Generalized Path Optimization Problem for a Weighted Digraph over an Additively Idempotent Semiring

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Abstract: In this paper, a generalized path optimization problem for a weighted digraph (i.e., directed graph) over an additively idempotent semiring was considered. First, the conditions for power convergence of a matrix over an additively idempotent semiring were investigated. Then we proved that the path optimization problem is associated with powers of the adjacency matrix of the weighted digraph. The classical matrix power method for the shortest path problem on the min-plus algebra was generalized to the generalized path optimization problem. The proposed generalized path optimization model encompasses different path optimization problems, including the longest path problem, the shortest path problem, the maximum reliability path problem, and the maximum capacity path problem. Finally, for the four special cases, we illustrate the pictorial representations of the graphs with example data and the proposed method.

Keywords: Semiring, Path optimization, Digraph, Incline, Partial order.

1. INTRODUCTION

Fuzzy matrices [1] are that with entries within the real interval [0, 1]. Theories of fuzzy sets and matrices have been applied to many fields such as pattern recognition and diagnosis [2], cluster analysis [3], neural networks [4], decision making [5], optimization [6], control [7], etc. Fan [1] gave a systemic investigation for the fuzzy matrix theory. Convergence, indices, and periods of power sequence of fuzzy matrices under the operations max and min were considered in [1,8-13]. In [8], the convergence of powers of a fuzzy matrix was proposed. In [9], the fuzzy matrix theory was used to investigate the fuzzy bidirectional associative memories, and in [12], an upper bound of indices of fuzzy matrices was given. In [13], the conditions of stabilization of power sequence for a given fuzzy matrix were discussed.

Some results about the Boolean algebra and the (max, min) fuzzy algebra were generalized to more extensive algebraic structures, such as the distributive lattices [14-17], inclines [18-30], additively idempotent semirings [31], etc. Thus, the theory of fuzzy matrices was generalized to matrices over these extensive algebraic structures.

In [14], the index and period for lattice matrices were estimated, necessary and sufficient conditions for convergence of the powers of a lattice matrix were obtained, and conditions for a nilpotent lattice matrix were discussed. In [15], necessary and sufficient conditions for a matrix over a distributive lattice to be nilpotent were given. In [16], the eigenproblem of matrices over distributive lattices was presented. In [17], lattice-ordered effect algebras were proposed.

Incline algebra, proposed by Cao *et al.* [18] in 1984, is a broader algebraic structure than a distributive lattice. Boolean algebra, max-min fuzzy algebra, and distributive lattices are examples of inclines. Han and Li [19] and Duan [20] studied the convergence of power sequence for the incline matrices. In [21], the invertible conditions for matrices over an incline were considered. In [22], standard bases of a finitely generated vector space over a linearly ordered commutative incline were studied. In [23], the group of all invertible matrices and the group of all permutation matrices over an incline were discussed. In [24], Cramer's rule over inclines was presented, and the group of invertible incline matrices was investigated.

In [25], the invertibility of incline matrices and the existence of various generalized inverses were discussed. In [26], necessary and sufficient conditions for an incline Hall matrix to be regular were given. In [27], the regularity of incline matrices was considered. In [28], the invertible incline matrices and their ranks and Schein ranks were discussed. In [29], conditions for an element in an incline to be regular were obtained. In [30], a review of the algebraic structure of inclines, matrix theory over inclines, topological and convergence results, and some applications were presented.

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Some ordered algebraic structures and their associated optimization problems were considered in, e.g., [31-49]. The combination of such algebraic structures with graph theory leads to some interesting applications in the optimization field.

In [31], the classification of the additively idempotent semirings was considered. In [32,33], max-plus algebra was introduced, and the walk model on it was presented. In [37, 38], the shortest path problems on digraphs, i.e., directed graphs, were investigated. Also, the shortest path models were simulated in water utilities [39] and transportation problems [40]. In [41], the shortest path problems on undirected graphs were examined.

In [42], the reliability of paths in communication networks was studied. Similar reliability problems were considered in [43-45]. These problems all belong to optimization problems in the framework of ordered algebra. More general discussions were considered in [46-49].

