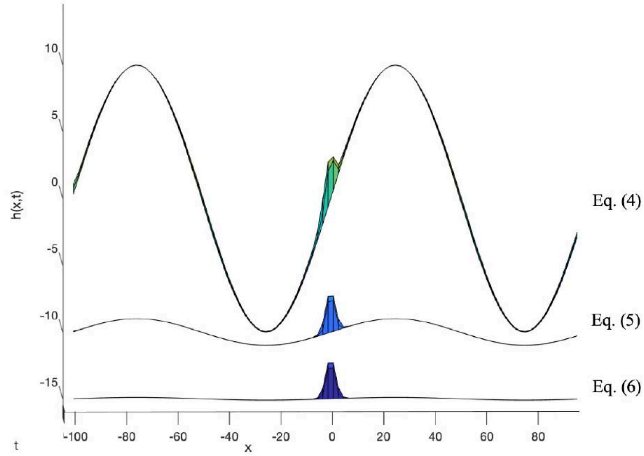

Figure 2: Solutions to (1) with Gaussian noise term with values (3) and initial conditions (4)-(6) for $t = 100$.

Figure **2** represents the so-called bell curve Gaussian surface. On the one hand, in the graph for the amplitude of the initial condition between 0.1 and 1 (in equations (5) and (6)) the same curves are obtained in spite of initial surface differences. Alternatively, while initial condition amplitude equals 10, the wavy surface affects the Gaussian noise term curve edge resulting in a decrease in the noise term appearance.

White Noise Term

The white noise term for $\eta(x,t)$ is not a regular function. The solution $h(x,t)$ partially inherits this roughness of the surface due to the noise and therefore, the function $(\partial h/\partial x)^2$ is not studied till now. Nevertheless, more reliable solutions can be formulated by suitable approximation schemes, which are explained in detail in [47]. The most direct one can be easily stated. One smoothens η to η_κ as

$$\eta_\kappa(x,t) = \int dx'\,\kappa\varphi\big(\kappa(x-x')\big)\eta(x',t) = \varphi_\kappa\cdot\eta(x,t) \quad , \quad \kappa \to \infty \qquad (14)$$

with some smooth function φ, a localized and normalized smearing function. Then $\eta_\kappa \to \eta$ as $\kappa \to \infty$ and equation (1) has well-defined solutions $h_\kappa(x,t)$ with noise terms η_κ. They move with a uniform background velocity v_κ along the h-direction. Then $v_\kappa \to \infty$ as $\kappa \to \infty$, but $h_\kappa(x,t) - v_\kappa t$ has a limit. Since v_κ sets merely the choice of a reference frame, the claim is that under this limit procedure, the fluctuation properties remain intact [45].

While understandable solutions are thus ensured, little is known about their properties. To make $\frac{\partial h(x,t)}{\partial x}$ stationary, one has to start the solution to the KPZ equation with two-sided Brownian motion. With this input, one argues that the height fluctuations will grow as $t^{\frac{1}{3}}$, while the transverse correlation length increases as $t^{\frac{2}{3}}$. Very recently, it has been proved that the variance of the stationary two-point function increases as $t^{\frac{4}{3}}$ by providing suitable upper and lower bounds [48].

If we choose as initial function the narrow wedge

$$h(x,0) = -|x|/\delta \quad (15)$$

with $\delta \ll 1$, it leads to following representation

$$h(x,t)\begin{cases} -\frac{x^2}{2\lambda t} \text{ for } |x| \le \frac{2\lambda t}{\delta}, \\ -\frac{|x|}{\delta} \text{ for } |x| > \frac{2\lambda t}{\delta}. \end{cases} \quad (16)$$

The initial condition may look artificial. In spite of short times, the nonlinearity dominates, and ignoring the other terms in the equation (16), h spreads very fast into the parabolic profile. It can be seen as the top part of a growing surface. Physically one thereby covers the case of macroscopically curved height profiles [48, 49, 50, 52].

