Vectorial Prabhakar Hardy Type Generalized Fractional Inequalities under Convexity

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ABSTRACT

We present a detailed great variety of Hardy type fractional inequalities under convexity and $L^p$ norm in the setting of generalized Prabhakar and Hilfer fractional calculi of left and right integrals and derivatives. The radial multivariate case of the above over a spherical shell is developed in detail to all directions. Many inequalities are of vectorial splitting rational $L^p$ type or of separating rational $L^p$ type, others involve ratios of functions and of fractional integral operators.

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1. Background

This work is inspired by [3-11].

Here we consider the Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [6], p. 97; [5])

\[ E_{\gamma}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k! (\alpha k + \beta)} z^k, \]  

(1)

where \( \Gamma \) is the gamma function; \( \alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0, z \in \mathbb{R} \), and \( (\gamma)_k = \gamma (\gamma + 1) \ldots (\gamma + k - 1) \). It is

\[ E_{\alpha, \beta}^0(z) = \frac{1}{\Gamma(\beta)}. \]

Here we follow [4].

Let \( a, b \in \mathbb{R}, a < b \) and \( x \in [a, b] \); \( f \in C([a, b]) \). Let also \( \varphi \in C^1([a, b]) \) which is increasing. The left and right Prabhakar fractional integrals with respect to \( \varphi \) are defined as follows:

\[ (e_{\rho, \mu, a, \varphi}^\varphi f)(x) = \int_a^x \varphi'(t)(\varphi(x) - \varphi(t))^{\mu-1} E_{\rho, \mu}^\varphi \left[ (\varphi(x) - \varphi(t))^{\nu} \right] f(t) dt, \]

(2)

and

\[ (e_{\rho, \mu, b, \varphi}^\varphi f)(x) = \int_x^b \varphi'(t)(\varphi(t) - \varphi(x))^{\mu-1} E_{\rho, \mu}^\varphi \left[ (\varphi(t) - \varphi(x))^{\nu} \right] f(t) dt, \]

(3)

where \( \rho, \mu > 0; \gamma, \omega \in \mathbb{R} \).

Functions (2) and (3) are continuous ([4]).

Next, additionally, assume that \( \varphi'(x) \neq 0 \) over \([a, b]\) and let \( \varphi, f \in C^1([a, b]) \), where \( N = \lceil \mu \rceil \), \( \lceil \cdot \rceil \) is the ceiling of the number), \( 0 < \mu \in \mathbb{N} \). We define the \( \varphi \)-Prabhakar-Caputo left and right fractional derivatives of order \( \mu \) ([4]) as follows (\( x \in [a, b] \)):

\[ (cD_{\rho, \mu, a, \varphi}^\varphi f)(x) = \int_a^x \varphi'(t)(\varphi(x) - \varphi(t))^{\nu-1} E_{\rho, N}^\varphi \left[ (\varphi(x) - \varphi(t))^{\nu} \right] \left( \frac{1}{\varphi(t)} \frac{d}{dt} \right)^N f(t) dt, \]

(4)

and

\[ (cD_{\rho, \mu, b, \varphi}^\varphi f)(x) = (-1)^N \int_x^b \varphi'(t)(\varphi(t) - \varphi(x))^{\nu-1} E_{\rho, N}^\varphi \left[ (\varphi(t) - \varphi(x))^{\nu} \right] \left( \frac{1}{\varphi(t)} \frac{d}{dt} \right)^N f(t) dt. \]

(5)

One can write these (see (4), (5)) as

\[ (cD_{\rho, \mu, a, \varphi}^\varphi f)(x) = \left( e_{\rho, N}^\varphi f(\varphi(x)) \right), \]

(6)

and
\[
\left( C D_{\rho,\mu,0,a,b}^{\gamma,\psi} f\right)(x) = (-1)^N \left( e^{-\gamma,\psi}_{\rho,\mu-N,0,a,b} f^{[N]} \right)(x),
\]
(7)

where

\[
f^{[N]}(x) = f^{(N)}(x) := \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)^N f(x),
\]
(8)

\[\forall \ x \in [a,b].\]

Functions (6) and (7) are continuous on \([a,b]\).

Next we define the \(\psi\)-Prabhakar-Riemann Liouville left and right fractional derivatives of order \(\mu\) ([4]) as follows \((x \in [a,b], \ f \in C([a,b])):\)

\[
\left( RL D_{\rho,\mu,0,a,b}^{\gamma,\psi} f\right)(x) = \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)^N \int_a^x \psi(t)(\psi(x)-\psi(t))^{\mu-1} E_{\rho,N-\mu}^{-\gamma} [\omega(\psi(t)-\psi(x))^{\mu}] f(t) dt,
\]
(9)

and

\[
\left( RL D_{\rho,\mu,0,b,a}^{\gamma,\psi} f\right)(x) = \left( - \frac{1}{\psi(x)} \frac{d}{dx} \right)^N \int_x^b \psi(t)(\psi(t)-\psi(x))^{\mu-1} E_{\rho,N-\mu}^{-\gamma} [\omega(\psi(t)-\psi(x))^{\mu}] f(t) dt.
\]
(10)

That is we have

\[
\left( RL D_{\rho,\mu,0,a,b}^{\gamma,\psi} f\right)(x) = \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)^N \left( e^{-\gamma,\psi}_{\rho,\mu-N,0,a,b} f \right)(x),
\]
(11)

and

\[
\left( RL D_{\rho,\mu,0,b,a}^{\gamma,\psi} f\right)(x) = \left( - \frac{1}{\psi(x)} \frac{d}{dx} \right)^N \left( e^{-\gamma,\psi}_{\rho,\mu-N,0,b,a} f \right)(x),
\]
(12)

\[\forall \ x \in [a,b].\]

We define also the \(\psi\)-Hilfer-Prabhakar left and right fractional derivatives of order \(\mu\) and type \(0 \leq \beta \leq 1\) ([4]), as follows

\[
\left( H D_{\rho,\mu,0,a,b}^{\gamma,\psi,\beta} f\right)(x) = e^{\gamma,\psi}_{\rho,\beta(N-\mu),0,a,b} \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)^N e^{-\gamma,\psi}_{\rho,\beta(1-\mu),0,a,b} f(x),
\]
(13)

and

\[
\left( H D_{\rho,\mu,0,b,a}^{\gamma,\psi,\beta} f\right)(x) = e^{\gamma,\psi}_{\rho,\beta(N-\mu),0,b,a} \left( - \frac{1}{\psi(x)} \frac{d}{dx} \right)^N e^{-\gamma,\psi}_{\rho,\beta(1-\mu),0,a,b} f(x),
\]
(14)

\[\forall \ x \in [a,b].\]
When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta(N - \mu)$, we have that $N - 1 < \mu \leq \mu + \beta(N - \mu) \leq \mu + N = N$, hence $[\xi] = N$.

We can easily write that

$$
\left( \mathcal{H} D_{\rho,\mu,\alpha}^{\gamma,\beta} f \right)(x) = e^{-\gamma \beta t^{\rho - 1}} \int_{-\infty}^{\infty} K_{\gamma,\beta}(t) dt,
$$

and

$$
\left( \mathcal{H} D_{\rho,\mu,\beta}^{\gamma,\alpha} f \right)(x) = e^{-\gamma \beta t^{\rho - 1}} \int_{-\infty}^{\infty} K_{\gamma,\beta}(t) dt,
$$

$\forall \ x \in [a,b]$.

In this work we develop a great variety of fractional inequalities of Hardy type involving convexity and engaging the above exposed: $\psi$ -Prabhakar fractional left and right fractional integrals, the $\psi$ -Prabhakar-Caputo left and right fractional derivatives, the $\psi$ -Riemann-Liouville left and right fractional derivatives, and the $\psi$ -Hilfer-Prabhakar left and right fractional derivatives. The radial multivariate case of all of the above over a spherical shell is studied in full detail. We involve ratios of functions and of integral operators and we produce among others vectorial splitting rational $L_p$ inequalities, as well as separating rational $L_p$ inequalities.

2. Prerequisites

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma$-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k(x, \cdot)$ measurable on $\Omega_2$, and

$$
K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y), \text{ for any } x \in \Omega_1.
$$

We suppose that $K(x) > 0$ a.e. on $\Omega_1$ and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g_i : \Omega_1 \to \mathbb{R}$, $i = 1, \ldots, n$, with the representation

$$
g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y),
$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, \ldots, n$.

Denote by $\bar{x} = x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\bar{g} := (g_1, \ldots, g_n)$ and $\bar{f} := (f_1, \ldots, f_n)$.

We consider here $\Phi : \mathbb{R}^n_+ \to \mathbb{R}$ a convex function, which is increasing per coordinate, i.e. if $x_j \leq y_j$, $i = 1, \ldots, n$, then

$$
\Phi(x_1, \ldots, x_n) \leq \Phi(y_1, \ldots, y_n).
$$

In [3], p. 588, we proved that

**Theorem 1** Let $u$ be a weight function on $\Omega_1$, and $k, K, g, f$, $i = 1, \ldots, n \in \mathbb{N}$, and $\Phi$ defined as above. Assume that the function $x \to u(x) \frac{k(x, y)}{K(x)}$ is integrable on $\Omega_1$ for each fixed $y \in \Omega_2$. Define $\nu$ on $\Omega_2$ by
\[ v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} \, d\mu_1(x) < \infty. \] (19)

Then
\[ \int_{\Omega_1} u(x) \Phi \left( \frac{g_1(x)}{K(x)}, \ldots, \frac{g_n(x)}{K(x)} \right) \, d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi \left( \|f_1(y)\|, \ldots, \|f_n(y)\| \right) \, d\mu_2(y), \] (20)

under the assumptions:

(i) \( f_i, \Phi(\|f_1\|,\ldots,\|f_n\|) \) are \( k(x,y)d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \), for all \( i = 1,\ldots,n \),

(ii) \( v(y) \Phi(\|f_1(y)\|,\ldots,\|f_n(y)\|) \) is \( \mu_2 \)-integrable.

**Notation 2** From now on we may write
\[ \tilde{g}(x) = \int_{\Omega_2} k(x,y) \tilde{f}(y) \, d\mu_2(y), \] (21)

which means
\[ (g_1(x),\ldots,g_n(x)) = \left( \int_{\Omega_2} k(x,y) f_1(y) \, d\mu_2(y), \ldots, \int_{\Omega_2} k(x,y) f_n(y) \, d\mu_2(y) \right). \] (22)

Similarly, we may write
\[ \|\tilde{g}(x)\| = \left\| \int_{\Omega_2} k(x,y) \tilde{f}(y) \, d\mu_2(y) \right\|. \] (23)

and we mean
\[ \left( |g_1(x)|, \ldots, |g_n(x)| \right) = \left( \left\| \int_{\Omega_2} k(x,y) f_1(y) \, d\mu_2(y) \right\|, \ldots, \left\| \int_{\Omega_2} k(x,y) f_n(y) \, d\mu_2(y) \right\| \right). \] (24)

We also can write that
\[ \|\tilde{g}(x)\| \leq \int_{\Omega_2} k(x,y) \|\tilde{f}(y)\| \, d\mu_2(y), \] (25)

and we mean the fact that
\[ |g_i(x)| \leq \int_{\Omega_2} k(x,y) \|f_i(y)\| \, d\mu_2(y), \] (26)

for all \( i = 1,\ldots,n \), etc.

**Notation 3** Next let \( (\Omega_1, \Sigma_1, \mu_1) \) and \( (\Omega_2, \Sigma_2, \mu_2) \) be measure spaces with positive \( \sigma \)-finite measures, and let \( k_j : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R} \) be a nonnegative measurable function, \( k_j(x,) \) measurable on \( \Omega_2 \) and
\[ K_j(x) = \int_{\Omega_2} k_j(x,y) \, d\mu_2(y), \] \( x \in \Omega_1, j = 1,\ldots,m. \) (27)
We suppose that \( K_j(x) > 0 \) a.e. on \( \Omega_1 \). Let the measurable functions \( g_{ji} : \Omega_1 \to \mathbb{R} \) with the representation

\[
g_{ji}(x) = \int_{\Omega_2} k_j(x, y)f_{ji}(y)d\mu_2(y),
\]

where \( f_{ji} : \Omega_2 \to \mathbb{R} \) are measurable functions, \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

Denote the function vectors \( \tilde{g}_i := (g_{j1}, g_{j2}, \ldots, g_{jn}) \) and \( \tilde{f}_j := (f_{j1}, \ldots, f_{jn}) \), \( j = 1, \ldots, m \).

We say \( \tilde{f}_j \) is integrable with respect to measure \( \mu \), iff all \( f_{ji} \) are integrable with respect to \( \mu \).

We also consider here \( \Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+, j = 1, \ldots, m \), convex functions that are increasing per coordinate. Again \( u \) is a weight function on \( \Omega_1 \).

We make

**Remark 4** Following Notation 3, let \( F_j : \Omega_2 \to \mathbb{R} \cup \{ \pm \infty \} \) be measurable functions, \( j = 1, \ldots, m \), with \( 0 < F_j(y) < \infty \) on \( \Omega_2 \). In (27) we replace \( k_j(x, y) \) by \( k_j(x, y)F_j(y) \), \( j = 1, \ldots, m \), and we have the modified \( K_j(x) \) as

\[
L_j(x) := \int_{\Omega_2} k_j(x, y)F_j(y)d\mu_2(y), x \in \Omega_1.
\]

We assume \( L_j(x) > 0 \) a.e. on \( \Omega_1 \).