This article considers matrices over an additively idempotent semiring, which is broader than incline algebra. The conditions for power convergence of a matrix over an additively idempotent semiring are investigated. Then we consider the generalized path optimization problem for a weighted digraph over an additively idempotent semiring. The problem is associated with the powers of the adjacency matrix of the weighted digraph. The classical matrix power method for the shortest path problem is proved to be still applicable for the generalized path optimization problem. The main results are presented in Section 3.

2. PRELIMINARIES

A semiring is an algebraic structure$(E,+,\cdot)$ such that $(E,+)$ is an Abelian monoid (identity o), $(E,\cdot)$ is a monoid (identity e), multiplication $\cdot$ distributes over addition + from either side, $a\cdot o=o\cdot a=o$ for all $a\in E$, and $o\neq e$. Usually, the semiring $(E,+,\cdot)$ is denoted by E briefly if the operations + and $\cdot$ are known, and we use the notation $ab=a\cdot b$. A semiring E is additively idempotent (alias path algebra) if $a+a=a$ for all $a\in E$.

In an additively idempotent semiring, we can define the partial order relation:

$a\leq b$ if and only if $a+b=b$.

Then the following propositions hold:

$$o\leq a, \tag{1}$$

$$a\leq a+b, b\leq a+b \text{ and } a+b=sup\{a,b\}, \tag{2}$$

$$\text{if } a\leq b \text{ then } a+c\leq b+c, ac\leq bc,\ ca\leq cb. \tag{3}$$

Proposition (2) means that if there is c such that $a\leq c, b\leq c$, then $a+b\leq c$.

We stipulate naturally the infinite sum $\sum_{i\in I}a_i$ exists if and only if $sup\{a_i|i\in I\}$ exists and in this case

$$\sum_{i\in I}a_i=sup\{a_i|i\in I\}. \tag{4}$$

An additively idempotent semiring is said to be an incline if $a+e=e$ for all $a\in E$. In an incline, the following relations hold:

$$a\leq e, \tag{5}$$

$$a+ab=a, b+ab=b, \tag{6}$$

$$ab\leq a, ab\leq b. \tag{7}$$

A semiring E is said to be selective if $a+b=a$ or b for all $a,b\in E$.

Let $G=(V,U)$ be a digraph without multiple arcs (network), where $V=\{1,2,\cdots,n\}$ and U are the vertex set and arc set, respectively. A path is a finite series of vertices $p=(k_0,k_1,\cdots,k_r)$, where $(k_s,k_{s+1})\in U,\ s=0,1,\cdots,r-1$. r is the arc number of the path. If $k_0=k_r$ then, the path p is called a circuit. If any two vertices in p are different except that $k_0=k_r$ possibly holds, then the path p is called elementary. The following notations are used:

P_{ij}: the set of all paths from vertex i to vertex j.

P_{ij}^m: the set of all paths from vertex i to vertex j, with exact m arcs.

$P_{ij}^{[m]}$: the set of all paths from vertex i to vertex j, with at most m arcs.

P_{ij}^E: the set of all elementary paths from vertex i to vertex j.

Among the four sets, only the first is possibly infinite. For every arc $(i,j)\in U$, associate a weight $w(i,j)\in E$. Such a digraph is called a weighted digraph over semiring E. For a path

$$p=(i,k_1,k_2,\cdots,k_r,j)\in P_{ij},$$

We define its weight as

$$w(p)=w(i,k_1)\cdot w(k_1,k_2)\cdot\cdots\cdot w(k_r,j). \tag{8}$$

Define the adjacency matrix of the *n*th-order weighted digraph over a semiring as $A=(a_{ij})_{n\times n}$, where

$$a_{ij}=\begin{cases} w(i,j), & (i,j)\in U,\\ o, & (i,j)\notin U.\end{cases} \tag{9}$$

Let $M_n(E)$ denote the set of all the *n*th-order square matrices over the semiring E. Define the addition $A+B$ and product AB of matrices in $M_n(E)$ like in a ring [50]. The (i,j) entry of A is denoted by a_{ij} or $[A]_{ij}$. We use the notations

$$A^{[m]} = A + A^2 + \cdots + A^m, \tag{10}$$

$$\underline{m} = \{1,2,\cdots,m\}. \tag{11}$$

Let $A \in M_n(E)$. A is called power-convergent if $A^k = A^{k+1}$ for some positive integer k. If E is additively idempotent, define

$$A \le B \text{ if and only if } a_{ij} \le b_{ij}, \tag{12}$$

for all $i,j \in \underline{n}$. Eq. (12) defines a partial order over $M_n(E)$.