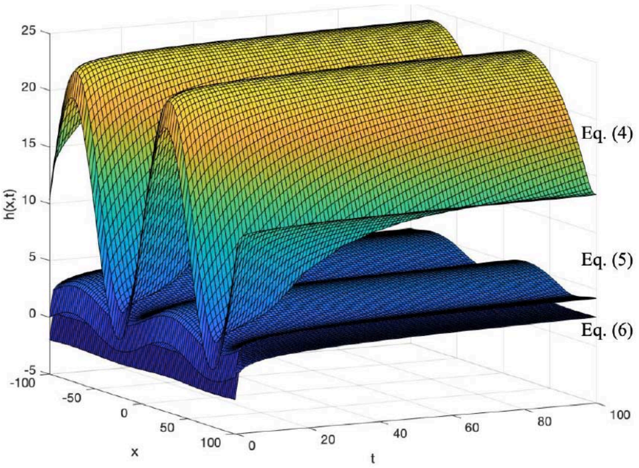

Figure 3: Solutions to (1) with white noise term with values (3) and initial conditions (4)-(6).

The solution (16) for the distribution of $h(x,t)$ for all $t > 0$ is exact in the properly normalized limit $\delta \to 0$. A further explanation has been presented in the paper [48, 49, 50, 52].

At the beginning of time *t*, there is an angled surface as an effect of the noise term. Despite the difference in amplitude of the initial condition, the primary form of the graph is the same. Figure **3** represents the initial condition amplitude effect only to the surface but not the noise itself.

DISCUSSION AND RESULTS

To introduce visible differences between the noise terms and amplitude, the impact of them is presented in Figure **4**. It is obvious that the initial condition effect is the same in all the simulations. However, at the beginning of the time t, the parts of each graph are differently formed due to the effect of the noise terms. Without noise term, the graph started with the straight wave surface and smoothed after a certain time. The second simulation result presented for the white noise term graph started with an edge surface at the beginning and took effect of the initial condition to itself. Also, in the third graph with Gaussian noise term, the surface has the same wavy surface formation. However, it can be seen in the middle of the graph that a bell curve shape appeared due to the Gaussian noise term.

Discussion and results

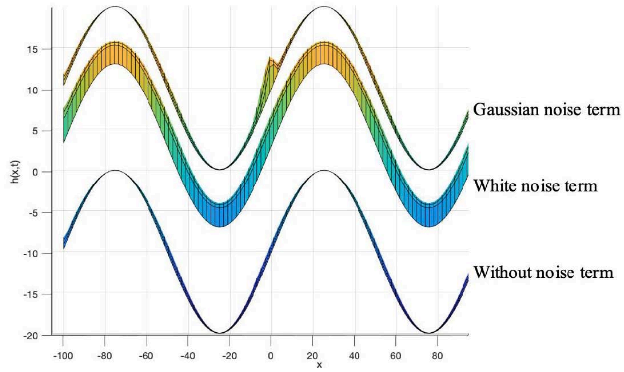

Figure 4: Cross-section of $h(x,t)$ and x with the effect of initial condition (4) to all three cases for $t = 100$.

CONCLUSION

The height distribution function of the one-dimensional KPZ equation with different initial condition amplitudes and different noise terms is numerically analysed. It can be concluded that for fixed parameters of KPZ equation (1), the noise terms represented similar shapes on the surface with the same initial condition. However, the noise term effect remained at the same level in every simulation despite different amplitudes. Moreover, when the initial condition was high enough, then the presence of Gaussian noise cannot be observed on the surface (see Figure **2**).