As new \( \tilde{f}_j \) we consider now \( \tilde{g}_j := \frac{\tilde{f}_j}{F_j} \), \( j = 1, \ldots, m \), where \( \tilde{f}_j = (f_{j1}, \ldots, f_{jn}) \); \( \tilde{g}_j = \left( \frac{f_{j1}}{F_j}, \ldots, \frac{f_{jn}}{F_j} \right) \).

Notice that

\[
g_{ji}(x) = \int_{\Omega_2} k_j(x, y)f_{ji}(y)d\mu_2(y) = \int_{\Omega_2} (k_j(x, y)F_j(y))\left( \frac{f_{ji}(y)}{F_j(y)} \right)d\mu_2(y),
\]

\[
x \in \Omega_1, \text{ all } j = 1, \ldots, m; i = 1, \ldots, n.
\]

So we can write

\[
\tilde{g}_i(x) = \int_{\Omega_2} (k_j(x, y)F_j(y))\tilde{g}_j(y)d\mu_2(y), j = 1, \ldots, m.
\]

We mention

**Theorem 5** ([3], p. 481) Here we follow Remark 4. Let \( \rho \in \{1, \ldots, m\} \) be fixed. Assume that the function
is integrable on $\Omega_1$, for each $y \in \Omega_2$. Define $U_m$ on $\Omega_2$ by

$$U_m(y) := \left( \prod_{j=1}^{m} F_j(y) \right) \int_{\Omega_1} \frac{u(x) \prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} L_j(x)} \, d\mu_1(x) < \infty. \quad (32)$$

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{g_j(x)}{L_j(x)} \right) \, d\mu_1(x) \leq \prod_{j \neq \rho} \int_{\Omega_2} \Phi_j \left( \frac{\tilde{f}_j(y)}{F_j(y)} \right) \, d\mu_2(y) \cdot \int_{\Omega_2} \left( \frac{\tilde{f}_\rho(y)}{F_\rho(y)} \right) U_m(y) \, d\mu_2(y), \quad (33)$$

under the assumptions:

(i) $\frac{\tilde{f}_j}{F_j}$, $\Phi_j \left( \frac{g_j(x)}{L_j(x)} \right)$ are both $k_j(x,y) F_j(y) \, d\mu_2(y)$ -integrable, $\mu_1$ -a.e. in $x \in \Omega_1$, $j = 1, ..., m$,

(ii) $U_m \Phi_\rho \left( \frac{\tilde{f}_\rho}{F_\rho} \right)$, $\Phi_1 \left( \frac{f_1}{F_1} \right)$, $\Phi_2 \left( \frac{f_2}{F_2} \right)$, ..., $\Phi_\rho \left( \frac{f_\rho}{F_\rho} \right)$, ..., $\Phi_m \left( \frac{f_m}{F_m} \right)$, are $\mu_2$ -integrable, where $\Phi_\rho \left( \frac{f_\rho}{F_\rho} \right)$ is absent.

We also mention

**Theorem 6** ([3], p. 519) Here all as in Notation 3 and Remark 4. Assume that the functions ($j = 1, 2, ..., m \in \mathbb{N}$)

$$x \mapsto \left( \frac{u(x) k_j(x,y) F_j(y)}{K_j(x)} \right)$$

are integrable on $\Omega_1$, for each fixed $y \in \Omega_2$. Define $W_j$ on $\Omega_2$ by

$$W_j(y) := \left( \int_{\Omega_1} \frac{u(x) k_j(x,y)}{K_j(x)} \, d\mu_1(x) \right) F_j(y) < \infty, \quad (34)$$

on $\Omega_2$. 
Let \( p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}^n \to \mathbb{R}_+ \), \( j = 1,...,m \), be convex and increasing per coordinate.

Then

\[
\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{|g_j(x)|}{L_j(x)} \right) d\mu(x) \leq \prod_{j=1}^{m} \int_{\Omega_2} W_j(y) \Phi_j \left( \frac{|f_j(y)|}{F_j(y)} \right) d\mu_2(y),
\]

under the assumptions:

(i) \( \frac{f_j}{F_j} \Phi_j \left( \frac{|f_j|}{F_j} \right)^{p_j} \) are both \( k_j(x,y)F_j(y)d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \), \( j = 1,...,m \),

(ii) \( W_j \Phi_j \left( \frac{|f_j|}{F_j} \right)^{p_j} \) is \( \mu_2 \)-integrable, \( j = 1,...,m \).

We make

Remark 7 Let \( (\Omega_1, \Sigma_1, \mu_1) \) and \( (\Omega_2, \Sigma_2, \mu_2) \) be measure spaces with positive \( \sigma \)-finite measures, and let \( k : \Omega_1 \times \Omega_2 \to \mathbb{R} \) be nonnegative measurable functions, \( k(x,.) \) measurable on \( \Omega_2 \), and

\[
K(x) = \int_{\Omega_2} k(x,y) d\mu_2(y), \text{ for any } x \in \Omega_1.
\]

We assume \( K(x) > 0 \) a.e. on \( \Omega_1 \) and the weight functions are nonnegative functions on the related set. We consider measurable functions \( g_i : \Omega_1 \to \mathbb{R} \), with the representation

\[
g_i(x) = \int_{\Omega_2} k(x,y)f_i(y) d\mu_2(y),
\]

where \( f_i : \Omega_2 \to \mathbb{R} \) are measurable functions, \( i = 1,...,n \). Here \( u \) stands for a weight function on \( \Omega_1 \). So we follow Notation 3 for \( j = m = 1 \). We write here \( \tilde{g} := (g_1,...,g_n) \), \( \tilde{f} := (f_1,...,f_n) \).

We set

\[
\left\| \tilde{f}(y) \right\|_\infty := \max \{ |f_1(y)|,...,|f_n(y)| \},
\]

and

\[
\left\| \tilde{f}(y) \right\|_q := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, \quad q \geq 1.
\]

We assume that
\[ 0 < \left\| \hat{f}(y) \right\|_q < \infty, \text{ a.e. on } (a, b), \tag{37} \]

\( 1 \leq q \leq \infty \) fixed.

Let

\[ L_q(x) := \int_{t_2} k(x, y) \left\| f(y) \right\|_q d\mu_q(y), \quad x \in \Omega_1, \tag{38} \]

\( 1 \leq q \leq \infty \) fixed.

We assume \( L_q(x) > 0 \) a.e. on \( \Omega_1 \).

We further assume that the function

\[ x \mapsto \left( \frac{u(x)k(x, y)\left\| f(y) \right\|_q}{L_q(x)} \right) \tag{39} \]

is integrable on \( \Omega_1 \), for almost each fixed \( y \in \Omega_2 \).

Define \( W_q \) on \( \Omega_2 \) by

\[ W_q(y) := \left( \int_{\Omega_1} \frac{u(x)k(x, y)}{L_q(x)} d\mu_q(x) \right) \left\| f(y) \right\|_q < \infty, \tag{40} \]

a.e. on \( \Omega_2 \).

Let

\[ \gamma = \left( \frac{f_1}{\left\| \hat{f}(y) \right\|_q}, \frac{f_2}{\left\| \hat{f}(y) \right\|_q}, \ldots, \frac{f_n}{\left\| \hat{f}(y) \right\|_q} \right), \tag{41} \]

i.e. \( \gamma = \frac{\hat{f}}{\left\| \hat{f}(y) \right\|_q} \).

Here \( \Phi : \mathbb{R}_+^n \to \mathbb{R} \) is a convex and increasing per coordinate function.

We mention

**Theorem 8** ([3], p. 536) Let all here as in Remark 7. Then
\[
\int_{\Omega_1} u(x) \Phi \left( \frac{g(x)}{L_q(x)} \right) \, d\mu_1(x) \leq \int_{\Omega_2} W_q(y) \Phi \left( \frac{f(y)}{f(y)_q} \right) \, d\mu_2(y),
\]

(42)

under the assumptions:

(i) \[ \frac{f(y)}{f(y)_q} \Phi \left( \frac{f(y)}{f(y)_q} \right) \] are both \( k(x,y) f(y) \, d\mu_2(y) \)-integrable, \( \mu_1 \)-a.e. in \( x \in \Omega_1 \),

(ii) \[ W_q(y) \Phi \left( \frac{f(y)}{f(y)_q} \right) \] is \( \mu_2 \)-integrable.

Theorem 8 comes directly from Theorem 1.

We will also use:

Let \((\Omega_1, \Sigma_1, \mu_1), (\Omega_2, \Sigma_2, \mu_2)\) measure spaces with positive \( \sigma \)-finite measures, and \( k_i : \Omega_1 \times \Omega_2 \to \mathbb{R} \) are nonnegative measurable functions, with \( k_i(x,\cdot) \) measurable on \( \Omega_2 \), and measurable functions \( g_{ji} : \Omega_1 \to \mathbb{R} : \)

\[ g_{ji}(x) = \int_{\Omega_2} k_i(x,y) f_{ji}(y) \, d\mu_2(y), \]

where \( f_{ji} : \Omega_2 \to \mathbb{R} \) are measurable functions, for all \( j = 1,2; i = 1,...,m \).

**Theorem 9** ([3], p. 552) Here \( 0 < f_{ji}(y) < \infty \), a.e., \( i = 1,...,m \). Assume that the functions \( (i = 1,...,m \in \mathbb{N}) \)

\[ x \mapsto \left( \frac{u(x) k_i(x,y) f_{ji}(y)}{g_{ji}(x)} \right) \]

are integrable on \( \Omega_1 \), for each fixed \( y \in \Omega_2 \); with \( g_{ji}(x) > 0 \), a.e. on \( \Omega_1 \).

Define \( \psi_i \) on \( \Omega_2 \) by

\[ \psi_i(y) = f_{ji}(y) \int_{\Omega_1} u(x) \frac{k_i(x,y)}{g_{ji}(x)} \, d\mu_1(x) < \infty, \]

(43)
a.e. on \( \Omega_2 \).

Let \( p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1,...,m \), be convex and increasing. Then
\[
\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left( \frac{g_{1i}(x)}{g_{2i}(x)} \right) d\mu_1(x) \leq \prod_{i=1}^{m} \left( \int_{\Omega_2} \psi_i(y) \Phi_i \left( \frac{f_{1i}(y)}{f_{2i}(y)} \right)^{\frac{1}{p_i}} d\mu_2(y) \right) \frac{1}{p_i},
\]  
(44)

under the assumptions:

(i) \( \frac{f_{1i}(y)}{f_{2i}(y)} \), \( \Phi_i \left( \frac{f_{1i}(y)}{f_{2i}(y)} \right)^{\frac{1}{p_i}} \) are both \( k_i(x,y)f_{2i}(y)d\mu_2(y) \) -integrable, \( \mu_1 \) -a.e. in \( x \in \Omega_1 \),

(ii) \( \psi_i(y) \Phi_i \left( \frac{f_{1i}(y)}{f_{2i}(y)} \right)^{\frac{1}{p_i}} \) is \( \mu_2 \)-integrable, \( i = 1, \ldots, m \).

3. Main Results

We make

**Remark 10** Here \( \rho_j, \mu_j, \gamma_j, \omega_j > 0; f_{ji} \in C([a,b]) \) and \( \psi \in C^1([a,b]) \) which is increasing; \( j = 1, \ldots, m \) and \( i = 1, \ldots, n \). Set

\[
\varphi_{j+}(y) := \left\| e_{\rho_j,\mu_j,\omega_j,a+f_j}(y) \right\| \quad := \max_{j=1,\ldots,n} \left\{ e_{\rho_j,\mu_j,\omega_j,a+f_j}(y) \right\},
\]  
(45)

and

\[
\varphi_{j+}(y) := \left( \int_{[a,b]} e_{\rho_j,\mu_j,\omega_j,a+f_j}(y) \right)^{\frac{1}{q}} \quad q \geq 1;
\]  
(46)

\( y \in [a,b] \), which \( \varphi_{j+} \) are continuous functions, \( j = 1, \ldots, m \). We have that

\[
0 < q \varphi_{j+}(y) < \infty \text{ in } [a,b],
\]  
(47)

\( j = 1, \ldots, m \); where \( 1 \leq q \leq \infty \) is fixed.

Here it is

\[
k_j^+(x,y) := k_j(x,y) = \begin{cases} 
\psi'(y)(\psi(x)-\psi(y))^{\gamma_j-1} E_{\rho_j,\mu_j}^{\gamma_j} [\omega_j(\psi(x)-\psi(y))]^\gamma_j & a < y \leq x, \\
0, x < y < b, 
\end{cases}
\]  
(48)

\( j = 1, \ldots, m \), and

\[
L_{jq}^+(x) := \int_a^x \psi'(y)(\psi(x)-\psi(y))^{\mu_j-1} E_{\rho_j,\mu_j}^{\gamma_j} [\omega_j(\psi(x)-\psi(y))]^\gamma \varphi_j^+(y) dy, 
\]  
(49)

\( \forall x \in [a,b], 1 \leq q \leq \infty \).
We have that $L_{jq}^+(x) > 0$ on $[a,b]$.