3. MAIN RESULTS

Let A be the adjacency matrix of an *n*th-order weighted digraph over a semiring E. It is easy to verify that

$$[A^m]_{ij} = \sum_{k_1,\cdots,k_{m-1}=1}^{n} a_{ik_1}a_{k_1k_2}\cdots a_{k_{m-1}j} = \sum_{p\in P_{ij}^m} w(p), \tag{13}$$

$$[A^{[m]}]_{ij} = \sum_{p\in P_{ij}^{[m]}} w(p), \tag{14}$$

where the empty sum is o. Further, we have the following lemma.

Lemma 1. Let A be the adjacency matrix of a weighted digraph over an additively idempotent semiring E. If A is power-convergent and $A^k = A^{k+1}$, then the sum $\sum_{p\in P_{ij}} w(p)$ exists, and

$$[A^{[k]}]_{ij} = \sum_{p\in P_{ij}} w(p). \tag{15}$$

In particular, if $A \le A^2 \le \cdots \le A^k = A^{k+1}$ holds, then

$$[A^k]_{ij} = \sum_{p\in P_{ij}} w(p). \tag{16}$$

Theorem 1. Let $A = (a_{ij})$ be an *n*th-order square matrix over an additively idempotent semiring E and satisfy $a_{ii} = e$ for all $i \in \underline{n}$, and $a_{ik_1}a_{k_1k_2}\cdots a_{k_ri} \le e$ for any integer r $(1 \le r \le n-1)$ and mutually distinct $i,k_1,k_2,\cdots,k_r \in \underline{n}$. Then

(i) $[A^m]_{ii} = e$ for any $i \in \underline{n}$ and $m \ge 2$,

(ii) $A \le A^2 \le \cdots \le A^{n-1} = A^n = A^{n+1} = \cdots$,

(iii) $[A^{n-1}]_{ij} = \sum a_{ij_1}a_{j_1j_2}\cdots a_{j_lj}$ if $i \ne j$,

where $\sum$ denotes the sum for all l $(0 \le l \le n-2)$ and mutually different $j_1,j_2,\cdots,j_l \in \underline{n}\backslash\{i,j\}$.

Proof: First, we can have more general inequality $a_{ik_1}a_{k_1k_2}\cdots a_{k_ri} \le e$ for any integer $r \ge 1$. From

$$a_{ij} = a_{ii}a_{ij} \le [A^2]_{ij},$$

it follows $A \le A^2 \le \cdots \le A^n \le \cdots$. These inequalities imply (i).

To prove (ii) we consider $[A^n]_{ij}$ for $i \ne j$. Since $[A^n]_{ij} = \sum_{k_1,\cdots,k_{n-1}=1}^{n} a_{ik_1}a_{k_1k_2}\cdots a_{k_{n-1}j}$, so, if $k_r = k_s$, $0 \le r < s \le n, k_0 = i, k_n = j$, we get

$$a_{ik_1}a_{k_1k_2}\cdots a_{k_{n-1}j} \le a_{ik_1}a_{k_1k_2}\cdots a_{k_{r-1}k_r}a_{k_sk_{s+1}}\cdots a_{k_{n-1}j} \le [A^{n-s+r}]_{ij} \le [A^{n-1}]_{ij}.$$

Thus $A^n \le A^{n-1}$ is derived, and (ii) is proved.

For (iii), from

$$a_{ij_1}a_{j_1j_2}\cdots a_{j_lj} \le [A^{l+1}]_{ij} \le [A^{n-1}]_{ij},$$

we have

$$\sum a_{ij_1}a_{j_1j_2}\cdots a_{j_lj} \le [A^{n-1}]_{ij},$$

where $\sum$ denotes the sum for all l $(0 \le l \le n-2)$ and mutually different $j_1,j_2,\cdots,j_l \in \underline{n}\backslash\{i,j\}$.