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REFERENCES

[1] Kardar K., Parisi G., Zhang Y.Z. Dynamic scaling of growing interfaces. Phys. Rev. Lett. 1986; 56:889–892. https://doi.org/10.1103/PhysRevLett.56.889

[2] Quastel J. Introduction to KPZ. Current developments in mathematics, 2011; 2011.1 https://doi.org/10.4310/CDM.2011.v2011.n1.a3

[3] Barna I. F., Bognár, G., Guedda, M., Mátyás, L., & Hriczó, K. Analytic self-similar solutions of the Kardar-Parisi-Zhang interface growing equation with various noise terms. Mathematical Modelling and Analysis, 2020; 25(2), 241-256. https://doi.org/10.3846/mma.2020.10459

[4] Comets F, Cosco C, Mukherjee C. Space-time fluctuation of the Kardar-Parisi-Zhang equation in $ d¥geq 3$ and the Gaussian free field. 2019; arXiv preprint arXiv:1905.03200 https://doi.org/10.1007/s10955-020-02539-7

[5] Carrasco I.S, Oliveira T.J. Universality and dependence on initial conditions in the class of the nonlinear molecular beam epitaxy equation. Physical Review E, 2016; 94(5): 050801. https://doi.org/10.1103/PhysRevE.94.050801

[6] Alexandre K., Le Doussal P. Exact short-time height distribution in the one-dimensional Kardar-Parisi-Zhang equation with Brownian initial condition. Physical Review E, 2017; 96.2 020102. https://doi.org/10.1103/PhysRevE.96.020102

[7] Vallee O. Functional sensitivity of Burgers and related equations to initial conditions. 2003; arXiv preprint nlin/0304014.

[8] Barabasi A.L., Stanley H.E. Fractal concepts in surface growth. Cambridge University Press, 1995. https://doi.org/10.1017/CBO9780511599798

[9] Hwa T. Frey E. Exact scaling function of interface growth dynamics. Physical Review A, 1991; 44(12): R7873. https://doi.org/10.1103/PhysRevA.44.R7873

[10] Frey E., Täuber U. C., Hwa T. Mode-coupling and renormalization group results for the noisy burgers equation. Physical Review E, 1996; 53(5):4424. https://doi.org/10.1103/PhysRevE.53.4424

[11] Lässig M. On growth, disorder, and field theory. Journal of Physics: Condensed Matter, 1998; 10(44):9905. https://doi.org/10.1088/0953-8984/10/44/003

[12] Kriecherbauer T. Krug J. A pedestrian's view on interacting particle systems, KPZ universality and random matrices. Journal of Physics A: Mathematical and Theoretical, 2010; 43(40):403001. https://doi.org/10.1088/1751-8113/43/40/403001

[13] Quastel, J., Remenik D. KP governs random growth off a one dimensional substrate. 2019; arXiv preprint arXiv:1908.10353.

[14] Prolhac, S., Spohn H. Height distribution of the Kardar-Parisi-Zhang equation with sharp-wedge initial condition: Numerical evaluations. Physical Review E, 2011; 84(1), 011119. https://doi.org/10.1103/PhysRevE.84.011119

[15] Imamura T., Sasamoto T., Spohn H. On the equal time two-point distribution of the one-dimensional KPZ equation by replica. Journal of Physics A: Mathematical and Theoretical, 2013; 46.35. 355002. https://doi.org/10.1088/1751-8113/46/35/355002

[16] Ferrari, Patrik L., Bálint V. Upper tail decay of KPZ models with Brownian initial conditions. 2020; arXiv preprint arXiv:2007.13496

[17] Calabrese, P., Le Doussal, P. Interaction quench in a Lieb–Liniger model and the KPZ equation with flat initial conditions. Journal of Statistical Mechanics: Theory and Experiment, 2014(5); 005004. https://doi.org/10.1088/1742-5468/2014/05/P05004

[18] Le Doussal P. Crossover between various initial conditions in KPZ growth: flat to stationary. Journal of Statistical Mechanics: Theory and Experiment, 2017(5); 053210. https://doi.org/10.1088/1742-5468/aa6f3e

[19] Krajenbrink A, Doussal PL. Exact short-time height distribution in 1D KPZ equation with Brownian initial condition. 2019; arXiv preprint arXiv:1705.04654.