Let $\rho \in \{1,\ldots,m\}$ be fixed. The weight function $u$ is chosen so that

$$U_m^+(y) := \left( \prod_{j=1}^{m} \varphi_{jq}(y) \right) \int_y^b \frac{u(x) \prod_{j=1}^{m} k_j^+(x,y)}{\prod_{j=1}^{m} L_{jq}^+(x)} \, dx < \infty,$$

(50)

\[ \forall \; y \in [a,b], \text{ and that } U_m^+ \text{ is integrable on } [a,b]. \]

A direct application of Theorem 5 gives:

**Theorem 11** It is all as in Remark 10. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+^+$, $j = 1,\ldots,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{\sum_{i=1}^{n} e^{\gamma_j \omega_j b_{i,j} y} f_i(x)}{\prod_{j=1}^{m} L_{jq}^+(x)} \right) \, dx \leq \int_a^b \prod_{j=1}^{m} \Phi_j \left( \frac{\sum_{i=1}^{n} e^{\gamma_j \omega_j b_{i,j} y} f_i(y)}{\prod_{j=1}^{m} L_{jq}^+(x)} \right) \, dy.$$

(51)

We make

**Remark 12** Here $\rho, \mu, \gamma, \omega > 0$; $f_{ji} \in C\left( [a,b] \right)$ and $\psi \in C^1\left( [a,b] \right)$ which is increasing; $j = 1,\ldots,m$ and $i = 1,\ldots,n$. Set

$$\varphi_{jq}^- := \left\{ \frac{e^{\gamma_j \omega_j b_{i,j} y} f_i(x)}{\prod_{j=1}^{m} L_{jq}^+(x)} \right\}_1 := \max_{j=1,\ldots,m} \left\{ \frac{e^{\gamma_j \omega_j b_{i,j} y} f_i(x)}{\prod_{j=1}^{m} L_{jq}^+(x)} \right\},$$

(52)

and

$$q \varphi_{jq}^- := \left( \sum_{i=1}^{n} e^{\gamma_j \omega_j b_{i,j} y} f_i(y) \right)^{1/q}, q \geq 1;$$

(53)

$y \in [a,b]$, which $q \varphi_{jq}^-$ are continuous functions, $j = 1,\ldots,m$. We have also that

$$0 < q \varphi_{jq}^- (y) < \infty \text{ in } [a,b],$$

(54)

$$j = 1,\ldots,m; \text{ where } 1 \leq q \leq \infty \text{ is fixed.}$$

Here it is
Let $\rho$ and $\varphi$, $j = 1, \ldots, m$, are convex functions increasing per coordinate. Then

\[ k_j(x, y) := k_j(x, y) = \begin{cases} \psi^j(y)(\psi(y) - \psi(x))^{\mu_j - 1} E^{\gamma_j}_{\rho_j, \mu_j} \left[ \omega_j(\psi(y) - \psi(x))^{\nu_j} \right] x \leq y < b, \\ 0, a < y < x, \end{cases} \tag{55} \]

\[ \forall \ x \in [a, b], \ 1 \leq q \leq \infty. \]

We have that $L_{j_0}^q(x) > 0$ on $[a, b]$.

Let $\rho \in \{1, \ldots, m\}$ be fixed. The weight function $u$ is chosen so that

\[ U_m^-(y) := \left( \prod_{j=1}^m \varphi_j(y) \right) \int_a^y \frac{u(x) \prod_{j=1}^m k_j(x, y)}{\prod_{j=1}^m L_{jq}^-(x)} \, dx < \infty, \tag{57} \]

\[ \forall \ y \in [a, b], \text{ and that } U_m^- \text{ is integrable on } [a, b]. \]

A direct application of Theorem 5 gives:

**Theorem 13** It is all as in Remark 12. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, are convex functions increasing per coordinate. Then

\[ \int_a^b \frac{u(x) \prod_{j=1}^m \left( \frac{f_j(x)}{\varphi_j(y)} \right)}{\prod_{j=1}^m L_{jq}^-(x)} \, dx \leq \left( \prod_{j=1}^m \Phi_j \right) \left( \int_{a}^{b} \frac{f_j(y)}{\prod_{j=1}^m \varphi_j(y)} \, dy \right) \int_{a}^{b} \frac{U_m^-(y)dy}{\prod_{j=1}^m L_{jq}^-(x)} \]. \tag{58} \]

We make

**Remark 14** Here $j = 1, \ldots, m; i = 1, \ldots, n$. Let $\rho_j, \mu_j, \omega_j > 0$, $\gamma_j < 0$, and $f_{ji} \in C_{\nu_j}^N([a, b]), N_j = \left[ \mu_j \right], \mu_j \in \mathbb{N}; \theta := \max(N_1, \ldots, N_m), \psi \in C^\theta([a, b]), \psi$ is increasing with $\psi(x) \neq 0$ over $[a, b]$. Set

\[ f_{ji}^{\psi}(x) = \left( \frac{1}{\psi(x) \, dx} \right)^{\nu_j} f_{ji}(x), \ x \in [a, b]. \]

Set

\[ \lambda_{ji}^\psi(y) := \left[ \frac{C_{\rho_j^\psi, \mu_j^\psi, \omega_j^\psi, a^\psi, f_{ji}(y)}}{\prod_{j=1}^m L_{jq}^-(x)} \right] := \max_{j=1, \ldots, m} \left\{ \frac{C_{\rho_j^\psi, \mu_j^\psi, \omega_j^\psi, a^\psi, f_{ji}(y)}}{\prod_{j=1}^m L_{jq}^-(x)} \right\}. \tag{59} \]

and
\[ q \lambda_{j^+}(y) := \left\| \sum_{i=1}^{n} D_{\rho_i,\mu_i,\omega_i,a,f_i}^{\gamma_i}(y) \right\|^q_p, \quad q \geq 1; \]

\( y \in [a,b], \) which all \( q \lambda_{j^+} \) are continuous functions, \( j = 1,\ldots,m. \) We also have that

\[ 0 < q \lambda_{j^+}(y) < \infty \text{ in } [a,b], \]

\( j = 1,\ldots,m; \) where \( 1 \leq q \leq \infty \) is fixed.

Here it is

\[ c_k^+ (x,y) := k_j(x,y) = \begin{cases} \psi'(y)(\psi(x)-\psi(y))^{N_j-\mu_j-1} E_{\rho_j,N_j-\mu_j}^{-\gamma_j} [\omega_j(\psi(x)-\psi(y))^{\rho_j}], \quad a < y \leq x, \\ 0, \quad x < y < b, \end{cases} \]

\( j = 1,\ldots,m, \) and

\[ c L^+_j(x) := \int_a^x \psi'(y)(\psi(x)-\psi(y))^{N_j-\mu_j-1} E_{\rho_j,N_j-\mu_j}^{-\gamma_j} [\omega_j(\psi(x)-\psi(y))^{\rho_j}], \quad \lambda_{j^+}(y)dy, \]

\[ \forall \ x \in [a,b], \ 1 \leq q \leq \infty, \ j = 1,\ldots,m. \]

We have that \( c L^+_j(x) > 0 \) on \( [a,b]. \)

Let \( \rho \in \{1,\ldots,m\} \) be fixed. The weight function \( u \) is chosen so that

\[ c U^+_m(y) := \left( \prod_{j=1}^{m} \lambda_{j^+}(y) \right) \frac{u(x) \prod_{j=1}^{m} c_k^+(x,y)}{\prod_{j=1}^{m} c L^+_j(x)} dx < \infty, \]

\( \forall \ y \in [a,b], \) and that \( c U^+_m \) is integrable on \( [a,b]. \)

A direct application of Theorem 11, see also (6), gives:

**Theorem 15** It is all as in Remark 14. Here \( \Phi_j : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+, \ j = 1,\ldots,m, \) are convex functions increasing per coordinate. Then
\[ \int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_j \left( \frac{c D_{\rho_j,\mu_j,\omega_j,\alpha_j} \cdot f_j(x)}{c L_{\bar{r}_j}(x)} \right) \, dx \leq \left\{ \begin{array}{l} \int_{a}^{b} \prod_{j=1}^{m} \Phi_j \left( \frac{f_{j,W}^{N_j}(y)}{q \lambda_{j,W}(y)} \right) \, dy \quad j \neq p, \\
 \int_{a}^{b} \Phi_p \left( \frac{f_{j,W}^{N_p}(y)}{q \lambda_{p,W}(y)} \right) \, U_m(y) \, dy. \end{array} \right. \] (65)

We make

**Remark 16** Here \( j = 1, \ldots, m; \ i = 1, \ldots, n. \) Let \( \rho_j, \mu_j, \omega_j > 0, \gamma_j < 0, \) and \( f_{j,W} \in C_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j}([a,b]), N_j = [\mu_j], \mu_j \in \mathbb{N}; \)
\( \theta := \max(N_1, \ldots, N_n), \psi \in C^0([a,b]), \psi \) is increasing with \( \psi'(x) \neq 0 \) over \([a, b].\) Set
\[ f_{j,W}^{N_j}(x) = \left( \frac{1}{\psi'(x)} \right)^{N_j} f_{j,W}(x), \; x \in [a, b]. \] Set
\[ \lambda_{j,-}(y) := \left\| c D_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j,\gamma_j,\omega_j} \cdot f_j(y) \right\|_r := \max_{i=1, \ldots, n} \left\{ c D_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j,\gamma_j,\omega_j} \cdot f_j(y) \right\}, \] (66)
and
\[ q \lambda_{j,-}(y) := \left\| c D_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j,\gamma_j,\omega_j} \cdot f_j(y) \right\|_q := \left( \sum_{i=1}^{n} c D_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j,\gamma_j,\omega_j} \cdot f_j(y)^q \right)^{\frac{1}{q}}, \; q \geq 1; \] (67)
\( y \in [a, b], \) which all \( q \lambda_{j,-} \) are continuous functions, \( j = 1, \ldots, m. \) We also have that
\[ 0 < q \lambda_{j,-}(y) < \infty \text{ in } [a, b], \] (68)
\( j = 1, \ldots, m; \) where \( 1 \leq q \leq \infty \) is fixed.

Here it is
\[ c k_{j}(x, y) := k_j(x, y) = \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{\gamma_j,\omega_j,\alpha_j} E_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j} [\omega_j (\psi(y) - \psi(x))^{\gamma_j}] \; x \leq y < b, \\
0, \; a < y < x, \end{cases} \] (69)
\( j = 1, \ldots, m, \) and
\[ c L_{j,W}(x) := \int_{b}^{a} \psi'(y)(\psi(y) - \psi(x))^{\gamma_j,\omega_j,\alpha_j} E_{\rho_j,\mu_j,\omega_j,\alpha_j}^{\gamma_j,\omega_j,\alpha_j} [\omega_j (\psi(y) - \psi(x))^{\gamma_j}] \; \lambda_{j,-}(y) \, dy, \] (70)
\( \forall \; x \in [a, b], \; 1 \leq q \leq \infty, \; j = 1, \ldots, m. \)

We have that \( c L_{j,W}(x) > 0 \) on \([a, b].\)
Let \( \rho \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that

\[
C U_m^{-}(y) := \left( \prod_{j=1}^{m} \lambda_{j-}(y) \right) \int_a^y \frac{u(x) \prod_{j=1}^{m} k_{j}(x, y)}{\prod_{j=1}^{m} L_{j}(x)} \, dx < \infty,
\]

\( \forall \ y \in [a, b] \), and that \( C U_m^{-} \) is integrable on \([a, b]\).

A direct application of Theorem 13, see also (7), gives:

**Theorem 17** It is all as in Remark 16. Here \( \Phi_j : \mathbb{R}^+ \to \mathbb{R}^+ \), \( j=1,\ldots,m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^{m} \Phi_j \left( \int_a^b \frac{\prod_{j=1}^{m} f_{j}(x)}{\prod_{j=1}^{m} \lambda_{j}(y)} \, dy \right) \, dx \leq \int_a^b \prod_{j=1}^{m} \Phi_j \left( \int_a^b \frac{\prod_{j=1}^{m} f_{j}(x)}{\prod_{j=1}^{m} \lambda_{j}(y)} \, dy \right) \, dy.
\]

We make

**Remark 18** Here \( j=1,\ldots,m \); \( i=1,\ldots,n \). Let \( \rho_j, \mu_j, \omega_j > 0 \), \( \gamma_j < 0 \), and \( f_{ji} \in C([a, b]) \), \( N_j = \begin{bmatrix} \mu_j \end{bmatrix} \), \( \mu_j \notin \mathbb{N} ; \theta := \max(N_1,\ldots,N_m) \), \( \psi \in C^\theta([a, b]) \), \( \psi \) is increasing with \( \psi'(x) \neq 0 \) over \([a, b]\). Here \( 0 \leq \beta_j \leq 1 \) and \( \xi_j = \mu_j + \beta_j (N_j - \mu_j) \). We assume that \( RL D_{\rho_j, \mu_j, \omega_j, a, f_{ji}} \in C([a, b]), j=1,\ldots,m, i=1,\ldots,n \). Set

\[
\alpha M_j(\gamma) := \left\| RL D_{\rho_j, \mu_j, \omega_j, a, f_{ji}}(\gamma) \right\|_{\infty} := \max_{j=1,\ldots,m} \left\{ \left\| RL D_{\rho_j, \mu_j, \omega_j, a, f_{ji}}(\gamma) \right\| \right\},
\]

and

\[
q M_j(\gamma) := \left\| RL D_{\rho_j, \mu_j, \omega_j, a, f_{ji}}(\gamma) \right\|_{q} := \left( \sum_{i=1}^{n} \left\| RL D_{\rho_j, \mu_j, \omega_j, a, f_{ji}}(\gamma) \right\|^{q} \right)^{\frac{1}{q}}, \quad q \geq 1;
\]

\( y \in [a, b] \), which all \( q M_j \) are continuous functions, \( j=1,\ldots,m \). We also have that

\[
0 < q M_j(\gamma) < \infty \text{ in } [a, b],
\]

\( j=1,\ldots,m \); where \( 1 \leq q \leq \infty \) is fixed.