Conversely, for all $k_1,k_2,\cdots,k_{n-2} \in \underline{n}$, repeatedly using the above deleting methods for the equal subscripts eventually results in

$$a_{ik_1}a_{k_1k_2}\cdots a_{k_{n-2}j} \le a_{ij_1}a_{j_1j_2}\cdots a_{j_lj},$$

where $0 \le l \le n-2$ and $j_1,j_2,\cdots,j_l \in \underline{n}\backslash\{i,j\}$ are mutually different. Accordingly,

$$[A^{n-1}]_{ij} = \sum_{k_1,\cdots,k_{n-2}=1}^{n} a_{ik_1}a_{k_1k_2}\cdots a_{k_{n-2}j} \le \sum a_{ij_1}a_{j_1j_2}\cdots a_{j_lj},$$

where $\sum$ denotes the sum for all l $(0 \le l \le n-2)$ and mutually different $j_1,j_2,\cdots,j_l \in \underline{n}\backslash\{i,j\}$. (iii) is proved. Hence, the theorem was proved. ∎

It follows from Theorem 1 that

Theorem 2. Let G be an *n*th-order weighted digraph over an additively idempotent semiring E, and A be its adjacency matrix satisfying $a_{ii} = e$ for all $i \in \underline{n}$. If $w(p) \le e$ holds for each elementary circuit p of G, then

$$[A^{n-1}]_{ij} = \sum_{p\in P_{ij}} w(p) = \sum_{p\in P_{ij}^E} w(p). \tag{17}$$

In particular, if the semiring E is selective, then $[A^{n-1}]_{ij}$ stands for the greatest element of the weights of paths in P_{ij}, and the greatest element is achieved on some elementary path.

If the additively idempotent semiring E is an incline, then the following two corollaries are obtained.

Corollary 1. Let $A = (a_{ij})$ be an *n*th-order square matrix over an incline $(E,+,\cdot)$. If $a_{ii} = e$ for all $i \in \underline{n}$, then

(i) $[A^m]_{ii} = e$ for all $i \in \underline{n}$ and $m \ge 2$,

(ii) $A \le A^2 \le \cdots \le A^{n-1} = A^n = A^{n+1} = \cdots$,

(iii) $[A^{n-1}]_{ij} = \sum a_{ij_1}a_{j_1j_2}\cdots a_{j_lj}$ if $i \ne j$,

where $\sum$ denotes the sum for all l $(0 \le l \le n-2)$ and mutually different $j_1,j_2,\cdots,j_l \in \underline{n}\backslash\{i,j\}$.

Corollary 2. Let G be an *n*th-order weighted digraph over an incline E and A be the adjacency matrix satisfying $a_{ii} = e$ for all $i \in \underline{n}$. Then

$$[A^{n-1}]_{ij} = \sum_{p\in P_{ij}} w(p) = \sum_{p\in P_{ij}^E} w(p). \qquad (18)$$

In particular, if the incline E is selective then $[A^{n-1}]_{ij}$ stands for the greatest element of the weights of paths in P_{ij}, and the greatest element is achieved on some elementary path.

4. SPECIAL CASES AND ILLUSTRATIVE EXAMPLES

The following four cases belong to the models discussed above. In each case, the weighted digraph $G = (V, U)$ over semiring E is *n*th-order with the vertex set $V = \{1,2,\cdots n\}$ and $(i,i) \in U$ and $a_{ii} = w(i,i) = e$ for all $i \in V$. Let R and R^+ denote the real number set and nonnegative real number set, respectively.

Case 1. The longest path problem [47,51,52] belongs to the path optimization for the weighted digraph over the additively idempotent semiring

$$E = (R \cup \{-\infty\}, max, +),\ o = -\infty, e = 0.$$

The weight $w(p)$ denotes the length of the path. If $w(p) \le 0$ for any elementary circuit p, then $[A^{n-1}]_{ij}$ denotes the maximum length of paths from vertex i to vertex j.

If the additively idempotent semiring is

$$E = (R^+ \cup \{-\infty\},\max,+),\ o = -\infty,\ e = 0,$$

the length is nonnegative in this case. The condition becomes for any elementary circuit p, it holds that $w(p) = 0$.

We note that in [32,33], the (max,+) linear algebra was discussed in detail.