[20] Quastel J, Remenik D. How flat is flat in random interface growth?. Transactions of the American Mathematical Society. 2019; 371(9), 6047-85. https://doi.org/10.1090/tran/7338

[21] Esipov S.E., Newman T.J. Interface growth and Burgers turbulence: the problem of random initial conditions. Physical Review E. 1993; 48(2), 1046. https://doi.org/10.1103/PhysRevE.48.1046

[22] Kozachenko Y., Orsingher E., Sakhno L., Vasylyk O. Estimates for functionals of solutions to Higher-Order Heat-Type equations with random initial conditions. Journal of Statistical Physics. 2018; 172(6), 1641-62. https://doi.org/10.1007/s10955-018-2111-0

[23] Blömker D., Cannizzaro G., Romito M. Random initial conditions for semi-linear PDEs. Proceedings of the Royal Society of Edinburgh Section A: Mathematics. 2020; 150(3) 1533-65. https://doi.org/10.1017/prm.2018.157

[24] Le Doussal, Pierre, Pasquale Calabrese. The KPZ equation with flat initial condition and the directed polymer with one free end. Journal of Statistical Mechanics: Theory and Experiment. 2012; 6001. https://doi.org/10.1088/1742-5468/2012/06/P06001

[25] Gueudré T, Le Doussal P, Rosso A, Henry A, Calabrese P. Short-time growth of a Kardar-Parisi-Zhang interface with flat initial conditions. Physical Review E. 2012; 86(4):041151. https://doi.org/10.1103/PhysRevE.86.041151

[26] Sarkar S, Virág B. Brownian absolute continuity of the KPZ fixed point with arbitrary initial condition. 2020; arXiv preprint arXiv:2002.08496.

[27] Meerson B, Schmidt J. Height distribution tails in the Kardar–Parisi–Zhang equation with Brownian initial conditions. Journal of Statistical Mechanics: Theory and Experiment. 2017(10);103207. https://doi.org/10.1088/1742-5468/aa8c12

[28] Bognár G. Roughening in Nonlinear Surface Growth Model. Applied Sciences, 2020; 10 (4), 1422 https://doi.org/10.3390/app10041422

[29] da Silva R.G., Lyra M.L., da Silva C.R., Viswanathan GM. Roughness scaling and sensitivity to initial conditions in a symmetric restricted ballistic deposition model. The European Physical Journal B-Condensed Matter and Complex Systems. 2000; 17(4), 693-7. https://doi.org/10.1007/s100510070110

[30] Barna I. F., Bognár, G., Guedda, M., Mátyás, L., Hriczó, K. Analytic self-similar solutions of the Kardar-Parisi-Zhang interface growing equation with various noise terms. Mathematical Modelling and Analysis, 2020; 25(2), 241-256. https://doi.org/10.3846/mma.2020.10459

[31] Barna I. F., Bognár, G., Guedda, M., Mátyás, L., Hriczó, K. Analytic traveling-wave solutions of the Kardar-Parisi-Zhang interface growing equation with different kind of noise terms, (2019). Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, 2019; 333, pp 239-253 https://doi.org/10.1007/978-3-030-56323-3_19

[32] Inc M. The approximate and exact solutions of the space-and time-fractional Burgers equations with initial conditions by variational iteration method. Journal of Mathematical Analysis and Applications. 2008; 345(1) 476-84. https://doi.org/10.1016/j.jmaa.2008.04.007

[33] Bukharev I.A, Kosterlitz J.M. Influence of initial conditions on KPZ growth. InAPS March Meeting Abstracts 1996; H33-04

[34] Fukai Y.T, Takeuchi K.A. Kardar-Parisi-Zhang interfaces with curved initial shapes and variational formula. Physical review letters. 2020; 124(6), 060601. https://doi.org/10.1103/PhysRevLett.124.060601

[35] Meerson B, Sasorov P.V, Vilenkin A. Nonequilibrium steady state of a weakly-driven Kardar–Parisi–Zhang equation. Journal of Statistical Mechanics: Theory and Experiment. 2018; 2018(5), 053201. https://doi.org/10.1088/1742-5468/aabbcc