Here it is
\[ p k_j^+(x, y) := k_j(x, y) = \begin{cases} \\
\psi'(y)(\psi(x) - \psi(y))^j_{j-\mu_j^{-1}} E^{\frac{j_{j-\mu_j^{-1}}}{\psi'(y)} j_{j-\mu_j^{-1}}} E^{\frac{j_{j-\mu_j^{-1}}}{\psi'(y)} j_{j-\mu_j^{-1}}} [\omega_j(\psi(x) - \psi(y))^{p_j}] & a < y \leq x, \\
0, & x < y < b, 
\end{cases} \tag{76} \]

\[ j = 1, \ldots, m, \text{ and} \]

\[ p L_{jq}^+(x) := \int_a^b \psi'(y)(\psi(x) - \psi(y))^{j_{j-\mu_j^{-1}}} E^{\frac{j_{j-\mu_j^{-1}}}{\psi'(y)} j_{j-\mu_j^{-1}}} [\omega_j(\psi(x) - \psi(y))^{p_j}] M_{j+}(y) dy, \tag{77} \]

\[ \forall \; x \in [a, b], \; 1 \leq q \leq \infty. \]

We have that \( p L_{jq}^+(x) > 0 \; \text{on} \; [a, b]. \)

Let \( \overline{\rho} \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that

\[ p U_{m}^+(y) := \left( \prod_{j=1}^m M_{j+}(y) \right) \int_a^b \frac{u(x) \prod_{j=1}^m p k_j^+(x, y)}{\prod_{j=1}^m p L_{jq}^+(x)} dx < \infty, \tag{78} \]

\[ \forall \; y \in [a, b], \text{and that} \; p U_{m}^+ \; \text{is integrable on} \; [a, b]. \]

A direct application of Theorem 11, see also (15), gives:

**Theorem 19** It is all as in Remark 18. Here \( \Phi_j : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m, \; j = 1, \ldots, m, \) are convex functions increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \Phi_j \begin{cases} \left( \frac{\mu D_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} f_j(x)}{p j_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} a f_j(x)} \right) \right] dx \leq \left( \prod_{j \neq \overline{\rho}} p L_{jq}^+ \right) \begin{cases} \left( \frac{RL D_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} f_j(y)}{q M_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} a f_j(y)} \right) \right] dy \end{cases} \tag{79} \]

\[ \int_a^b \Phi_{-p} \begin{cases} \left( \frac{RL D_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} f_j(y)}{q M_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} a f_j(y)} \right) \right] \begin{cases} \left( \frac{p U_{m}^+(y)}{q M_{j_1, j_2, \ldots, j_m}^{(1-\beta_j)} a f_j(y)} \right) \right] dy \end{cases}. \]

We make
Remark 20 Here \( j = 1, \ldots, m; i = 1, \ldots, n \). Let \( \rho_j, \mu_j, \omega_j > 0, \gamma_j < 0, \) and \( f_{ji} \in C([a, b]), N_j = \lfloor \mu_j \rfloor, \mu_j \notin \mathbb{N} \); \( \theta := \max(N_1, \ldots, N_m), \psi \in C^\theta([a, b]), \psi \) is increasing with \( \psi(x) \neq 0 \) over \([a, b]\). Here \( 0 \leq \beta_j \leq 1 \) and \( \xi_j = \mu_j + \beta_j(N_j - \mu_j) \). We assume that \( \int_0^r D_{\rho_j, \beta_j, \omega_j, b-f_{ji}} \in C([a, b]), j = 1, \ldots, m, i = 1, \ldots, n \). Set
\[
\mu_{j-}(y) = \max_{j=1, \ldots, m} \left\{ \int_0^r D_{\rho_j, \beta_j, \omega_j, b-f_{ji}}(y) \right\},
\]
and
\[
\nu_{j-}(y) = \left( \sum_{i=1}^n \int_0^r D_{\rho_j, \beta_j, \omega_j, b-f_{ji}}(y) \right)^{\frac{1}{q}}, q \geq 1;
\]
y \( \in [a, b] \), which all \( \nu_{j-} \) are continuous functions, \( j = 1, \ldots, m \). We also have that
\[
0 < \nu_{j-}(y) < \infty \text{ in } [a, b],
\]
\( j = 1, \ldots, m \), where \( 1 \leq q \leq \infty \) is fixed.

Here it is
\[
pk_j(x, y) := k_j(x, y) = \begin{cases} 
\psi'(y)(\psi(y) - \psi(x))^{j-\mu_1-1} E_{\rho_j, \beta_j, \omega_j} \left[ \omega_j(\psi(y) - \psi(x))^{\gamma_j} \right], & x \leq y < b, \\
0, & a < y < x,
\end{cases}
\]
\( j = 1, \ldots, m \), and
\[
pL_{jq}(x) := \int_x^b \psi'(y)(\psi(y) - \psi(x))^{j-\mu_1-1} E_{\rho_j, \beta_j, \omega_j} \left[ \omega_j(\psi(y) - \psi(x))^{\gamma_j} \right] M_{j-}(y) dy,
\]
\( \forall x \in [a, b], 1 \leq q \leq \infty \).

We have that \( pL_{jq}(x) > 0 \) on \([a, b]\).

Let \( \bar{\rho} \in \{1, \ldots, m\} \) be fixed. The weight function \( u \) is chosen so that
\[
pU_m^-(y) := \left( \prod_{j=1}^m M_{j-}(y) \right) \int_a^y \frac{u(x) \prod_{j=1}^m pk_j(x, y)}{\prod_{j=1}^m pL_{jq}(x)} dx < \infty,
\]
\( \forall y \in [a, b], \) and that \( pU_m^- \) is integrable on \([a, b]\).

A direct application of Theorem 13, see also (16), gives:
Theorem 21 It is all as in Remark 20. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{H D_{\rho_j, \mu_j, a_j, b_j} (f_j(x))}{L_{\frac{1}{\rho_j}}(x)} \right) dx \leq \prod_{j=1}^m \Phi_j \left( \frac{RL D_{\rho_j, \mu_j, a_j, b_j} (f_j(y))}{M_j(y)} \right) dy,$$

(86)

We make

Remark 22 The basic background here is as in Remark 10. Also $q \varphi_{j+}(y)$, $1 \leq q \leq \infty$, $y \in [a, b]$ is as in (45), (46), (47); $k_j^+(x, y)$ is as (48) and $L_{\frac{1}{\rho_j}}(x)$ as in (49), where $x, y \in [a, b]$. Here it is

$$K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^{\mu_j} E_{\rho_j, \mu_j} \left[ \omega_j (\psi(x) - \psi(a))^{\nu_j} \right],$$

(87)

$\forall$ $x \in [a, b]$, $j = 1, \ldots, m$. Indeed it is

$$\frac{k_j^+(x, y)}{K_j^+(x)} = \frac{\chi(x, y) \nu_j \left( \frac{\psi(x) - \psi(y)}{\psi(x) - \psi(a)} \right)^{\nu_j-1}}{\left( \frac{\psi(x) - \psi(y)}{\psi(x) - \psi(a)} \right)^{\nu_j}} \frac{E_{\rho_j, \mu_j} \left[ \omega_j (\psi(x) - \psi(y))^{\nu_j} \right]}{E_{\rho_j, \mu_j} \left[ \omega_j (\psi(x) - \psi(a))^{\nu_j} \right]}.,$$

(88)

$\forall$ $x, y \in [a, b]$, $j = 1, \ldots, m$; $\chi$ is the characteristic function.

We define $q W_{j+}$ on $[a, b]$, with appropriate choice of weight function $u$, by

$$q W_{j+} (y) := q \varphi_{j+} (y) \left( \int_y^b u(x) \frac{k_j^+(x, y)}{K_j^+(x)} dx \right) < \infty,$$

(89)

$\forall$ $y \in [a, b]$, and that $q W_{j+}$ is integrable on $[a, b]$; $j = 1, \ldots, m$.

A direct application of Theorem 6, see also (2), follows:

Theorem 23 It is all as in Remark 22. Let $p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, be convex and increasing per coordinate. Then
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Theorem 25

A direct application of Theorem 6, see also (3), follows:

We define \( W_{j-} \) on \([a, b]\), with appropriate choice of weight function \( u \), by

\[
W_{j-}(y) := \phi_{j-}(y) \left( \int_a^y u(x) k_{j-}(x, y) \, dx \right) < \infty,
\]

\( \forall \ y \in [a, b] \), and that \( W_{j-} \) is integrable on \([a, b]; j = 1,...,m.\)

A direct application of Theorem 6, see also (3), follows:

Theorem 25

It is all as in Remark 24. Let \( p_j > 1; \sum_{j=1}^m \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \), \( j = 1,...,m \), be convex and increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\int_{L_{j-}^+} e^{\gamma_{j-}^{p_j} p_j f_j(x)}}{L_{j-}^+} dx \right) \leq \prod_{j=1}^m \int_a^b W_{j-}(y) \Phi_j \left( \frac{f_j(y)}{q \phi_{j-}(y)} \right) dy \right)^{1/p_j}. \tag{94}
\]

We need

Remark 26

The basic background here is as in Remark 14. Also \( \lambda_{j+}(y), 1 \leq q \leq \infty, y \in [a, b] \) is as in (59), (60), (61); \( k_j^+(x, y) \) is as (62) and \( L_{j+}^+(x) \) as in (63), where \( x, y \in [a, b] \). Here it is
\[ C K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^{\rho_j - \mu_j} E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(a))^{\rho_j} \right] \]  
(95)

\[ \forall \ x \in [a, b], \ j = 1, \ldots, m. \]  
Indeed it is

\[ \frac{C}{C} \frac{k_j^+(x, y)}{K_j^+(x)} = \left( \frac{\chi_{[a, b]}(y) \psi'(y)}{\psi(x) - \psi(a)} \right)^{\rho_j - \mu_j} \left( \frac{E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(a))^{\rho_j} \right]}{E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(x) - \psi(a))^{\rho_j} \right]} \right). \]  
(96)

\[ \forall \ x, y \in [a, b], \ j = 1, \ldots, m. \]

We define \( q W_{j+} \) on \([a, b]\), with appropriate choice of weight function \( u \), by

\[ q W_{j+}(y) := q \lambda_{j+}(y) \left( \int y u(x) C k_j^+(x, y) dx \right) < \infty, \]  
(97)

\[ \forall \ y \in [a, b], \text{ and that } q W_{j+} \text{ is integrable on } [a, b]; \ j = 1, \ldots, m. \]

A direct application of Theorem 23, see also (6), follows:

**Theorem 27** It is all as in Remark 26. Let \( p_j > 1; \sum \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+, \ j = 1, \ldots, m \), be convex and increasing per coordinate. Then

\[ \int_a^b u(x) \prod_{j=1}^m \left[ \frac{D_{\rho_j, \mu_j, \omega_j, a+f_j}(x)}{L_{f_j}(x)} \right] dx \leq \prod_{j=1}^m \int_{a_q}^{q W_{j+}(y)} \Phi_j \left( \frac{f_{j+}^{[N_j]}(y)}{q \lambda_{j+}(y)} \right)^{\frac{1}{p_j}} dy. \]  
(98)

We need

**Remark 28** The basic background here is as in Remark 16. Also \( q \lambda_{j-}(y), \ 1 \leq q \leq \infty, \ y \in [a, b] \) is as in (66), (67), (68); \( C k_j^-(x, y) \) is as (69) and \( C L_{f_j}(x) \) as in (70), where \( x, y \in [a, b] \). Here it is

\[ C K_j^-(x) := K_j(x) = (\psi(b) - \psi(x))^{\rho_j - \mu_j} E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(b) - \psi(x))^{\rho_j} \right] \]  
(99)

\[ \forall \ x \in [a, b], \ j = 1, \ldots, m. \]  
Indeed it is

\[ \frac{C}{C} \frac{k_j^-(x, y)}{K_j^-(x)} = \left( \frac{\chi_{[a, b]}(y) \psi'(y)}{(\psi(b) - \psi(x))^{\rho_j - \mu_j}} \right)^{\rho_j - \mu_j} \left( \frac{E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(b) - \psi(x))^{\rho_j} \right]}{E_{\rho_j, N_j - \mu_j + 1}^{-\tau_j} \left[ \omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right). \]  
(100)

\[ \forall \ x, y \in [a, b], \ j = 1, \ldots, m. \]
We define \( q^c W_j \) on \([a,b]\), with appropriate choice of weight function \( u \), by

\[
c^q W_j(y) := \lambda_j(y) \left( \int_a^y u(x) c^q k_j(x, y) \frac{dx}{c K_j(x)} \right) < \infty,
\]

(101)

\( \forall \ y \in [a,b] \), and that \( q^c W_j \) is integrable on \([a,b] \); \( j = 1, \ldots, m \).