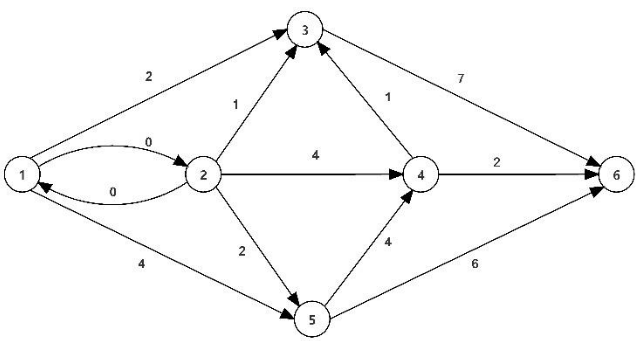

Figure 1: Weighted digraph used in Examples 1 and 2.

Example 1. We consider the weighted digraph over semiring $E = (R^+ \cup \{-\infty\},\max,+)$ in Figure **1**. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 2 & -\infty & 4 & -\infty \\ 0 & 0 & 1 & 4 & 2 & -\infty \\ -\infty & -\infty & 0 & -\infty & -\infty & 7 \\ -\infty & -\infty & 1 & 0 & -\infty & 2 \\ -\infty & -\infty & -\infty & 4 & 0 & 6 \\ -\infty & -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}.$$

We checked that $A^5 = A^6$ and

$$A^5 = \begin{pmatrix} 0 & 0 & 9 & 8 & 4 & 16 \\ 0 & 0 & 9 & 8 & 4 & 16 \\ -\infty & -\infty & 0 & -\infty & -\infty & 7 \\ -\infty & -\infty & 1 & 0 & -\infty & 8 \\ -\infty & -\infty & 5 & 4 & 0 & 12 \\ -\infty & -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}.$$

For example, the maximum length of the path from vertex 2 to vertex 6 in Figure 1 is 16, which corresponds to the path p=(2,1,5,4,3,6).

Case 2. The shortest path problem [53,54] belongs to the path optimization for the weighted digraph over the additively idempotent semiring

$$E = (R \cup \{+\infty\}, min, +),\ o = +\infty, e = 0.$$

If $w(p) \ge 0$ for any elementary circuit p then $[A^{n-1}]_{ij}$ denotes the minimum length of paths from vertex i to vertex j.

If the semiring is

$$E = (R^+ \cup \{+\infty\}, min, +),\ o = +\infty,\ e = 0,$$

which is an incline, the length is nonnegative in this case.

Example 2. We consider the weighted digraph over the incline $E = (R^+ \cup \{+\infty\}, min, +)$ in Figure 1. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 2 & +\infty & 4 & +\infty \\ 0 & 0 & 1 & 4 & 2 & +\infty \\ +\infty & +\infty & 0 & +\infty & +\infty & 7 \\ +\infty & +\infty & 1 & 0 & +\infty & 2 \\ +\infty & +\infty & +\infty & 4 & 0 & 6 \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 \end{pmatrix}.$$

We checked that $A^3 = A^4$ and

$$A^3 = \begin{pmatrix} 0 & 0 & 1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 4 & 2 & 6 \\ +\infty & +\infty & 0 & +\infty & +\infty & 7 \\ +\infty & +\infty & 1 & 0 & +\infty & 2 \\ +\infty & +\infty & 5 & 4 & 0 & 6 \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 \end{pmatrix}.$$

For example, the shortest path from vertex 1 to vertex 6 in Figure 1 is 6, which corresponds to the path *p*= (1,2,4,6).

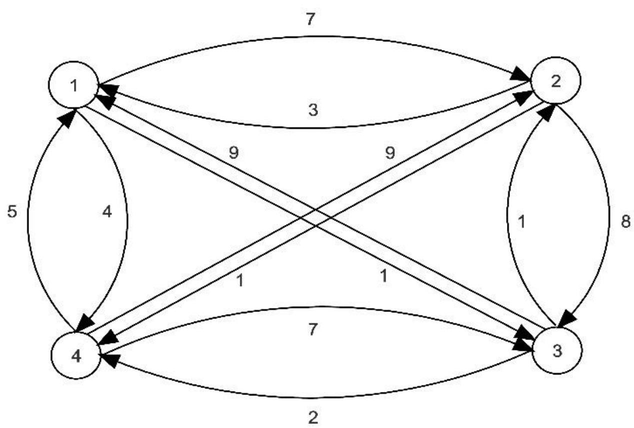

Figure 2: Weighted digraph used in Examples 3 and 5.