[36] Sayfidinov O., Bognár G. Numerical Solutions of the Kardar-Parisi-Zhang Interface Growing Equation with Different Noise Terms. In: Jármai K., Voith K. (eds) Vehicle and Automotive Engineering 3. VAE 2020. Lecture Notes in Mechanical Engineering. Springer, Singapore. 2020; 302-311. https://doi.org/10.1007/978-981-15-9529-5_27

[37] Corwin I, Ghosal P. KPZ equation tails for general initial data. Electronic Journal of Probability, 2020; 25. https://doi.org/10.1214/20-EJP467

[38] Corwin I., Ghosal P. Lower tail of the KPZ equation. Duke Mathematical Journal. 2020; 169(7): 1329-95. https://doi.org/10.1215/00127094-2019-0079

[39] Korutcheva E., Cuerno, R. (editors): Advances in Condensed Matter and Statistical Physics, Nova Scientific Publishing. Inc., 2004; 237-259.

[40] Quastel J., Spohn H. The one-dimensional KPZ equation and its universality class. Journal of Statistical Physics, 2015; 160.4. 965-984. https://doi.org/10.1007/s10955-015-1250-9

[41] Takeuchi K. A., Sano, M. Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals. Physical Review Letters, 2010; 104(23), 230601. https://doi.org/10.1103/PhysRevLett.104.230601

[42] Halpin-Healy T., Zhang Y. C. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics. Physics Reports, 1995; 254(4-6), 215-414. https://doi.org/10.1016/0370-1573(94)00087-J

[43] Hoshino, M., KPZ equation with fractional derivatives of white noise. Stochastics and Partial Differential Equations: Analysis and Computations, 2016; 4(4), 827-890. https://doi.org/10.1007/s40072-016-0078-x

[44] Squizzato D, Canet L. Kardar-Parisi-Zhang equation with temporally correlated noise: A nonperturbative renormalization group approach. Physical Review E. 2019 Dec 30;100(6):062143. https://doi.org/10.1103/PhysRevE.100.062143

[45] Kardar M. Statistical physics of fields. Cambridge University Press, 2007. https://doi.org/10.1017/CBO9780511815881

[46] Fogedby H. C. Kardar-Parisi-Zhang equation in the weak noise limit: Pattern formation and upper critical dimension. Physical Review E, 2006; 73(3), 031104. https://doi.org/10.1103/PhysRevE.73.031104

[47] Prolhac S., Spohn H. Height distribution of the Kardar-Parisi-Zhang equation with sharp-wedge initial condition: Numerical evaluations. Physical Review E, 2011; 84(1), 011119. https://doi.org/10.1103/PhysRevE.84.011119

[48] Sasamoto T., Spohn H. Exact height distributions for the KPZ equation with narrow wedge initial condition. Nuclear Physics B, 2010; 834(3), 523-542. https://doi.org/10.1016/j.nuclphysb.2010.03.026

[49] Cosco C., Nakajima, S., Nakashima M. Law of large numbers and fluctuations in the sub-critical and $ L^ 2$ regions for SHE and KPZ equation in dimension $ d¥geq 3$. 2020; arXiv preprint arXiv:2005.12689. https://doi.org/10.1214/17-AAP1338

[50] Chhita S, Ferrari P.L, Spohn H. Limit distributions for KPZ growth models with spatially homogeneous random initial conditions. Annals of Applied Probability, 2018; 28(3):1573-603.

[51] Sasamoto, Tomohiro. Spatial correlations of the 1D KPZ surface on a flat substrate. Journal of Physics A: Mathematical and General, 2005; 38-33, L549. https://doi.org/10.1088/0305-4470/38/33/L01

[52] Baik J, Liu Z. Periodic TASEP with general initial conditions. Probability Theory and Related Fields. 2020; 1-98. https://doi.org/10.1007/s00440-020-01004-6

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