A direct application of Theorem 25, see also (7), follows:

**Theorem 29** It is all as in Remark 28. Let \( p_j > 1: \sum_{j=1}^m \frac{1}{p_j} = 1 \). Let the functions \( \Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+, \ j = 1, \ldots, m \), be convex and increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \left[ \frac{c D^j \varphi}{\rho_j \mu_j^j} \right] \left[ \frac{c L^j_k(x)}{\rho_j \mu_j^j(x)} \right] \right) \ dx \leq \prod_{j=1}^m \int_a^b W_j(y) \Phi_j \left( \left[ \frac{\rho_j^q \lambda_j^j(y)}{\rho_j^q \lambda_j^j(y)} \right] \right) ^{p_j} \left[ \frac{1}{\rho_j} \right] ^{p_j} \ dy.
\]

(102)

We need

**Remark 30** The basic background here is as in Remark 18. Also \( q M_j^+(y), \ 1 \leq q \leq \infty, \ y \in [a,b] \) is as in (73), (74), (75); \( p k_j^+(x, y) \) is as (76) and \( p L^j_k(x) \) as in (77), where \( x, y \in [a,b] \). Here it is

\[
p K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^\eta_j \mu_j^j \left[ \alpha_j (\psi(x) - \psi(a))^\eta_j \right]
\]

(103)

\( \forall \ x \in [a,b] \), \( j = 1, \ldots, m \). Indeed it is

\[
\frac{p k_j^+(x, y)}{p K_j^+(x)} = \frac{\varphi^+ (y) (\psi(x) - \psi(y))^\eta_j \mu_j^j \left[ \alpha_j (\psi(x) - \psi(y))^\eta_j \right]}{(\psi(x) - \psi(a))^\eta_j \mu_j^j \left[ \alpha_j (\psi(x) - \psi(a))^\eta_j \right]},
\]

(104)

\( \forall \ x, y \in [a,b] \), \( j = 1, \ldots, m \).

We define \( q^p W_j \) on \([a,b]\), with appropriate choice of weight function \( u \),

\[
p^q W_j(y) := q M_j^+(y) \left( \int_y^b u(x) p^q k_j^+(x, y) \frac{dx}{p K_j^+(x)} \right) < \infty,
\]

(105)

\( \forall \ y \in [a,b] \), and that \( q^p W_j \) is integrable on \([a,b] \); \( j = 1, \ldots, m \).

A direct application of Theorem 23, see also (15), follows:
Theorem 31 It is all as in Remark 30. Here \( \Phi_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\mu \mathbf{D}^{\gamma_j, \beta_j \mu \omega_j \alpha_j, b_j - f_j(x)}}{p_j L_{jq}(x)} \right) \, dx \leq \prod_{j=1}^m \left( \int_a^b W_{j+}(y) \Phi_j \left( \frac{\mathbb{R} L_{jq}(x)}{\mathbb{R} M_{jq}(y)} \right)^{\frac{1}{p_j}} \right)^{\frac{1}{p_j}}.
\]  

(106)

We need

Remark 32 The basic background here is as in Remark 20. Also \( q M_{j-}(y), 1 \leq q \leq \infty, y \in [a, b] \) is as in (80), (81), (82); \( p_j k_j(x, y) \) is as in (83) and \( p_j L_{jq}(x) \) as in (84), where \( x, y \in [a, b] \). Here it is

\[
p_j K_j(x) := K_j(x) = (\psi(b) - \psi(x))^{\gamma_j-1} \beta_j E^{\gamma_j, \beta_j}_{\rho_j, \gamma_j+1} \left[ \omega_j \left( \psi(b) - \psi(x) \right)^{\beta_j} \right].
\]  

(107)

\( \forall \ x \in [a, b], j = 1, \ldots, m \). Indeed it is

\[
p_j k_j(x, y) = \left( \chi_{x, b}(y) \psi \left( \frac{\psi(y) - \psi(x)}{\psi(b) - \psi(x)} \right)^{\gamma_j-1} \beta_j \right) \left( \frac{E^{\gamma_j, \beta_j}_{\rho_j, \gamma_j+1} \left[ \omega_j \psi(y) - \psi(x) \right]^{\beta_j}}{E^{\gamma_j, \beta_j}_{\rho_j, \gamma_j+1} \left[ \omega_j \psi(b) - \psi(x) \right]^{\beta_j}} \right).
\]  

(108)

\( \forall \ x, y \in [a, b], j = 1, \ldots, m \).

We define \( p_j W_{j-} \) on \([a, b]\), with appropriate choice of weight function \( u \),

\[
p_j W_{j-}(y) := q M_{j-}(y) \left( \int_a^b u(x) p_j k_j(x, y) \frac{1}{p_j K_j(x)} \, dx \right) < \infty,
\]  

(109)

\( \forall \ y \in [a, b] \), and that \( p_j W_{j-} \) is integrable on \([a, b] \); \( j = 1, \ldots, m \).

A direct application of Theorem 25, see also (16), follows:

Theorem 33 It is all as in Remark 32. Here \( \Phi_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \), \( j = 1, \ldots, m \), are convex functions increasing per coordinate. Then

\[
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\mu \mathbf{D}^{\gamma_j, \beta_j \mu \omega_j \alpha_j, b_j - f_j(x)}}{p_j L_{jq}(x)} \right) \, dx \leq \prod_{j=1}^m \left( \int_a^b W_{j+}(y) \Phi_j \left( \frac{\mathbb{R} L_{jq}(x)}{\mathbb{R} M_{jq}(y)} \right)^{\frac{1}{p_j}} \right)^{\frac{1}{p_j}}.
\]  

(110)
We make

**Remark 34** Let \( f_i \in C([a, b]) \), \( i = 1, \ldots, n \), and \( \bar{f} = (f_1, \ldots, f_n) \). We set

\[
\left\| \bar{f}(y) \right\|_q := \max\{\|f_1(y)\|, \ldots, \|f_n(y)\|\},
\]

and

\[
\left\| \bar{f}(y) \right\|_q := \left( \sum_{i=1}^n |f_i(y)|^q \right)^{1/q}, q \geq 1; y \in [a, b].
\]

Clearly it is \( \|\bar{f}(y)\|_q \in C([a, b]) \), for all \( 1 \leq q \leq \infty \). We assume that \( \|\bar{f}(y)\|_q > 0 \), a.e. on \((a, b)\), for \( q \in [1, \infty] \) being fixed.

Let

\[
L_q^+(x) := \int_a^x k^+(x, y) \left\| \bar{f}(y) \right\|_q \, dy, x \in [a, b],
\]

\( 1 \leq q \leq \infty \) fixed.

We assume \( L_q^+(x) > 0 \) a.e. on \((a, b)\).

Here we considered

\[
k^+(x, y) := k(x, y) := \begin{cases} \psi(y) \left( \psi(x) - \psi(y) \right)^{\mu-1} E_{\rho, \mu}^+ \left[ \omega(\psi(x) - \psi(y)) \right], & a < y \leq x, \\ 0, & 0 < y < b, \end{cases}
\]

where \( \rho, \mu, \gamma, \omega > 0; \psi \in C^1([a, b]) \) which is increasing.

The weight function \( u \) is chosen so that

\[
W_q^+(y) := \left\| \bar{f}(y) \right\|_q \left( \int_y^b u(x) \frac{k^+(x, y)}{L_q^+(x)} \, dx \right) < \infty,
\]

a.e. on \((a, b)\) and that \( W_q^+ \) is integrable on \([a, b]\).

A direct application of Theorem 8 produces:

**Theorem 35** Let all as in Remark 34. Here \( \Phi: \mathbb{R}^+ \rightarrow \mathbb{R} \) is a convex and increasing per coordinate function. Then
\[ \int_a^b u(x) \Phi \left( \frac{e^{x^+W_{q} - H}}{L_q(x)} \right) dx \leq \int_a^b W_q^+ (y) \Phi \left( \frac{f(y)}{[f(y)]_q} \right) dy. \quad (115) \]

We make

**Remark 36** Let \( f_i \in C([a, b]), \ i = 1, ..., n \), and \( \overrightarrow{f} = (f_1, ..., f_n) \). We set

\[ \|\overrightarrow{f}(y)\|_q := \max \{\|f_1(y)\|, ..., |f_n(y)|\}, \]

\[ \|\overrightarrow{f}(y)\| := \left( \sum_{i=1}^n |f_i(y)|^q \right)^{1/q}, \ q \geq 1; \ y \in [a, b]. \quad (116) \]

Clearly it is \( \|\overrightarrow{f}(y)\| \in C([a, b]) \), for all \( 1 \leq q \leq \infty \). We assume that \( \|\overrightarrow{f}(y)\| > 0 \), a.e. on \( (a, b) \), for \( q \in [1, \infty] \) being fixed.

Let

\[ L_q^-(x) := \int_x^b k^{-} (x, y) \|\overrightarrow{f}(y)\|_q \ dy, \ x \in [a, b], \quad (117) \]

\( 1 \leq q \leq \infty \) fixed.

We assume \( L_q^-(x) > 0 \) a.e. on \( (a, b) \).

Here we considered

\[ k^-(x, y) := \begin{cases} k(x, y) := & \psi'(y)(\psi(y) - \psi(x))^{q-1} E_{\rho, \mu, \omega}^{q} \left[ \phi(\psi(y) - \psi(x)) \right] \quad x \leq y < b, \\ 0, & a < y < x, \end{cases} \quad (118) \]

where \( \rho, \mu, \gamma, \omega > 0; \ \psi \in C^1([a, b]) \) which is increasing.

The weight function \( u \) is chosen so that

\[ W_q^-(y) := \|\overrightarrow{f}(y)\| \left( \int_a^y \frac{u(x)k^-(x, y)}{L_q(x)} dx \right) < \infty, \quad (119) \]

a.e. on \( (a, b) \) and that \( W_q^- \) is integrable on \([a, b]\).

A direct application of Theorem 8 produces:

**Theorem 37** Let all as in Remark 36. Here \( \Phi : \mathbb{R}^n_+ \to \mathbb{R} \) is a convex and increasing per coordinate function. Then
Next we deal with the spherical shell:

Background 38 We need:

Let $N \geq 2$, $S^{N-1} := \{ x \in \mathbb{R}^N : |x| = 1 \}$ the unit sphere on $\mathbb{R}^N$, where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Also denote the ball $\mathcal{B}(0,R) := \{ x \in \mathbb{R}^N : |x| < R \} \subseteq \mathbb{R}^N$, $R > 0$, and the spherical shell

$$A := \mathcal{B}(0,R_2) - \mathcal{B}(0,R_1), 0 < R_1 < R_2.$$  

(121)

For the following see [12, pp. 149-150], and [13, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$, and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Clearly here

$$\mathbb{R}^N - \{0\} = (0, \infty) \times S^{N-1},$$  

and

$$\overline{A} = [R_1, R_2] \times S^{N-1}.$$  

(123)

We will be using

**Theorem 39** ([1, p. 322]) Let $f: \overline{A} \to \mathbb{R}$ be a Lebesgue integrable function. Then

$$\int_{\overline{A}} f(x) \, dx = \int_{S^{N-1}} \left( \int_{R_1}^{R_2} f(r\omega) r^{N-1} dr \right) d\omega.$$  

(124)

So we are able to write an integral on the shell in polar form using the polar coordinates $(r, \omega)$.

We need

**Definition 40** Let $\rho, \mu, \gamma, w > 0$; $f \in C(\overline{A})$ and $\psi \in C^1([R_1, R_2])$ which is increasing. The left and right radial Prabhakar fractional integrals with respect to $\psi$ are defined as follows:

$$\left( e^{\gamma w}_{\rho,\mu,w,R_1} f \right)(x) = \int_{R_1}^{r} \psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho,\mu}^{\gamma} \left[ w(\psi(r) - \psi(t))^\mu \right] f(t\omega) \, dt,$$  

(125)

and

$$\left( e^{\gamma w}_{\rho,\mu,w,R_2} f \right)(x) = \int_{R_2}^{r} \psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho,\mu}^{\gamma} \left[ w(\psi(t) - \psi(r))^\mu \right] f(t\omega) \, dt,$$  

(126)

where $x \in \overline{A}$, that is $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.  


Based on [1], p. 288 and [2, 4], we have that (125), (126) are continuous functions over \( \overline{A} \) when \( \mu \geq 1 \).

We make

**Remark 41** Let \( f_i \in C(\overline{A}) \), where the shell \( A \) is as in (121), \( i = 1, \ldots, n \), and \( \overline{f} = (f_1, \ldots, f_n) \). We set

\[
\begin{align*}
\| f(y) \|_\infty &:= \max \{ |f_1(y)|, \ldots, |f_n(y)| \}, \\
\| f(y) \|_q &:= \left( \sum_{i=1}^n |f_i(y)|^q \right)^{\frac{1}{q}}, \quad q \geq 1; \ y \in \overline{A}.
\end{align*}
\]

(127)

Clearly it is \( \| f(y) \|_q \in C(\overline{A}) \), \( 1 \leq q \leq \infty \). One can write that

\[
\| f(y) \|_q = \| f(t\omega) \|_q, \quad 1 \leq q \leq \infty,
\]

(128)

where \( t \in [R_1, R_2] \), \( \omega \in S^{N-1} \); \( y = t\omega \), by Background 38.