Example 3. Consider the weighted digraph over the incline $E=(R^{+}\cup\{+\infty\},min,+)$ in Figure 2. The adjacency matrix is

$$A=\begin{pmatrix}0&7&1&4\\3&0&8&1\\9&1&0&2\\5&9&7&0\end{pmatrix}.$$

We checked that $A^3=A^4$ and

$$A^3=\begin{pmatrix}0&2&1&3\\3&0&4&1\\4&1&0&2\\5&7&6&0\end{pmatrix}.$$

For example, the shortest path from vertex 4 to vertex 2 in Figure 2 is 7, which corresponds to the path *p*= (4,1,3,2).

Case 3. The maximum reliability path [42,55] problem corresponds to the path optimization for the weighted digraph over the incline

$$E=([0,1],max,\ \times),\ o=0,e=1.$$

The weight $w(p)$ denotes the reliability of the path. Then $[A^{n-1}]_{ij}$ denotes the maximum reliability of the paths from vertex i to vertex j.

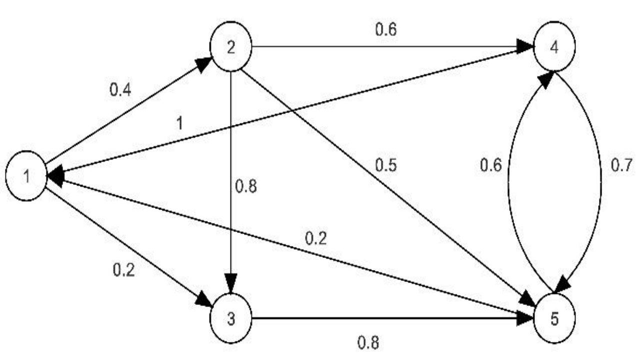

Figure 3: Weighted digraph used in Example 4.

Example 4. Consider the weighted digraph over the incline $E=([0,1],max,\ \times)$ in Figure 3. The adjacency matrix is

$$A=\begin{pmatrix}1&0.4&0.2&0&0\\0&1&0.8&0.6&0.5\\0&0&1&0&0.8\\1&0&0&1&0.7\\0.2&0&0&0.6&1\end{pmatrix}.$$

We checked that $A^4=A^5$ and

$$A^4=\begin{pmatrix}1&0.4&0.32&0.24&0.256\\0.6&1&0.8&0.6&0.64\\0.48&0.192&1&0.48&0.8\\1&0.4&0.32&1&0.7\\0.6&0.24&0.192&0.6&1\end{pmatrix}.$$

For example, the maximum reliability of the path from vertex 3 to vertex 2 in Figure 3 is 0.192, which corresponds to the path *p*= (3,5,4,1,2).

Case 4. The maximum capacity path [51,56] problem corresponds to the path optimization for the weighted digraph over the incline

$$E=(R^{+}\cup\{+\infty\},max,min),\ o=0,e=+\infty.$$

The weight $w(p)$ denotes the capacity of the path. Then $[A^{n-1}]_{ij}$ denotes the maximum capacity of the paths from vertex i to vertex j.

Example 5. Consider the weighted digraph over the incline $E=(R^{+}\cup\{+\infty\},max,min)$ in Figure 2. The adjacency matrix is

$$A=\begin{pmatrix}+\infty&7&1&4\\3&+\infty&8&1\\9&1&+\infty&2\\5&9&7&+\infty\end{pmatrix}.$$

We checked that $A^3=A^4$ and

$$A^3=\begin{pmatrix}+\infty&7&7&4\\8&+\infty&8&4\\9&7&+\infty&4\\8&9&8&+\infty\end{pmatrix}.$$

For example, the maximum capacity path from vertex 4 to vertex 1 in Figure 2 is 8, which corresponds to the path *p*= (4,2,3,1).

5. CONCLUSION

It was shown that the different path optimization problems could be united in a generalized path optimization model based on an additively idempotent semiring. The classical matrix power method for the shortest path problem on the min-plus algebra was generalized to the generalized path optimization problem. For the adjacency matrix A of the *n*th-order weighted digraph, the entry $[A^{n-1}]_{ij}$ gives the corresponding optimal value of paths from vertex i to vertex j. The proposed generalized path optimization model includes the longest path problem, the shortest path problem, the maximum reliability path problem, and the maximum capacity path problem.

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