We assume that \( \| f(y) \|_q > 0 \) on \( \overline{A} \), \( 1 \leq q \leq \infty \) fixed.

Consider the kernel

\[
k^*_r(r; t) := k(r, t) := \chi_{[R_1, r]}(t) \gamma(t) (\rho \gamma(t) - \rho(t))^\mu E_{\nu, \mu}^\gamma \left[ \omega(\rho(t) - \rho(t))^\nu \right]
\]

(129)

where \( \rho, \mu, \gamma, w > 0; \gamma \in C^1([R_1, R_2]) \) which is increasing.

Let

\[
L_{q*}(x) = L_{q*}(r\omega) = \int_{R_1}^{R_2} k^*_r(r, t) \| f(t\omega) \|_q \, dt,
\]

(130)

\( x = r\omega \in \overline{A} \), \( 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2] \), \( \omega \in S^{N-1} \).

We have that \( L_{q*}(r\omega) > 0 \) for \( r \in (R_1, R_2) \), for every \( \omega \in S^{N-1} \).

Here we choose the weight \( u(x) = u(r\omega) = L_{q*}(r\omega) \).

Consider the function

\[
W_{q*}(y) = W_{q*}(t\omega) = \| f(t\omega) \|_q \left( \int_{R_1}^{R_2} k^*_r(r, t) \, dr \right) < \infty,
\]

(131)

\( \forall \ t \in [R_1, R_2] \), \( \omega \in S^{N-1} \); and \( W_{q*}(t\omega) \) is integrable over \( [R_1, R_2] \), \( \forall \ \omega \in S^{N-1} \).

Here \( \Phi : R^n_+ \to R \) is a convex and increasing per coordinate function. By (115) we obtain
\[ \int_{R_1}^{R_2} L_{q^*}(r \omega) \Phi \left( \frac{e^{r \omega}}{\rho, \mu, w, R_1^+ + f(r \omega)} \right) dr \leq \int_{R_1}^{R_2} W_{q^*}^+(t \omega) \Phi \left( \frac{f(t \omega)}{\| f(t \omega) \|_q} \right) dt, \quad (132) \]

\forall \ \omega \in S^{N-1}.

Here we have \( R_1 \leq r \leq R_2 \), and \( R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1} \), and \( R_1^{1-N} \leq r^{1-N} \leq R_2^{1-N} \), also \( r^{N-1} r^{1-N} = 1 \). Thus by (132), we have

\[ \int_{R_1}^{R_2} L_{q^*}(r \omega) \Phi \left( \frac{e^{r \omega}}{\rho, \mu, w, R_1^+ + f(r \omega)} \right) r^{N-1} dr \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{R_1}^{R_2} W_{q^*}^+(r \omega) \Phi \left( \frac{f(r \omega)}{\| f(r \omega) \|_q} \right) r^{N-1} dr, \quad (133) \]

\forall \ \omega \in S^{N-1}.

Therefore it holds

\[ \int_{S^{N-1}} \left( \int_{R_1}^{R_2} L_{q^*}(r \omega) \Phi \left( \frac{e^{r \omega}}{\rho, \mu, w, R_1^+ + f(r \omega)} \right) r^{N-1} dr \right) d\omega \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{S^{N-1}} \left( \int_{R_1}^{R_2} W_{q^*}^+(r \omega) \Phi \left( \frac{f(r \omega)}{\| f(r \omega) \|_q} \right) r^{N-1} dr \right) d\omega. \quad (134) \]

Using Theorem 39 we derive:

**Theorem 42** All as in Remark 41. Then

\[ \int A L_{q^*}(x) \Phi \left( \frac{e^{r \omega}}{\rho, \mu, w, R_1^+ + f(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int A W_{q^*}^+(x) \Phi \left( \frac{f(x)}{\| f(x) \|_q} \right) dx, \quad (135) \]

where \( e^{r \omega} = (e^{r \omega}, e^{r \omega} f_1(x), ..., e^{r \omega} f_n(x)) \) and coordinates are assumed to be continuous functions on \( \overline{A} \).

We make

**Remark 43** Let \( f_i \in C(\overline{A}) \), where the shell \( A \) is as in (121), \( i = 1, ..., n \), and \( \bar{f} = (f_1, ..., f_n) \). We set

\[ \left\| \bar{f}(y) \right\|_\infty := \max \| f_i(y) \| \],

and

\[ \left\| \bar{f}(y) \right\|_q := \left( \sum_{i=1}^n \| f_i(y) \|^q \right)^{\frac{1}{q}}, \quad q \geq 1; \quad y \in \overline{A}. \quad (136) \]

Clearly it is \( \left\| \bar{f}(y) \right\|_q \in C(\overline{A}) \), \( 1 \leq q \leq \infty \). One can write that
\[
\left\| \overline{f}(y) \right\| = \left\| f(t\omega) \right\|, \quad 1 \leq q \leq \infty,
\]

where \( t \in [R_1, R_2], \omega \in S^{N-1}; \ y = t\omega \), by Background 38.

We assume that \( \left\| \overline{f}(y) \right\| > 0 \) on \( \bar{A} \), \( 1 \leq q \leq \infty \) fixed.

Consider the kernel
\[
k_{\gamma}(r,t) := k(r,t) := \chi_{(r,R_2)}(t)\psi'(t)(\psi(t) - \psi(r))^{1-\mu} E_{\rho,\mu}[w(\psi(t) - \psi(r))^\rho]
\]
where \( \rho, \mu, \gamma, w > 0; \psi \in C^1([R_1, R_2]) \) which is increasing.

Let
\[
L_{q*}(x) = L_{q*}(r\omega) = \int_{R_1}^{R_2} k_{\gamma}(r,t) \left\| f(t\omega) \right\| dt,
\]
\( x = r\omega \in \bar{A}, \ 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2], \omega \in S^{N-1} \).

We have that \( L_{q*}(r\omega) > 0 \) for \( r \in (R_1, R_2) \), for every \( \omega \in S^{N-1} \).

Here we choose the weight \( u(x) = u(r\omega) = L_{q*}(r\omega) \).

Consider the function
\[
W_{q*}(y) = W_{q*}(t\omega) = \left\| f(t\omega) \right\| \left( \int_{R_1}^{R_2} k_{\gamma}(r,t) dr \right) < \infty,
\]
\( \forall \ t \in [R_1, R_2], \omega \in S^{N-1}; \) and \( W_{q*}(t\omega) \) is integrable over \([R_1, R_2], \forall \omega \in S^{N-1}\).

Here \( \Phi: R_+^{n} \to R \) is a convex and increasing per coordinate function. By (120) we obtain
\[
\int_{R_1}^{R_2} L_{q*}(r\omega) \Phi \left( \frac{e^{y'y} \gamma \mu, w, R_2 - f(r\omega)}{L_{q*}(r\omega)} \right) dr \leq \int_{R_1}^{R_2} W_{q*}(r\omega) \Phi \left( \frac{f(t\omega)}{f(t\omega)} \right) dt,
\]
\( \forall \omega \in S^{N-1} \).

Here we have \( R_1 \leq r \leq R_2 \), and \( R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1} \), and \( R_2^{N-1} \leq r^{1-N} \leq R_1^{1-N} \), also \( r^{N-1} r^{1-N} = 1 \). Thus by (141), we have
\[
\int_{R_1}^{R_2} L_{q*}(r\omega) \Phi \left( \frac{e^{y'y} \gamma \mu, w, R_2 - f(r\omega)}{L_{q*}(r\omega)} \right) r^{N-1} dr \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{R_1}^{R_2} W_{q*}(r\omega) \Phi \left( \frac{f(r\omega)}{f(r\omega)} \right) r^{N-1} dr,
\]
\( \forall \omega \in S^{N-1} \).
Therefore it holds
\[
\int_{S^{N-1}} \left( \int_{R_1}^{R_2} L_{q*}(r\omega) \Phi \left( \frac{e^{\frac{1}{2}} r^{N-1} f(r\omega)}{L_{q*}(r\omega)} \right) dr \right) d\omega \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{S^{N-1}} \left( \int_{R_1}^{R_2} W^{-}_{q*}(r\omega) \Phi \left( \frac{f(r\omega)}{L_{q*}(r\omega)} \right) dr \right) d\omega. 
\] (143)

Using Theorem 39 we derive:

**Theorem 44** All as in Remark 43. Then
\[
\int_{A} L_{q*}(x) \Phi \left( \frac{e^{\frac{1}{2}} x f(x)}{L_{q*}(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A} W^{-}_{q*}(x) \Phi \left( \frac{f(x)}{L_{q*}(x)} \right) dx, 
\] (144)

where \( e^{\frac{1}{2}} r^{N-1} f(x) = (e^{\frac{1}{2}} r^{N-1} f_1(x), \ldots, e^{\frac{1}{2}} r^{N-1} f_n(x)) \) and coordinates are assumed to be continuous functions on \( \overline{A} \).

We need

**Definition 45** Let \( \rho, \mu, w > 0, \gamma < 0, \ N = [\mu], \ \mu \in \mathbb{N}; f \in C^{N}(\overline{A}) \) and \( \psi \in C^{N}([R_1, R_2], \psi'(r) \neq 0, \ \forall \ r \in [R_1, R_2], \) and \( \psi \) is increasing. We define the \( \psi \)-Prabhakar-Caputo radial left and right fractional derivatives of order \( \mu \) as follows (\( x \in \overline{A}, \ x = r\omega, \ r \in [R_1, R_2], \omega \in S^{N-1} \))
\[
\left( \mathcal{C}_x D^{\frac{\rho}{\mu}, \gamma}_{\rho, \mu, w, R_2} f \right)(x) = \left( \mathcal{C}_x D^{\frac{\rho}{\mu}, \gamma}_{\rho, \mu, w, R_1} f \right)(r\omega) := \int_{R_1}^{R_2} \psi'(t)(\psi(r) - \psi(t))^{N-\mu-1} E_{\rho, \mu, w, R_2}^{\frac{\rho}{\mu}, \gamma} \left[w(\psi(r) - \psi(t))^{\rho} \left( \frac{1}{\psi'(r)} \right) \right] f(t\omega) dt 
\] (125)
\[
= (e^{\frac{1}{2}} r^{N-1} f_{\psi}(x)), 
\]
where
\[
f_{\psi}^{[N]}(x) = f_{\psi}(r\omega) := \left( \frac{1}{\psi'(r)} \right)^{\frac{1}{\mu}} f(r\omega), 
\] (126)
is the \( N \) th order \( \psi \)-radial derivative of \( f \),
and
\[
\left( \mathcal{C}_x D^{\frac{\rho}{\mu}, \gamma}_{\rho, \mu, w, R_2} f \right)(x) = \left( \mathcal{C}_x D^{\frac{\rho}{\mu}, \gamma}_{\rho, \mu, w, R_1} f \right)(r\omega) := \left( -1 \right)^{N} \int_{R_1}^{R_2} \psi'(t)(\psi(r) - \psi(t))^{N-\mu-1} E_{\rho, \mu, w, R_2}^{\frac{\rho}{\mu}, \gamma} \left[w(\psi(t) - \psi(r))^{\rho} \left( \frac{1}{\psi'(r)} \right) \right] f(t\omega) dt 
\] (147)
\[
= (e^{\frac{1}{2}} r^{N-1} f_{\psi}(x)), 
\]
\( \forall \ x \in \overline{A} \).
In this work we assume that \( f_n(r, t, \rho, \mu, w, R_1, R_2) \) and \( f_n^*(r, t, \rho, \mu, w, R_1, R_2) \) are continuous functions over \( \overline{A} \).

We make

**Remark 46** Let \( \rho, \mu, w > 0, \gamma < 0, N = [\mu], \mu \notin \mathbb{N}; f_i \in C^N(\overline{A}) \) \( i = 1, \ldots, n \), and \( \overline{f} = (f_1, \ldots, f_n) \), and \( \psi \in C^N([R_1, R_2]), \psi'(r) \neq 0, \forall r \in [R_1, R_2] \) and \( \psi \) is increasing. We follow Definition 45 and we set:

\[
\left\| f^{[N]}_{\psi}(y) \right\| := \max \left\{ f^{[N]}_{\psi}(y), \ldots, f^{[N]}_{\psi}(y) \right\}
\]

and

\[
\left\| f^{[N]}_{\psi}(y) \right\| := \left( \sum_{i=1}^{n} f^{[N]}_{\psi}(y)^{1/n} \right)^{q}, q \geq 1; y \in \overline{A}.
\]

One can write that

\[
\left\| f^{[N]}_{\psi}(y) \right\| = \left\| f^{[N]}_{\psi}(t \omega) \right\|, 1 \leq q \leq \infty,
\]

where \( t \in [R_1, R_2], \omega \in S^{N-1}; y = t \omega \).

Notice that \( \left\| f^{[N]}_{\psi}(y) \right\| \in C(\overline{A}), 1 \leq q \leq \infty. \)

We assume that \( \left\| f^{[N]}_{\psi}(y) \right\| > 0 \) on \( \overline{A}, 1 \leq q \leq \infty \) fixed.

Consider the kernel

\[
k^{+}(r, t) = k(r, t) := \chi_{(R_1, R_2)}(t) \psi'(t)(\psi(r) - \psi(t))^{N-\mu-1} F_{\rho, w}^{-1} \left[ \psi(\psi(r) - \psi(t))^{\gamma} \right]
\]

Let

\[
L^{+}_{q}(x) = \int_{R_1}^{R_2} k^{+}(r, t)\left\| f^{[N]}_{\psi}(t \omega) \right\| dt,
\]

\( x = r \omega \in \overline{A}, 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2], \omega \in S^{N-1}. \)

We have that \( L^{+}_{q}(r \omega) > 0 \) for \( r \in (R_1, R_2), \forall \omega \in S^{N-1}. \)

Here we choose the weight \( u(x) = u(r \omega) = L^{+}_{q}(r \omega). \)

Consider the function

\[
W^{+}_{q}(y) = \int_{R_1}^{R_2} k^{+}(r, t)\left\| f^{[N]}_{\psi}(t \omega) \right\| dt < \infty,
\]

\( \forall t \in [R_1, R_2], \omega \in S^{N-1}; \) and \( W^{+}_{q}(t \omega) \) is integrable over \( [R_1, R_2], \forall \omega \in S^{N-1}. \)
Here $\Phi : \mathbb{R}^n_+ \to \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (145) follows:

**Theorem 47** All as in Remark 46. Then

$$
\int_{A}^{C} L_q(x) \Phi \left( \frac{C D_{\rho,w}^{\gamma + \frac{\gamma}{\mu} + \frac{\gamma}{\mu + w} + f_{n + 1}}(x)}{C L_q(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A}^{C} W_q(x) \Phi \left( \frac{f_{n+1}^{[N]}(x)}{f_{n+1}^{[N]}(y)} \right) dx,
$$

(153)

where $\left( C D_{\rho,w}^{\gamma + \frac{\gamma}{\mu} + \frac{\gamma}{\mu + w} + f_{n + 1}}(x) \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We make

**Remark 48** Let $\rho, \mu, w > 0$, $\gamma < 0$, $N = \lceil \mu \rceil$, $\mu \not\in \mathbb{N}$, $f_i \in C^N(\overline{A})$, $i = 1, \ldots, n$, and $\overline{f} = (f_1, \ldots, f_n)$, and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall \ r \in [R_1, R_2]$, and $\psi$ is increasing. We follow Definition 45 and we set:

$$
\left\| f_{\psi}^{[N]}(y) \right\|_q := \max \left\| f_{\psi}^{[N]}(y) \right\|,
$$

and

$$
\left\| f_{\psi}^{[N]}(y) \right\|_q := \left( \sum_{i=1}^{n} \left\| f_{\psi}^{[N]}(y) \right\|^q \right)^{1/q}, q \geq 1, y \in \overline{A}.
$$

One can write that

$$
\left\| f_{\psi}^{[N]}(y) \right\|_q = \left\| f_{\psi}^{[N]}(t \omega) \right\|_q, 1 \leq q \leq \infty,
$$

(154)

(155)

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$, $y = t \omega$.

Notice that $\left\| f_{\psi}^{[N]}(y) \right\|_q \in C(\overline{A})$, $1 \leq q \leq \infty$.

We assume that $\left\| f_{\psi}^{[N]}(y) \right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$$
c_k(r, t) := k(r, t) := \chi_{[r, R_2]}(t) \psi(t)(\psi(t) - \psi(r))^{\frac{N-1}{\mu} - \gamma} E_{\rho, N-\mu}^{-\gamma} \left[ w(t) \psi(t) - \psi(r) \right]^{\gamma}
$$

(156)

Let

$$
c L_q(x) = c L_q(r \omega) = \int_{R_1}^{R_2} c k(r, t) \left\| f_{\psi}^{[N]}(t \omega) \right\|_q dt,
$$

(157)

where $r \omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.
We have that $C L_q(r \omega) > 0$ for $r \in (R_1, R_2]$, $\forall \omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r \omega) = C L_q(r \omega)$.

Consider the function

$$C W_q(y) = \frac{C}{C L_q(x)} \left[ \int_{A}^{R_2} k^r(r, t) \, dr \right] < \infty,$$

$\forall t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $C W_q(t \omega)$ is integrable over $[R_1, R_2]$, $\forall \omega \in S^{N-1}$.

Here $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (147) follows:

**Theorem 49** All as in Remark 48. Then

$$\int_{A}^{C L_q(x)} \left[ \left( \frac{C}{C L_q(x)} \int_{R_1}^{R_2} x \right) \right] dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A}^{C L_q(x)} \left[ \left( \frac{C}{C L_q(x)} \int_{R_1}^{R_2} x \right) \right] dx,$$

(159)

where $\left( \frac{C}{C L_q(x)} \int_{R_1}^{R_2} x \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We need

**Definition 50** Let $\rho, \mu > 0$, $\gamma < 0$, $N = \left\lfloor \mu \right\rfloor$, $\mu \notin \mathbb{N}$, $f \in C(A)$ and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall r \in [R_1, R_2]$ and $\psi'$ is increasing. The $\psi'$-Prabhakar-Riemann Liouville left and right radial fractional derivatives of order $\mu$ are defined as follows (see also Definition 40)

$$\left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_1} f \right)(x) = \left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_1} f \right)(r \omega) := \left( \frac{1}{\psi'(r)} \right)^N \left( \frac{\rho}{\rho, N - \mu, w, R_1} f \right)(x),$$

(160)

and

$$\left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_2} f \right)(x) = \left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_2} f \right)(r \omega) := \left( -\frac{1}{\psi'(r)} \right)^N \left( \frac{\rho}{\rho, N - \mu, w, R_2} f \right)(x),$$

(161)

$\forall x \in \overline{A}$; where $x = r \omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

In this work we assume that $\left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_1} f \right)$, $\left( \frac{RL}{R} D^{\gamma, \psi}_{\rho, \mu, w, R_2} f \right) \in C(A)$.

Next we define the $\psi'$-Hilfer-Prabhakar left and right radial fractional derivatives of order $\mu$ and type $\beta \in [0, 1]$, as follows ($\xi := \mu + \beta(N - \mu)$, see also Definition 40):
Consider the kernel

\[ k^+(r,t) := k(r,t) := \chi_{[R_1,R_2]}(t) \psi(t) - \psi(t) \int_{R_1}^{R_2} E^{-\beta \rho \xi - \mu \omega} \left[ w(\psi(t) - \psi(t)) \right] dt \]

Let

\[ L_q^p(x) = \int_{R_1}^{R_2} k^+(r,t) \left\| R^y \mathcal{D}^{(1-\beta)\psi}_{\rho,\xi,w,R_1+} f(x) \right\|_q dt, \]

where \( x \in \overline{A} \), and \( x = r \omega, r \in [R_1,R_2], \omega \in S^{N-1} \).

We make

**Remark 51** Let \( \rho, \mu, w > 0, \gamma < 0, N = \mu, \mu \not\in \mathbb{N}; 0 \leq \beta \leq 1, \xi = \mu + \beta(N-\mu), f_i \in C(\overline{A}), i = 1,...,n \), and \( \psi \in C^\gamma([R_1,R_2]), \psi'(r) \neq 0, \forall r \in [R_1,R_2] \) and \( \psi \) is increasing. We follow Definition 50, especially (162) and we set:

\[
\left\| R^y \mathcal{D}^{(1-\beta)\psi}_{\rho,\xi,w,R_1+} t \omega \right\|_q := \max \left\{ \left\| R^y \mathcal{D}^{(1-\beta)\psi}_{\rho,\xi,w,R_1+} f_i(x) \right\| \right\}, \quad i = 1,...,n
\]

and

\[
\left\| R^y \mathcal{D}^{(1-\beta)\psi}_{\rho,\xi,w,R_1+} f(x) \right\|_q := \left( \sum_{i=1}^{n} \left\| R^y \mathcal{D}^{(1-\beta)\psi}_{\rho,\xi,w,R_1+} f_i(x) \right\|_{q}^{-\gamma} \right)^{\frac{1}{\gamma}}, \quad q \geq 1; y \in \overline{A}.
\]
We have that $P L^+_q(r \omega) > 0$ for $r \in (R_1, R_2)$, $\forall \omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r \omega) = P L^+_q(r \omega)$.

Consider the function

$$P W^+_q(y) = P W^+_q(t \omega) = \left\| \frac{RL}{R} D_{\rho, \zeta, w, R_1 + f} \right\|_q \left( \int_{R_1}^{R_2} P k^+(r, t) dr \right) < \infty,$$

(168)

$\forall t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $P W^+_q(t \omega)$ is integrable over $[R_1, R_2]$, $\forall \omega \in S^{N-1}$.

Here $\Phi : R_1^d \to R$ is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (162) follows:

**Theorem 52** All as in Remark 51. Then

$$\int_A P L^+_q(x) \Phi \left( \frac{RL}{R} D_{\rho, \zeta, w, R_1 + f}(x) \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_A P W^+_q(x) \Phi \left( \frac{RL}{R} D_{\rho, \zeta, w, R_1 + f}(x) \right) dx,$$

(169)

where $\frac{RL}{R} D_{\rho, \zeta, w, R_1 + f}(x) = \left( \frac{RL}{R} D_{\rho, \zeta, w, R_1 + f_1}(x), \ldots, \frac{RL}{R} D_{\rho, \zeta, w, R_1 + f_n}(x) \right)$ and the coordinates are assumed to be continuous on $A$.

We make

**Remark 53** Let $\rho, \mu, w > 0$, $\gamma < 0$, $N = \lfloor \mu \rfloor$, $\mu \notin \mathbb{N}$; $0 \leq \beta \leq 1$, $\xi = \mu + \beta(N - \mu)$, $f_1 \in C(\overline{A})$, $i = 1, \ldots, n$, and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall r \in [R_1, R_2]$, and $\psi$ is increasing. We follow Definition 50, especially (163) and we set:

$$\left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f_1}(y) \right\|_q := \max \left\{ \left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f_1}(y) \right\|_q, \ldots, \left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f_n}(y) \right\|_q \right\},$$

(170)

and

$$\left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f_1}(y) \right\|_q := \left( \sum_{i=1}^{n} \left( \left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f_1}(y) \right\|_q \right)^q \right)^{1/2}, q \geq 1; y \in \overline{A}.$$

One can write that

$$\left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f}(y) \right\|_q = \left\| \frac{RL}{R} D_{\rho, \zeta, w, R_2 - f}(t \omega) \right\|_q, 1 \leq q \leq \infty,$$

(171)
where \( t \in [R_1, R_2] \), \( \omega \in S^{N-1}, \ y = t\omega \).

Notice that \( \left\| \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) y \right\|_{q} \in C(A), \ 1 \leq q \leq \infty. \)

We assume that \( \left\| \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) y \right\|_{q} > 0 \) on \( A, \ 1 \leq q \leq \infty \) fixed.

Consider the kernel
\[
p_k(r, t):= k(r, t):= \chi_{[r_1, r_2]}(t) \psi(t) \left( \psi(t) - \psi(r) \right)^{\mu - 1} E^{\gamma \beta}_{\rho, \xi + \mu} \left[ w \left( \psi(t) - \psi(r) \right) \right]
\]
(172)

Let
\[
p L_q(x) = p L_q(r \omega) = \int_{R_1}^{R_2} p_k(r, t) \left\| \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) t \omega \right\|_{q} dt,
\]
(173)

\( x = r \omega \in A, \ 1 \leq q \leq \infty \) fixed; \( r \in [R_1, R_2], \ \omega \in S^{N-1}. \)

We have that \( p L_q(r \omega) > 0 \) for \( r \in (R_1, R_2), \ \forall \ \omega \in S^{N-1}. \)

Here we choose the weight \( u(x) = u(r \omega) = p L_q(r \omega). \)

Consider the function
\[
p W_q^{-} (y) = p W_q^{-} (r \omega) = \left\| \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) t \omega \right\|_{q} \left( \int_{R_1}^{R_2} p_k(r, t) dt \right) < \infty,
\]
(174)

\( \forall \ t \in [R_1, R_2], \ \omega \in S^{N-1}; \ \text{and} \ p W_q^{-} (r \omega) \text{ is integrable over } [R_1, R_2], \ \forall \ \omega \in S^{N-1}. \)

Here \( \Phi : R_+^n \rightarrow R \) is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (163) follows:

**Theorem 54** All as in Remark 53. Then

\[
\int_{A} p L_q^{-}(x) \Phi \left( \frac{\left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) x}{p L_q^{-}(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A} p W_q^{-}(x) \Phi \left( \frac{\left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) x}{p W_q^{-}(x)} \right) dx,
\]
(175)

where \( \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} \right) x = \left( \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} f_1(x), \ldots, \frac{RL}{R} D_{\rho, \xi, w, R_2}^{(1-\beta)\gamma} f_n(x) \right) \) and the coordinates are assumed to be continuous on \( A \).

We make

**Remark 55** Let \( f_{ji} \in C([a, b]), \ j = 1, 2; \ i = 1, \ldots, m; \ \psi \in C^1([a, b]) \) which is increasing. Let also \( \rho_i, \mu_i, \gamma_i, \omega_i > 0 \).
and \( \{ e_{p_i,\mu_i,\sigma_i,a+f_{ji}}(x) \}, \ x \in [a,b] \) as in (2). We assume here that \( 0 < f_{ji}(y) < \infty \) on \([a,b]\), \( i = 1,\ldots,m \).

Here we consider the kernel

\[
k^{-}_i(x,y) := k_i(x,y) = \begin{cases} 
\psi'(y)(\psi(x) - \psi(y))^{\mu_i-1} E_{\mu_i}^\psi \left[ \omega_i(\psi(x) - \psi(y))^{\rho_i} \right] a < y \leq x, \\
0, \ a < y < b, 
\end{cases}
\]

\( i = 1,\ldots,m. \)

Choose weight \( u(x) \geq 0 \), so that

\[
\psi_i(y) := f_{ji}(y) \int_y^b u(x) \frac{k^{-}_i(x,y)}{e_{p_i,\mu_i,\sigma_i,a+f_{ji}}(x)} \, dx < \infty,
\]

a.e. on \([a,b]\), and that \( \psi_i \) is integrable on \([a,b]\), \( i = 1,\ldots,m \).

Theorem 9 immediately implies:

**Theorem 56** All as in Remark 55. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+ \), \( i = 1,\ldots,m \), be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \left\{ e_{p_i,\mu_i,\sigma_i,a+f_{ji}}(x) \right\} \, dx \leq \prod_{i=1}^m \left\{ \int_a^b \psi_i(y) \Phi_i \left( \frac{f_{ji}(y)}{f_{ji}(y)} \right) \, dy \right\}^{\frac{1}{p_i}}.
\]

We make

**Remark 57** Let \( f_{ji} \in C([a,b]) \), \( j = 1,2; i = 1,\ldots,m \); \( \psi \in C^1([a,b]) \) which is increasing. Let also \( \rho_i, \mu_i, \gamma_i, \omega_i > 0 \) and \( \{ e_{p_i,\mu_i,\sigma_i,a+f_{ji}}(x) \}, \ x \in [a,b] \) as in (3). We assume here that \( 0 < f_{ji}(y) < \infty \) on \([a,b]\), \( i = 1,\ldots,m \).

Here we consider the kernel

\[
k^{-}_i(x,y) := k_i(x,y) = \begin{cases} 
\psi'(y)(\psi(y) - \psi(x))^{\mu_i-1} E_{\mu_i}^\psi \left[ \omega_i(\psi(y) - \psi(x))^{\rho_i} \right] x \leq y < b, \\
0, \ a < y < x, 
\end{cases}
\]

\( i = 1,\ldots,m. \)

Choose weight \( u(x) \geq 0 \), so that
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\[ \bar{\psi}_i(y) := f_{2i}(y) \int_a^y u(x) \frac{k_{ij}(x, y)}{\psi_{ij}(x)} \, dx < \infty, \quad (180) \]

a.e. on \([a, b]\), and that \( \bar{\psi}_i \) is integrable on \([a, b] \), \( i = 1, \ldots, m \).

Theorem 9 immediately implies:

**Theorem 58** All as in Remark 57. Let \( p_i > 1 \); \( \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R} \rightarrow \mathbb{R}, \quad i = 1, \ldots, m \), be convex and increasing. Then

\[ \int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{k_{ij}(x)}{\psi_{ij}(x)} \right) \, dx \leq \prod_{i=1}^m \left( \int_a^y \psi_i(y) \Phi_i \left( \frac{f_{ii}(y)}{f_{22i}(y)} \right) \, dy \right)^{\frac{1}{p_i}}. \quad (181) \]

We make

**Remark 59** Let \( j = 1, 2; i = 1, \ldots, n; \ \rho_i, \mu_i, \omega_i > 0, \ \gamma_i < 0, \ N_i = \left[ \mu_i \right], \ \mu_i \notin \mathbb{N}; \ \theta := \max(N_1, \ldots, N_m), \ \psi \in C^2([a, b]), \ \psi'(x) \neq 0 \) over \([a, b] \), \( \psi \) is increasing; \( f_{ji} \in C^N([a, b]) \) and \( f^{[N]}_{ji}(x) = \left( \frac{1}{\psi(x)} \frac{dy}{dx} \right)^{\mu_i} f_{ji}(x), \ \forall \ x \in [a, b] \). Here

\[ \left( c D_{\rho_i, \mu_i, \omega_i, a+b}^{\gamma_i} f_{ji} \right)(x) = \left( e^{\gamma_i} D_{\rho_i, \mu_i, \omega_i, a+b}^{\gamma_i} f^{[N]}_{ji} \right)(x), \quad (182) \]

\( \forall \ x \in [a, b], \ j = 1, 2; i = 1, \ldots, m. \)

We assume that \( 0 < f_{2i}^{[N]}(y) < \infty \) on \([a, b] \), \( i = 1, \ldots, m. \)

Here we consider the kernel

\[ c k_i(x, y) := k_i(x, y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{N_i - \mu_i - 1} E_{\rho_i, \mu_i, \omega_i}^{\gamma_i} [\omega_i(\psi(x) - \psi(y))^{\mu_i}], & a < y \leq x, \\ 0, & x < y < b, \end{cases} \quad (183) \]

\( i = 1, \ldots, m. \)

Choose weight \( u \geq 0 \), so that

\[ c \psi_i(y) := f_{2i}^{[N]}(y) u(x) \int_a^y \frac{c k_i(x, y)}{c D_{\rho_i, \mu_i, \omega_i, a+b}^{\gamma_i} f_{2i}(x)} \, dx < \infty, \quad (184) \]

a.e. on \([a, b]\), and that \( c \psi_i \) is integrable on \([a, b] \), \( i = 1, \ldots, m. \).
Theorem 56 immediately produces:

**Theorem 60** All as in Remark 59. Let $p_i > 1; \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+, i = 1, \ldots, m$, be convex and increasing. Then

$$
\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left( \frac{c D_{\gamma_i,\mu_i,\omega_i} f_i(x)}{c D_{\gamma_i,\mu_i,\omega_i} f_i(x)} \right) dx \leq \prod_{i=1}^{m} \int_a^b \psi_i(y) \Phi_i \left( \frac{f_i^{[N_i]}(y)}{f_i^{[N_i]}(y)} \right) dy \left( \frac{1}{p_i} \right) \frac{1}{p_i}.
$$

(185)

We make

**Remark 61** Let $j = 1, 2; i = 1, \ldots, n; \rho_i, \mu_i, \omega_i > 0, \gamma_i < 0, N_i = [\mu_i], \mu_i \in \mathbb{N}; \theta := \max(N_1, \ldots, N_m), \psi \in C^\alpha([a, b]), \psi'(x) \neq 0$ over $[a, b], \psi$ is increasing; $f_{ji} \in C^{N_i}([a, b])$ and $f_{ji}^{[N_i]}(x) = \left( \frac{1}{\psi'(x)} \right) f_{ji}^{(N_i)}(x) \forall x \in [a, b]$. Here

$$
\left( c D_{\rho_i,\mu_i,\omega_i, b} f_{ji}^{(N_i)}(x) \right)^{(7)} = (1)^{N_i} \left( e^{-\gamma_i} \right) f_{ji}^{(N_i)}(x),
$$

(186)

$$
\forall x \in [a, b], j = 1, 2; i = 1, \ldots, m.
$$

We assume that $0 < f_{ji}^{[N_i]}(y) < \infty$ on $[a, b], i = 1, \ldots, m$.

Here we consider the kernel

$$
k_j^{-}(x, y) = \begin{cases} 
\psi'(y)(\psi'(y) - \psi'(x))^{N_i - \mu_i - 1} E_{\rho_i,\mu_i,\omega_i, b}^{-\gamma_i} [\omega_i(\psi'(y) - \psi'(x))^{\rho_i}], x < y < b, \\
0, a < y < x,
\end{cases}
$$

(187)

$i = 1, \ldots, m$.

Choose weight $u \geq 0$, so that

$$
c \overline{\psi}_i(y) := f_{ji}^{[N_i]}(y) \frac{c k_j^{-}(x, y)}{c D_{\rho_i,\mu_i,\omega_i, b} f_{ji}^{(N_i)}(x)} dx < \infty,
$$

(188)

a.e. on $[a, b]$, and that $c \overline{\psi}_i$ is integrable on $[a, b], i = 1, \ldots, m$.

Theorem 58 immediately produces:

**Theorem 62** All as in Remark 61. Let $p_i > 1; \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+, i = 1, \ldots, m$, be convex and increasing. Then
\[
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{\left[ c D_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{i1}}(x) \right]}{c D_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{2i}}(x)} \right) \frac{1}{r_i} \leq \prod_{i=1}^m \int_a^b \psi_i(y) \Phi_i \left( \frac{\left[ f_{i1}(y) \right]}{f_{2i}(y)} \right) dy \tag{189} \]

We make

**Remark 63** Let \( j = 1, 2; i = 1, \ldots, m; \rho_i, \mu_i, \omega_i > 0, \gamma_i < 0, N_i = \lceil \mu_i \rceil, \mu_i \in \mathbb{N}; \theta := \max(N_1, \ldots, N_m), \psi \in C^0([a, b]), \psi^3(x) \neq 0 \) over \([a, b], \psi\) is increasing; \( f_{ji} \in C([a, b])\). Let \( 0 \leq \beta_i \leq 1 \) and \( \xi_i = \mu_i + \beta_i(N_i - \mu_i), i = 1, \ldots, m \). We assume that \( RL_{\rho_1, \eta_1, \alpha_1, a} f_{ji} \in C([a, b]) \) and \( 0 < RL_{\rho_1, \eta_1, \alpha_1, a} f_{ji} < \infty \) on \([a, b], i = 1, \ldots, m\). Here we have

\[
\psi^i(x) = e^{-\gamma_i/\rho_i} \rho_i^{-\gamma_i} \left[ \omega_i \psi(x) - \psi(y) \right]^{\xi_i - \mu_i} a < y \leq x, \tag{190} \]

\[
0, \quad x < y < b, \tag{191} \]

\( i = 1, \ldots, m \).

Choose weight \( u \geq 0 \), so that

\[
\psi_i(y) := RL_{\rho_1, \eta_1, \alpha_1, a} f_{ji}(y) \int_a^y u(x) k_i^+(x, y) \frac{dx}{u(x) k_i^+(x, y)} dx < \infty, \tag{192} \]

a.e. on \([a, b]\), and that \( \psi_i \) is integrable on \([a, b], i = 1, \ldots, m\).

Theorem 56 immediately produces:

**Theorem 64** All as in Remark 63. Let \( p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1 \). Let the functions \( \Phi_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, \ldots, m \), be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{\left[ RL_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{i1}}(x) \right]}{RL_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{2i}}(x)} \right) \frac{1}{r_i} \leq \prod_{i=1}^m \int_a^b \psi_i(y) \Phi_i \left( \frac{\left[ RL_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{i1}}(y) \right]}{RL_{\rho_i, \eta_i, \alpha_i, \beta_i, f_{2i}}(y)} \right) dy \tag{193} \]

We make
Remark 65 Let $j=1,2; i=1,...,m; \rho_j, \mu_i, \alpha_i > 0, \gamma_i < 0, N_i = \lceil \mu_i \rceil, \mu_i \notin \mathbb{N}; \theta := \max(N_1,...,N_m), \psi \in C^0([a,b]), \psi'(x) \neq 0$ over $[a,b], \psi$ is increasing; $f_{ji} \in C([a,b])$. Let $0 \leq \beta_i \leq 1$ and $\xi_i = \mu_i + \beta_i(N_i - \mu_i), i=1,...,m$. We assume that $RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \in C([a,b])$ and $0 < RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi(y) \leq \infty$ on $[a,b], i=1,...,m$. Here we have

\[
\left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(x) = e^{-\gamma_i \rho_j \psi} \left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(x),
\]

\[
\forall \ x \in [a,b], \ j=1,2; i=1,...,m.
\]

Here we consider the kernel

\[
p^{\rho_j}(x,y) := k_i(x,y) = \begin{cases} 
\psi'(y)(\psi(y) - \psi(x))^{\xi_{j-i-1}} e^{-\gamma_i \rho_j \psi} \left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(x), & x \leq y < b, \\
0, & a < y < x,
\end{cases}
\]

\[
i=1,...,m.
\]

Choose weight $u \geq 0$, so that

\[
p^{-\psi_i}(y) := \left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(y) \int_a^y u(x) p^{\rho_j}(x,y) \, dx \leq \infty,
\]

\[
a.e. \ on \ [a,b], \ and \ that \ \ u^{-\psi_i} \ is \ integrable \ on \ [a,b], \ i=1,...,m.
\]

Theorem 58 immediately produces:

**Theorem 66** All as in Remark 65. Let $p_i > 1; \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : R_+ \rightarrow R_+, i=1,...,m$, be convex and increasing. Then

\[
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(x) \right) dx \leq \prod_{i=1}^m \int_a^b u^{-\psi_i}(y) \Phi_i \left( \left( RL_{\rho_j,\xi_i,\alpha_i,b-f_{ji}} D_{\rho_j,\xi_i,\alpha_i,b-f_{ji}}^{\gamma_i} \psi \right)(y) \right) dy \prod_{i=1}^m \frac{1}{p_i}.
\]

**References**


